THE GROUND STATE PROBLEM FOR A QUANTUM HAMILTONIAN MODEL DESCRIBING FRICTION

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Abstract: In this paper, we consider the quantum version of a hamiltonian model describing friction. This model consists of a particle which interacts with a bosonic reservoir representing a homogeneous medium through which the particle moves. We show that if the particle is confined, then the Hamiltonian admits a ground state if and only if a suitable infrared condition is satisfied. The latter is violated in the case of linear friction, but satisfied when the friction force is proportional to a higher power of the particle speed.

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1 Introduction

In [BDB], together with S. De Bièvre, we introduced a classical Hamiltonian model of a particle moving through a homogeneous dissipative medium at zero temperature in such a way that the particle experiences an effective *linear* friction force proportional to its velocity. The medium consists at each point in the space of a vibration field with which the particle exchanges energy and momentum. More precisely the Hamiltonian is given by

$$\begin{aligned} H(q,p,\phi,\pi) &= \frac{p^2}{2} + V(q) + \frac{1}{2}\int_{\mathbb{R}^d} \mathrm{d}x \int_{\mathbb{R}^n} \mathrm{d}y\, c^2|\nabla_y \phi(x,y)|^2 + |\pi(x,y)|^2 \\ &\quad + \int_{\mathbb{R}^d} \mathrm{d}x \int_{\mathbb{R}^n} \mathrm{d}y\, \rho_1(x-q)\rho_2(y)\phi(x,y), \end{aligned} \tag{1.1}$$

where V is an external potential, c represents the speed of the wave propagation in the "membranes" and the functions ρ_1 and ρ_2 determine the coupling between the particle and the field and are smooth radial functions with compact support.

We studied the asymptotic behaviour of the particle motion for two categories of potentials: linear ones (which means constant external force) and confining ones. We proved that under suitable assumptions (on the initial conditions), for c sufficiently large and, most importantly, $n = 3$, the particle behaves asymptotically as if its motion was governed by the effective equation

$$\ddot{q}(t) + \gamma\dot{q}(t) = -\nabla V(q(t)),$$

where the friction coefficient γ is non negative and is explicit in terms of the parameters of the model:

$$\gamma := \frac{\pi}{c^3}|\hat{\rho}_2(0)|^2 \int_{\mathbb{R}^n} \mathrm{d}\xi \int_{\mathbb{R}^{d-1}} \mathrm{d}\eta \, |\hat{\rho}_1(|\xi|, \eta)|^2. \tag{1.2}$$

If $V = -F \cdot q$, which means that we apply a constant external force F to the particle, then this particle reaches exponentially fast (with rate γ) an asymptotic velocity $v(F) = \frac{F}{\gamma}$ which is proportional to the applied force (for small forces). This is, in particular, at the origin of Ohm's law. On the other hand, if V is confining, the particle stops at one of the critical points of the potential, the convergence rate being still exponential (but with rate $\frac{\gamma}{2}$ as expected from the effective equation).

In [BDB] we mostly concentrated on linear friction. This is why the $n = 3$ assumption was required. However, for other values of $n\,(> 3)$, our model still describes friction. Indeed, the reaction force of the environment on a particle moving with velocity v takes the form $-\gamma|v|^{n-3}v$ (for small v and where γ is defined in (1.2)). So one can see that we have linear friction when $n = 3$, and otherwise a friction force which is proportional to some other power of the particle velocity.

Such models, where a small system interacts with a *large* environment, are called open systems. The reason for studying those models is usually to have a Hamiltonian description of dissipative phenomena. There exist several mechanisms leading to dissipation. Among them, two important, and very different, mechanisms are radiation damping and friction (which can be linear or not). As far as radiation damping is concerned, there exist many models, which are more or less related to electromagnetism. One example is the "classical Nelson model"

$$H_{\text{nels}} = \frac{p^2}{2} + V(q) + \frac{1}{2}\int_{\mathbb{R}^d} \mathrm{d}x \left(|\nabla\phi(x)|^2 + |\pi(x)|^2\right) + \int_{\mathbb{R}^d} \rho(x-q)\phi(x)\mathrm{d}x,$$

which has been studied in [KKS] (except for the kinetic energy of the particle which was $\sqrt{p^2+1}$ instead of $\frac{p^2}{2}$). This model describes a particle interacting with a scalar radiation field, and exhibits radiation damping. Concerning friction, although there exist various Hamiltonian models in the literature, ours is the only one we are aware of that describes the friction produced by the motion of the particle through a homogeneous medium. Despite the formal similarity between our model and the classical Nelson model, we want to stress once again that they describe physically totally different phenomena. This is reflected in mathematical differences that will become apparent below.

Our goal in this paper is to begin the study of the quantum version of the model (1.1). Since the speed of the wave propagation will not play any role in our paper, we take it equal to 1. The quantum Hamiltonian then writes as follows

$$\begin{aligned} H &= (-\Delta + V)\otimes \mathbb{1} + \mathbb{1}\otimes \int \mathrm{d}x\,\mathrm{d}k\,\omega(x,k)a^*(x,k)a(x,k) \\ &\qquad + \int \mathrm{d}x\,\mathrm{d}k\,\frac{\rho_1(x-Q)\hat{\rho}_2(k)}{\sqrt{2\omega(x,k)}} \otimes a^*(x,k) + h.c., \end{aligned}$$

where a and a^* are the usual annihilation and creation operators on the bosonic Fock space $\mathcal{F}(L^2(\mathbb{R}^{d+n}, dx\, dk))$, and $\omega(x, k) = |k|$ is the bosons dispersion relation. In this paper, we start with the study of confining potentials which are less difficult. More precisely, we deal with the question of existence of a ground state. If a Hamiltonian is bounded from below, we say that it admits a ground state if the infimum of its spectrum is an eigenvalue. We call ground state energy this infimum and ground state any corresponding eigenvector if it exists. We will prove that such a ground state exists provided the following *infrared condition* is satisfied (Theorem 3.1):

$$\int_{\mathbb{R}^n} \mathrm{d}k\, \frac{|\hat{\rho}_2(k)|^2}{|k|^3} < +\infty.$$

Let us suppose that $\hat{\rho}_2(0) \neq 0$. Indeed, this is the only interesting case since the friction coefficient γ vanishes together with $\hat{\rho}_2(0)$ (see (1.2)). One can see that the infrared condition is fulfilled when the friction is non-linear ($n \geq 4$). On the other hand, for linear friction, there is generically no ground state (Proposition 3.2). Thus, we have a class of models, depending on a parameter n, describing friction phenomena, linear or proportional to a power of the velocity of the particle, for which we are able to say whether they admit a ground state or not.

We will describe precisely the quantum version of the model in Sect. 2, and we state our main results in Sect. 3.

To prove the existence of a ground sate, we follow the standard strategy: we first prove the result for coupling to a massive field and then we let the mass tend to zero. We study the massive case in Sect. 4 along the lines of [BFS1, BFS2, GJ] and the "zero mass" limit in Sect. 5 adapting the proof of [G] to our model. In the two parts of the proof, the main mathematical difference (and difficulty) with the models for radiation damping comes from the fact that the dispersion relation ω does not depend on x. Hence we have no *a priori* control on the momentum of the bosons in the "x-direction". A second difficulty which arises comes from the fact that, in the interaction term, the norm of $\rho_1(x - Q)$ as an operator on $L^2(\mathbb{R}^d)$ does not depend on x. In order to control this problem, we will need to use the exponential decay of the spectral projectors in the Q variable. The proof of Proposition 3.2 is also given in Sect. 5. Some of the proof are omitted or only briefly sketched: see [B1, B2] for more details.

2 Description of the Model

In this section, we introduce the quantum version of the model introduced in Sect. 1. The dynamics of the particle is given by the Schrödinger operator $H_p = -\Delta + V$ on $L^2(\mathbb{R}^d)$. Troughout this paper we will only consider confining potentials, so that H_p has a compact resolvent and purely discrete spectrum.

The Hilbert space for the environment will be the bosonic Fock space over $L^2(\mathbb{R}^{d+n}, dx\, dk)$. In what follows, we will just write $\mathcal{F} := \mathcal{F}(L^2(\mathbb{R}^{d+n}, dx\, dk))$. The Hamiltonian of the field is given by $H_f := \mathrm{d}\Gamma(\omega)$, where ω is the multiplication operator on $L^2(\mathbb{R}^{d+n}, dx\, dk)$ by the function $\omega(x,k) = |k|$. The function ω depends only on k, so we will write $\omega(k)$ for $\omega(x,k)$. It is well known that one can rewrite H_f using the creation and annihilation operators as follows:

$$H_f = \int_{\mathbb{R}^{d+n}} \mathrm{d}x\, \mathrm{d}k\, \omega(k) a^*(x,k) a(x,k). \tag{2.1}$$

We can now describe the full system. The Hilbert space is the tensor product of the particle space and of the environment one, namely $\mathcal{H} := L^2(\mathbb{R}^d) \otimes \mathcal{F}$, and the free Hamiltonian (*i*.e. without interaction) is given by $H_0 := H_p \otimes \mathbb{1} + \mathbb{1} \otimes H_f$. The interaction term is given by

$$H_I := \int \mathrm{d}x\, \mathrm{d}k\ \rho_1(x-Q) \left(\frac{\hat{\rho}_2(k)}{\sqrt{2\omega(k)}} \otimes a^*(x,k) + \frac{\bar{\hat{\rho}}_2(k)}{\sqrt{2\omega(k)}} \otimes a(x,k) \right), \tag{2.2}$$

where ρ_1 and ρ_2 are two smooth functions with compact support and spherical symmetry, and $\rho_1(x-Q)$ is the multiplication operator on $L^2(\mathbb{R}^d)$ by the function $\rho_1(x-\cdot)$. Finally, the full Hamiltonian of the interacting system is therefore

$$H := H_0 + H_I. \tag{2.3}$$

3 Main Results

3.1 *Selfadjointness*

From now, we will suppose that $n \geq 3$. We first give the precise condition we impose on the potential V :

(C) $\quad V \in L^2_{loc}(\mathbb{R}^d), \lim_{|q|\to\infty} V(q) = +\infty.$

This hypothesis ensures that H_p is well defined and is selfadjoint on $\mathcal{D}(H_p) = \{\psi \in L^2(\mathbb{R}^d) | H_p\psi \in L^2(\mathbb{R}^d)\}$ ([RS2], Theorem X.28). We also know that H_f is selfadjoint on its domain $\mathcal{D}(H_f)$ ([RS1], Chapter VIII.10). One then easily proves that H_0 is essentially selfadjoint on $\mathcal{D}(H_p) \otimes \mathcal{D}(H_f)$ ([RS1], Chapter VIII.10). We now have the following result

Proposition 3.1. *Suppose that $n \geq 3$, and V satisfies condition (C). Then H is selfadjoint on $\mathcal{D}(H) = \mathcal{D}(H_0)$. Moreover, H is essentially selfadjoint on any core for H_0, and it is bounded from below.*

This is in the standard way a consequence of the Kato-Rellich theorem ([RS2], Theorem X.12). The only ingredient needed is that H_I is infinitesimally H_0-bounded, which follows from the following lemma.

Lemma 3.1. *Under the hypothesis of Proposition 3.1, for all $\Psi \in \mathcal{D}(H_0)$, we have:*

$$\| \int \mathrm{d}x\, \mathrm{d}k \frac{\hat{\rho}_2(k)}{\sqrt{\omega(k)}} \rho_1(x-Q) \otimes a(x,k) \Psi \|^2_{\mathcal{H}} \leq \Big[\int \mathrm{d}x\, \mathrm{d}k |\rho_1(x)|^2 \frac{|\hat{\rho}_2(k)|^2}{\omega(k)^2} \Big] \| (\mathbb{1} \otimes H_f^{\frac{1}{2}}) \Psi \|^2_{\mathcal{H}}.$$

and

$$\| \int \mathrm{d}x\, \mathrm{d}k \frac{\hat{\rho}_2(k)}{\sqrt{\omega(k)}} \rho_1(x-Q) \otimes a^*(x,k)\Psi\|^2_{\mathcal{H}} \leq \Big[\int \mathrm{d}x \mathrm{d}k |\rho_1(x)|^2 \frac{|\hat{\rho}_2(k)|^2}{\omega(k)^2} \Big] \|(\mathbb{1} \otimes H_f^{\frac{1}{2}})\Psi\|^2_{\mathcal{H}}$$

$$+ \Big[\int \mathrm{d}x\, \mathrm{d}k |\rho_1(x)|^2 \frac{|\hat{\rho}_2(k)|^2}{\omega(k)} \Big] \|\Psi\|^2_{\mathcal{H}}.$$

Remark 3.1. *Such kind of estimates are well known [A1, BFS1, DJ] and are sometimes called $N_\tau-$ estimates. The $n \geq 3$ hypothesis ensures that the integrals on the right-hand side of both inequalities converge.*

Proof of Lemma 3.1: We use the fact that $\mathcal{H}$ is isomorphic to $L^2(\mathbb{R}^d, dq, \mathcal{F})$. We then have:

$$\| \int \mathrm{d}x\, \mathrm{d}k \frac{\hat{\rho}_2(k)}{\sqrt{\omega(k)}} \rho_1(x-Q) \otimes a(x,k)\Psi\|^2_{\mathcal{H}} = \int_{\mathbb{R}^d} \mathrm{d}q \|a(g_q)\Psi(q)\|^2_{\mathcal{F}}, \tag{3.1}$$

where g_q is the function $g_q(x,k) = \frac{\hat{\rho}_2(k)}{\sqrt{\omega(k)}} \rho_1(x-q)$. By a standard computation, one has (see [BFS1], Lemma I.6):

$$\|a(g_q)\Psi(q)\|^2_{\mathcal{F}} \leq \Big[\int \mathrm{d}x\, \mathrm{d}k |\rho_1(x)|^2 \frac{|\hat{\rho}_2(k)|^2}{\omega(k)^2} \Big] \|H_f^{1/2}\Psi(q)\|^2_{\mathcal{F}},$$

which, together with (3.1), proves the first inequality. One proves the second one in a similar way. □

3.2 *Existence of a ground state*

Let E_0 denote the ground state energy of H. It is well known that one of the main obstacles to the existence of a ground state, in those models where a particle interacts with a field, comes from the so-called *infrared catastrophe*, which is due to the behaviour of $\omega(k)$ for small k and in particular to the fact that $\omega(0) = 0$. We will then need the following "infrared condition" on the coupling:

(IR) $\quad \int_{\mathbb{R}^n} \mathrm{d}k \, \frac{|\hat{\rho}_2(k)|^2}{\omega(k)^3} < +\infty.$

The main result of our paper is the following.

Theorem 3.1. *Suppose $n \geq 3$, V satisfies hypothesis (C), and $\hat{\rho}_2$ satisfies (IR). Then H has a ground state.*

As we said in the introduction, this (IR) condition is satisfied when the friction is non-linear but not if it is linear. On the other way, in the case of the Nelson model, the same kind of condition is necessary and sufficient to have a ground state [G, LMS]. It is then reasonable to think this is also true for our model. Indeed, we will prove that if the infrared condition is violated, then there is no ground state but provided the additional condition $\hat{\rho}_1(0) \neq 0$ is satisfied, which means that the total charge of the particle does not vanish.

Proposition 3.2. *Suppose $n \geq 3$, V satisfies hypothesis (C), $\hat{\rho}_2$ does not satisfy (IR) and $\hat{\rho}_1(0) \neq 0$, then H has no ground state.*

To prove Theorem 3.1, we will need to study some "intermediate" models, and in particular to consider *massive* bosons and to "discretize" space. The term *massive* means that, instead of $\omega(k)$, we will consider a function $\omega_m(k)$ satisfying

$$(H_\omega) \qquad \nabla\omega_m \in L^\infty(\mathbb{R}^n), \quad \lim_{|k|\to\infty} \omega_m(k) = +\infty, \quad \inf \omega_m(k) = m > 0.$$

Our proof will use different methods developed in the literature [BFS1, BFS2, DG1, G, GJ]. For more detailed proofs, we also refer the reader to [B1].

Finally, we would like to emphasize that all the Hamiltonians we will deal with have the same structure as (2.3) and so, a similar result to the one of Proposition 3.1 is available for each of them.

4 Ground State for Massive Bosons

Our goal in this section is to prove a first result similar to Theorem 3.1 but in the case of massive bosons (Theorem 4.2, Sect. 4.2). The idea is first to consider a finite box ($|x| < L$) and then to control the remainder as L goes to infinity. We will see, in Sect. 4.2, that the "cutoff" model so obtained can be written in the form (4.1). We therefore first study models of this latter type (Theorem 4.1).

4.1 *Discrete models*

4.1.1 *Description*

We consider Hamiltonians of the form

$$\begin{aligned} H^{\mathrm{d}} &:= H_p \otimes \mathbb{1} + \mathbb{1} \otimes \sum_{l\in\mathbb{Z}^d} \int_{\mathbb{R}^n} \mathrm{d}k\, \omega_m(k) a_l^*(k) a_l(k) \\ &\qquad + \sum_{l\in\mathbb{Z}^d} \int_{\mathbb{R}^n} \mathrm{d}k\, (\beta_l(k) \otimes a_l^*(k) + \bar{\beta}_l(k) \otimes a_l(k)) \\ &= H_0^{\mathrm{d}} + W^{\mathrm{d}}, \end{aligned} \tag{4.1}$$

on the space $\mathcal{H}^{\mathrm{d}} := L^2(\mathbb{R}^d) \otimes \mathcal{F}\left(l^2(\mathbb{Z}^d) \otimes L^2(\mathbb{R}^n)\right)$, and where the $\beta_l(k)$ satisfy

(C_β) $\quad \beta_l(k) = \zeta_l \frac{\hat{\rho}_2(k)}{\sqrt{2\omega_m(k)}}$ where ζ_l is a multiplication operator on $L^2(\mathbb{R}^d)$ such that $\sup_l \||l|^s \zeta_l\| < +\infty$ for all $s > 0$,

$a_l(k)$ and $a_l^*(k)$ are the annihilation and creation operators on the space $\mathcal{F}\left(l^2(\mathbb{Z}^d)\otimes L^2(\mathbb{R}^n)\right)$, and for $l=(l_1,\ldots,l_d)\in\mathbb{Z}^d$, $|l|:=\sup_i|l_i|$.

Let E_0^{d} denote the ground state energy of H^{d}. We will prove the following:

Theorem 4.1. $\sigma_{ess}(H^{\mathrm{d}})\subset\left[E_0^{\mathrm{d}}+m,+\infty\right[$. *In particular, H^{d} has a ground state.*

4.1.2 *Cutoff models*

In the following, M will be a non negative number. On $\mathcal{H}^{\mathrm{d}}$, we define

$$H^{\mathrm{d}}(M):=H_0^{\mathrm{d}}+\sum_{|l|\le M}\int_{\mathbb{R}^n}\mathrm{d}k\left(\beta_l(k)\otimes a_l^*(k)+\bar{\beta}_l(k)\otimes a_l(k)\right)\;=H_0^{\mathrm{d}}+W^{\mathrm{d}}(M).$$

We also define

$$\tilde{H}^{\mathrm{d}}(M):=H_p\otimes\mathbb{1}+\mathbb{1}\otimes\sum_{|l|\le M}\int_{\mathbb{R}^n}\mathrm{d}k\,\omega_m(k)a_l^*(k)a_l(k)+W^{\mathrm{d}}(M),\qquad(4.2)$$

as an operator on the space $\mathcal{H}_M^{\mathrm{d}}:=L^2(\mathbb{R}^d)\otimes\mathcal{F}\left(l^2(\Lambda_M)\otimes L^2(\mathbb{R}^n)\right)$, where $\Lambda_M=\{l\in\mathbb{Z}^d,|l|\le M\}$. Let $E_0^{\mathrm{d}}(M)$ (resp. $\tilde{E}_0^{\mathrm{d}}(M)$) be the ground state energy for $H^{\mathrm{d}}(M)$ (resp. $\tilde{H}^{\mathrm{d}}(M)$). Our goal is to get information on H^{d} from information $H^{\mathrm{d}}(M)$ (as $M\to+\infty$). Thus, we first prove a result similar to Theorem 4.1, but for $H^{\mathrm{d}}(M)$.

Proposition 4.1. $\sigma_{ess}(H^{\mathrm{d}}(M))\subset\left[E_0^{\mathrm{d}}(M)+m,+\infty\right[$. *In particular, $H^{\mathrm{d}}(M)$ has a ground state $\phi_0^{\mathrm{d}}(M)$. Moreover, $E_0^{\mathrm{d}}(M)=\tilde{E}_0^{\mathrm{d}}(M)$.*

Lemma 4.1. $\sigma_{ess}(\tilde{H}^{\mathrm{d}}(M))\subset\left[\tilde{E}_0^{\mathrm{d}}(M)+m,+\infty\right[$. *In particular, $\tilde{H}^{\mathrm{d}}(M)$ has a ground state $\tilde{\phi}_0^{\mathrm{d}}(M)$.*

Proof of Lemma 4.1: The set Λ_M is finite. If its cardinal was one, we would have exactly the model studied in [DG1], and the lemma would correspond to their Theorem 4.1. The same proof works in the general case. $\square$

Proof of Proposition 4.1: The proposition follows from the preceding lemma using an identification between $\mathcal{H}_M^{\mathrm{d}}$ and some subspace of $\mathcal{H}^{\mathrm{d}}$, [GJ]. Indeed, one can write $l^2(\mathbb{Z}^d)\simeq l^2(\Lambda_M)\oplus l^2(\Lambda_M^c)$, where $\Lambda_M^c=\mathbb{Z}^d\backslash\Lambda_M$, so one has

$$\mathcal{F}\left(l^2(\mathbb{Z}^d)\otimes L^2(\mathbb{R}^n)\right)\simeq\mathcal{F}\left(l^2(\Lambda_M)\otimes L^2(\mathbb{R}^n)\right)\otimes\mathcal{F}\left(l^2(\Lambda_M^c)\otimes L^2(\mathbb{R}^n)\right).$$

And finally, $\mathcal{H}^{\mathrm{d}}\simeq\mathcal{H}_M^{\mathrm{d}}\otimes\mathcal{F}\left(l^2(\Lambda_M^c)\otimes L^2(\mathbb{R}^n)\right)$.

One can then identify $\mathcal{H}_M^{\mathrm{d}}$ with $\mathcal{H}_M^{\mathrm{d}}\otimes\Omega_M^c$ where Ω_M^c is the vacuum of $\mathcal{F}\left(l^2(\Lambda_M^c)\otimes L^2(\mathbb{R}^n)\right)$. We can rewrite $\mathcal{H}^{\mathrm{d}}$ as

$$\mathcal{H}^{\mathrm{d}}=\bigoplus_{j=0}^{+\infty}\left(\mathcal{H}_M^{\mathrm{d}}\otimes_s^j\left(l^2(\Lambda_L^c)\otimes L^2(\mathbb{R}^n)\right)\right)=\bigoplus_{j=0}^{+\infty}\mathcal{H}^{(j)}.$$

Actually, we have $\mathcal{H}^{\mathrm{d}}_M = \mathcal{H}^{(0)}$ and $(\mathcal{H}^{\mathrm{d}}_M)^\perp = \bigoplus_{j=1}^{+\infty} \mathcal{H}^{(j)}$. One sees that the $\mathcal{H}^{(j)}$ are invariants for $H^{\mathrm{d}}(M)$. But, on $\mathcal{H}^{(j)}$, one has

$$H^{\mathrm{d}}(M) = \tilde{H}^{\mathrm{d}}(M) \otimes \mathbb{1} + \mathbb{1} \otimes \sum_{|l|>L} \int_{\mathbb{R}^n} \mathrm{d}k\, \omega_m(k) a_l^*(k) a_l(k) \ \geq \ \tilde{H}^{\mathrm{d}}(M) \otimes \mathbb{1} + mj,$$

and on $\mathcal{H}^{(0)}$, $H^{\mathrm{d}}(M) = \tilde{H}^{\mathrm{d}}(M) \otimes \mathbb{1}$. Hence, we have

$$\sigma\left(H^{\mathrm{d}}(M)|_{\mathcal{H}^{\mathrm{d}}_M}\right) = \sigma\left(\tilde{H}^{\mathrm{d}}(M)\right) \quad \text{and} \quad \sigma_{ess}\left(H^{\mathrm{d}}(M)|_{\mathcal{H}^{\mathrm{d}}_M}\right) = \sigma_{ess}\left(\tilde{H}^{\mathrm{d}}(M)\right),$$

and also

$$\sigma_{ess}\left(H^{\mathrm{d}}(M)|_{(\mathcal{H}^{\mathrm{d}}_M)^\perp}\right) \subset \sigma\left(H^{\mathrm{d}}(M)|_{(\mathcal{H}^{\mathrm{d}}_M)^\perp}\right) \subset \left[\tilde{E}^{\mathrm{d}}_0(M) + m, +\infty\right[,$$

which ends the proof. Moreover, one can remark that $\phi^{\mathrm{d}}_0(M) = \tilde{\phi}^{\mathrm{d}}_0(M) \otimes \Omega^c_M$. □

4.1.3 *Removing the cutoff*

We first prove some convergence results as M goes to infinity.

Proposition 4.2. *$H^{\mathrm{d}}(M)$ converges to H^{d} in the strong resolvent sense.*

Proof : We have

$$H^{\mathrm{d}} - H^{\mathrm{d}}(M) = W^{\mathrm{d}} - W^{\mathrm{d}}(M) = \sum_{|l|>M} \int_{\mathbb{R}^n} \mathrm{d}k\, \beta_l(k) \otimes a_l^*(k) + \bar{\beta}_l(k) \otimes a_l(k).$$

Let $\psi \in D(H^{\mathrm{d}}_0)$. Using condition (C_β), one proves, in the same way as Lemma 3.1,

$$\|H^{\mathrm{d}}\psi - H^{\mathrm{d}}(M)\psi\| \leq \frac{2C(s)}{1+M^s} \|(H^{\mathrm{d}}_0)^{\frac{1}{2}}\psi\| + \left(\sum_{|l|>M} \int_{\mathbb{R}^n} \mathrm{d}k\, |\beta_l(k)|^2\right)^{\frac{1}{2}} \|\psi\|.$$

Using condition (C_β) once more, one shows that the right hand side tends to zero as M goes to infinity. So, $H^{\mathrm{d}}(M)$ converges strongly to H^{d} and hence also in the strong resolvent sense ([RS1], Theorem VIII.25). □

Proposition 4.3. *$E^{\mathrm{d}}_0(M)$ is a decreasing function of M which tends to E^{d}_0.*

Proof : We know that, if $\phi^{\mathrm{d}}_0(M)$ is a ground state for $H^{\mathrm{d}}(M)$, then $\phi^{\mathrm{d}}_0(M) = \tilde{\phi}^{\mathrm{d}}_0(M) \otimes \Omega^c_M$, and so, $\forall l \in \Lambda^c_M, \forall k \in \mathbb{R}^n, a_l(k)\phi^{\mathrm{d}}_0(M) = 0$.

As a consequence, it is easy to see that the function $E^{\mathrm{d}}_0(M)$ decreases with M and satisfies $E^{\mathrm{d}}_0(M) \geq E^{\mathrm{d}}_0$. Thus $E^{\mathrm{d}}_0(M)$ converges to some $E_\infty \geq E^{\mathrm{d}}_0$. But

$E_0^d \in \sigma(H^{\mathrm{d}})$ and $H^{\mathrm{d}}(M)$ converges to H^{d} in the strong resolvent sense, so ([RS1], Theorem VIII.24),

$$\forall M > 0, \exists E(M) \in \sigma(H^{\mathrm{d}}(M))/E(M) \to E_0^d.$$

Since $E_0^{\mathrm{d}}(M)$ is the ground state energy of $H^{\mathrm{d}}(M)$, we finally get $E_\infty = E_0^{\mathrm{d}}$. □

Proposition 4.4. *Let Δ be an interval bounded from above. For all $s > 0$, there exists $K(s, \Delta) > 0$ such that*

$$\|\chi_\Delta(H^{\mathrm{d}})(W^{\mathrm{d}} - W^{\mathrm{d}}(M))\chi_\Delta(H^{\mathrm{d}})\| \leq \frac{K(s, \Delta)}{1 + M^s}.$$

Proof : Let $\phi, \psi \in \mathcal{H}^{\mathrm{d}}$. Using condititon (C_β), we get

$$\begin{aligned} & \left|\langle \phi; \chi_\Delta(H^{\mathrm{d}})(W^{\mathrm{d}} - W^{\mathrm{d}}(M))\chi_\Delta(H^{\mathrm{d}})\psi\rangle\right| \\ \leq \quad & \frac{C(s)}{1 + M^s}\left(\|\phi\| \times \|(\mathbb{1} \otimes N^{\mathrm{d}})^{\frac{1}{2}}\chi_\Delta(H^{\mathrm{d}})\psi\| + \|\psi\| \times \|(\mathbb{1} \otimes N^{\mathrm{d}})^{\frac{1}{2}}\chi_\Delta(H^{\mathrm{d}})\phi\|\right). \end{aligned}$$

But Δ is bounded from above, $\mathbb{1} \otimes N^{\mathrm{d}} \leq \frac{1}{m} H_0^{\mathrm{d}}$ and W^{d} is relatively H_0^{d} bounded, so $(\mathbb{1} \otimes N^{\mathrm{d}})^{\frac{1}{2}}\chi_\Delta(H^{\mathrm{d}})$ is a bounded operator. Finally, one has

$$\left|\langle \phi; \chi_\Delta(H^{\mathrm{d}})(W^{\mathrm{d}} - W^{\mathrm{d}}(M))\chi_\Delta(H^{\mathrm{d}})\psi\rangle\right| \leq \frac{2C(s)\|(N^{\mathrm{d}})^{\frac{1}{2}}\chi_\Delta(H^{\mathrm{d}})\|}{1 + M^s}\|\phi\| \times \|\psi\|,$$

which ends the proof. □

Proof of Theorem 4.1: The proof goes in the same way as the one of [BFS2] (Theorem II.2.), i.e. we prove, using the results of Propositions 4.1, 4.3 and 4.4, that $\mathrm{Tr}\{[H^{\mathrm{d}} - E_0^{\mathrm{d}} - m + \epsilon]_-\} > -\infty$ for all $\epsilon > 0$ and where $[A]_-$ denotes the negative part of an operator A. □

4.2 *Continuous models*

In this section, we are interested in the model introduced in Sect. 2 but for massive bosons. We thus consider, on $\mathcal{H}$, the same Hamiltonian as in Section 2 but with $\omega(k)$ replaced by $\omega_m(k)$ satisfying (H_ω). We denote it $H_m = H_m^0 + W_m$ where H_m^0 denotes the free part and W_m the interaction. Let E_m denote the ground state energy of H_m. The main result of this section is the following:

Theorem 4.2. *$\sigma_{ess}(H_m) \subset [E_m + m, +\infty[$. In particular, H_m has a ground state.*

The strategy of the proof is very similar to the one of the previous section. However, one has to be more careful with the estimates when removing the cutoff because the norm of $\rho_1(x - Q)$ as an operator on $L^2(\mathbb{R}^d)$ does not decrease with x. Even worse, it does not depend on it. To control this problem, we will use the exponential decay of the spectral projectors in the Q variable, which are obtained via the Agmon method (see Sect. 4.2.2)

4.2.1 *Cutoff models*

Let j be a smooth function with compact support on $\mathbb{R}^d$ such that

$$0 \leq j(x) \leq 1, \quad j(x) = 1 \text{ for } |x| \leq 1/2, \quad \text{and} \quad j(x) = 0 \text{ for } |x| \geq 3/4.$$

For all $L > 0$, we define $j_L(x) = j(\frac{x}{L})$. We then define

$$\begin{aligned} H_m(L) &:= H_m^0 + \int_{\mathbb{R}^d} \mathrm{d}x \int_{\mathbb{R}^n} \mathrm{d}k\, \rho_1(x-Q) j_L(x) \frac{\hat{\rho}_2(k)}{\sqrt{2\omega_m(k)}} \otimes a^*(x,k) \\ &\qquad + \rho_1(x-Q) j_L(x) \frac{\bar{\hat{\rho}}_2(k)}{\sqrt{2\omega_m(k)}} \otimes a(x,k) \qquad (4.3) \\ &= H_m^0 + W_m(L) \end{aligned}$$

on $\mathcal{H}$. Using the definition of j_L, one can, in $W_m(L)$, replace $\int_{\mathbb{R}^d} \mathrm{d}x$ by $\int_{[-L,L]^d} \mathrm{d}x$. Finally, we define

$$\tilde{H}_m(L) := H_p \otimes \mathbb{1} + \mathbb{1} \otimes \int_{[-L,L]^d} \mathrm{d}x \int_{\mathbb{R}^n} \mathrm{d}k\, \omega_m(k) a^*(x,k) a(x,k) + W_m(L), \quad (4.4)$$

as an operator on $L^2(\mathbb{R}^d) \otimes \mathcal{F}\left(L^2([-L,L]^d) \otimes L^2(\mathbb{R}^n)\right)$. We denote by $E_m(L)$ and $\tilde{E}_m(L)$ the ground state energies of $H_m(L)$ and $\tilde{H}_m(L)$ respectively.

We have "cut" the Hamiltonian H_m in the x variable. We are now in a finite volume box. If we consider the variable p, conjugate to x, this is equivalent to "discretizing" the problem. One has to note that here $p \in \mathbb{Z}^d$. If

$$a_p^*(k) = \frac{1}{(2L)^{\frac{d}{2}}} \int_{[-L,L]^d} \mathrm{d}x\, e^{ipx} a^*(x,k), \quad a_p(k) = \frac{1}{(2L)^{\frac{d}{2}}} \int_{[-L,L]^d} \mathrm{d}x\, e^{-ipx} a(x,k),$$

and

$$\beta_p = \frac{1}{(2L)^{\frac{d}{2}}} \int_{[-L,L]^d} \mathrm{d}x\, \rho_1(x-Q) j_L(x)$$

denote the Fourier coefficients of $a^*(x,k), a(x,k)$ and $\rho_1(x-Q)j_L(x)$ respectively, the Hamiltonian can now be written as follows

$$\begin{aligned} \tilde{H}_m(L) &= H_p \otimes \mathbb{1} + \mathbb{1} \otimes \sum_{p \in \mathbb{Z}^d} \int_{\mathbb{R}^n} \mathrm{d}k\, \omega_m(k) a_p^*(k) a_p(k) \\ &\qquad + \sum_{p \in \mathbb{Z}^d} \int_{\mathbb{R}^n} \mathrm{d}k\, (\beta_p \frac{\hat{\rho}_2(k)}{\sqrt{2\omega_m(k)}} \otimes a_p^*(k) + \bar{\beta}_p \frac{\bar{\hat{\rho}}_2(k)}{\sqrt{2\omega_m(k)}} \otimes a_p(k)), \end{aligned}$$

which has the form (4.1). If the β_p satisfy (C_β), we will then have the following:

Proposition 4.5. $\forall L > 0, \sigma_{ess}(\tilde{H}_m(L)) \subset [\tilde{E}_m(L) + m, +\infty[$.

Finally, a splitting of $L^2(\mathbb{R}^d)$ into $L^2([-L,L]^d) \oplus L^2(\mathbb{R}^d \backslash [-L,L]^d)$ together with the argument of the previous section will lead to the

Proposition 4.6. *$\sigma_{ess}(H_m(L)) \subset [E_m(L)+m, +\infty[$. In particular, $H_m(L)$ has a ground state $\phi_m(L)$.*

So, it remains to check that the β_p satisfy the condition (C_β). The function j_L is zero for $|x| > L$ and ρ_1 has compact support (in a ball of radius R_1), so

$$\forall |q| > L + R_1, \forall x \in \mathbb{R}^d,\ \rho_1(x-q) j_L(x) = 0.$$

Then, for all p in $\mathbb{Z}^d$, β_p is a multiplication operator by a compactly supported function. Moreover, the function $\rho_1(x-q)j_L(x)$ is C^∞, so its Fourier coefficients decay faster than any power of p. Those two facts ensure that $\sup_p \sup_q \left|\beta_p(q)|p|^n\right| < C_n(L) < +\infty$ and so condition (C_β) is satisfied. To prove Theorem 4.2, it remains to control the limit $L \to +\infty$.

4.2.2 *Exponential bounds*

Proposition 4.7. *Let Δ be a bounded from above interval. For any $\alpha > 0$, there exists $M(\alpha, \Delta) > 0$ such that, for all L and m,*

1. $\|(e^{\alpha|Q|} \otimes \mathbb{1})\chi_\Delta(H_m(L))\| \le M(\alpha, \Delta)$.
2. $\|(e^{\alpha|Q|} \otimes \mathbb{1})\chi_\Delta(H_m)\| \le M(\alpha, \Delta)$.
3. $\|(e^{\alpha|Q|} \otimes \mathbb{1})\chi_\Delta(H)\| \le M(\alpha, \Delta)$.

The proof is exactly the same as the one of Theorem II.1 of [BFS1]. The only difference is that $\sigma_{ess}(H_p) = \emptyset$, which makes things easier and in particular one does not need any condition on α or on the supremum of the interval Δ.

For any $R > 0$, we now define

$$N(|x| > R) := \int_{|x|>R} \mathrm{d}x \int_{\mathbb{R}^n} \mathrm{d}k\, a^*(x,k) a(x,k). \tag{4.5}$$

$N(|x| > R)$ is the number of bosons outside the ball centered at the origin and of radius R (in the x variable). We will prove that the number of these "far away" bosons decays exponentially fast with R. More precisely, we have

Proposition 4.8. *For any $\alpha > 0$, there exists $C(\alpha) > 0$ such that, for all L,*

$$\langle \phi_m(L); \mathbb{1} \otimes N(|x| > R) \phi_m(L) \rangle \le C(\alpha) e^{-\alpha R}. \tag{4.6}$$

The idea is to adapt the proof of [BFS1]. What is new in our model is that we need an explicit control on the number of "far away" bosons in the x direction, even for massive bosons. For that purpose, we use the following lemma which comes from the well known pullthrough formula (see *e.g.* [G]):

Lemma 4.2. $\|\mathbb{1}\otimes a(x,k)\phi_m(L)\| \le \frac{1}{\omega_m(k)}\|\rho_1(x-Q)j_L(x)\frac{\hat\rho_2(k)}{\sqrt{2\omega_m(k)}}\otimes\mathbb{1}\phi_m(L)\|$.

Proof of Proposition 4.8 : Let $\alpha>0$, using Lemma 4.2, we have

$$\begin{aligned} & \langle\phi_m(L);\mathbb{1}\otimes N(|x|>R)\phi_m(L)\rangle \\ \le\ & \int_{|x|>R}\mathrm{d}x\int_{\mathbb{R}^n}\mathrm{d}k\,\frac{|\hat\rho_2(k)|^2}{2\omega_m^3(k)}\|\rho_1(x-Q)j_L(x)e^{-\alpha|Q|}\|^2_{\mathcal{B}(L^2)}\times\|e^{\alpha|Q|}\otimes\mathbb{1}\phi_m(L)\|^2. \end{aligned}$$

The function $\hat\rho_2$ is a Schwartz function and ω_m is bounded from below by $m>0$, so the integral with respect to the k variable converges. Recall that the function ρ_1 has compact support in the ball of radius R_1, so, for any given $x\in\mathbb{R}^d$, we have $\|\rho_1(x-Q)e^{-\alpha|Q|}\|_{\mathcal{B}(L^2)}\le\|\rho_1\|_\infty e^{\alpha R_1}e^{-\alpha|x|}$. Thus

$$\int_{|x|>R}\mathrm{d}x\,\|\rho_1(x-Q)e^{-\alpha|Q|}\|^2_{\mathcal{B}(L^2)}\le\|\rho_1\|^2_\infty e^{2\alpha R_1}\int_{|x|>R}\mathrm{d}x\,e^{-2\alpha|x|}\le K(\alpha)e^{-\alpha R}.$$

And hence, $\langle\phi_m(L);\mathbb{1}\otimes N(|x|>R)\phi_m(L)\rangle\le K'(\alpha)e^{-\alpha R}\|e^{\alpha|Q|}\otimes\mathbb{1}\phi_m(L)\|^2$.

Now, for any L, we have $E_m(L)\le E_p^0$ where E_p^0 is the ground state energy of H_p. Indeed, if ψ_p^0 is the ground state of H_p, we have

$$E_m(L)\le\langle\psi_p^0\otimes\Omega;H_m(L)\ \psi_p^0\otimes\Omega\rangle=E_p^0.$$

Take finally $\Delta=]-\infty,E_p^0]$. Then, one has $\phi_m(L)=\chi_\Delta(H_m(L))\phi_m(L)$, and hence

$$\|e^{\alpha|Q|}\otimes\mathbb{1}\phi_m(L)\|^2\ \le\ \|e^{\alpha|Q|}\otimes\mathbb{1}\,\chi_\Delta(H_m(L))\|^2\|\phi_m(L)\|\ \le\ M(\alpha,\Delta)^2,$$

which ends the proof. □

We finally give an estimate similar to the one of Proposition 4.4.

Proposition 4.9. *Let Δ and α be as in Proposition 4.7, then there exists $K(\alpha,\Delta)$ such that*

$$\|\chi_\Delta(H_m)(W_m-W_m(L))\chi_\Delta(H_m)\|\le K(\alpha,\Delta)e^{-\alpha L}.$$

Proof : The proof goes in the same way as the one of Proposition 4.4. As we already mentioned, the main difference comes from the fact that $\|\rho_1(x-Q)\|_{\mathcal{B}(L^2)}$ does not decay with x. This difficulty is overcome using Proposition 4.7, and writing

$$\begin{aligned} & |\langle\phi;\chi_\Delta(H_m)(W_m-W_m(L))\chi_\Delta(H_m)\psi\rangle| \\ =\ & |\langle(e^{2\alpha|Q|}\otimes\mathbb{1})\chi_\Delta(H_m)\phi;(e^{-2\alpha|Q|}\otimes\mathbb{1})(W_m-W_m(L))\chi_\Delta(H_m)\psi\rangle|. \end{aligned}$$

4.2.3 *Removing the cutoff*

Proposition 4.10. *$H_m(L)$ converges to H_m in the strong resolvent sense.*

Proof : The proof is similar to the one of Proposition 4.2. □

Proposition 4.11. *$E_m(L)$ converges to E_m as L goes to infinity.*

Proof : Remember that $\phi_m(L)$ is a ground state of $H_m(L)$. We have

$$\begin{aligned} E_m &\leq \langle \phi_m(L); H_m \phi_m(L) \rangle \leq E_m(L) + \langle \phi_m(L); (W_m - W_m(L)) \phi_m(L) \rangle \\ &\leq E_m(L) + 2\mathcal{R}e\big(\langle e^{\alpha |Q|} \otimes \mathbb{1} \phi_m(L); \int_{|x| > \frac{L}{2}} \mathrm{d}x \int_{\mathbb{R}^n} \mathrm{d}k\, e^{-\alpha |Q|} \rho_1(x - Q) \\ &\qquad \times (1 - j_L(x)) \frac{\hat{\rho}_2(k)}{\sqrt{2\omega_m(k)}} \otimes a(x,k) \phi_m(L) \rangle \big). \end{aligned}$$

Then, the same computation as in Proposition 4.2 leads to

$$\begin{aligned} E_m &\leq E_m(L) + K(\alpha) e^{-\frac{\alpha L}{2}} \langle \phi_m(L); \mathbb{1} \otimes N(|x| > \frac{L}{2}) \phi_m(L) \rangle \\ &\leq E_m(L) + C(\alpha) e^{-\alpha L}. \end{aligned}$$

Hence, the function $E_m(L)$ is bounded from below (and from above by E_p^0), so there exists a sequence L_n and E_∞ such that $\lim\limits_{n \to +\infty} E_m(L_n) = E_\infty$. For the same reason as in the proof of Proposition 4.3, we have $E_m = E_\infty$. The function $E_m(L)$ is then bounded with only one accumulating point E_m, which proves that the function converges to it. □

Proof of Theorem 4.2 : The proof is identical to the one of Theorem 4.1.

5 Proof of the Main Results

The goal of this section is to prove the results of Sect. 3. We start with Theorem 3.1. We adapt the method of [G]. We will insist on the differences with this paper. The idea is to approach (in a way which has to be made precise) H with Hamiltonians for which we know that they have a ground state and then to obtain the same result for H. More precisely, we will use the following lemma:

Lemma 5.1. *([AH], Lemma 4.9) Let $H, H_n (n \in \mathbb{N})$ be selfadjoint operators on a Hilbert space $\mathcal{H}$. We suppose that*

(i) $\forall n \in \mathbb{N}, H_n$ *has a ground state* ψ_n *with ground state energy* E_n*,*

(ii) H_n *tends to* H *in the strong resolvent sense,*

(iii) $\lim_{n \to +\infty} E_n = E$*,*

(iv) $\mathrm{w-}\lim_{n \to +\infty} \psi_n = \psi \neq 0$.

Then ψ is a ground state of H with ground state energy E.

5.1 *Infrared cutoff*

We denote by $\chi_{\sigma\leq\omega(k)}$ the caracteristic function of the set $\{k\in\mathbb{R}^n|\sigma\leq\omega(k)\}$. For any $\sigma>0$, we then define

$$\begin{aligned} H^\sigma &:= H_0+\int_{\mathbb{R}^d}\mathrm{d}x\int_{\mathbb{R}^n}\mathrm{d}k\,\rho_1(x-Q)\frac{\hat{\rho}_2(k)}{\sqrt{2\omega(k)}}\chi_{\sigma\leq\omega(k)}(k)\otimes a^*(x,k)\\ &\qquad\qquad +\rho_1(x-Q)\frac{\bar{\hat{\rho}}_2(k)}{\sqrt{2\omega(k)}}\chi_{\sigma\leq\omega(k)}(k)\otimes a(x,k)\\ &= H_0+H_{I,\sigma}, \end{aligned} \tag{5.1}$$

where H_0 is the free Hamiltonian defined in Section 2. We want to use Lemma 5.1 with H and H^{σ_n} where σ_n is some sequence going to zero.
We consider a function $\tilde{\omega}_\sigma(k)$ satisfying

$$\nabla\tilde{\omega}_\sigma\in L^\infty(\mathbb{R}^n),\quad \tilde{\omega}_\sigma(k)=\omega(k)\quad \text{if}\quad \omega(k)\geq\sigma,\quad \inf\tilde{\omega}_\sigma(k)\geq\frac{\sigma}{2}>0,$$

and we define

$$\tilde{H}^\sigma=H_p\otimes\mathbb{1}+\mathbb{1}\otimes\mathrm{d}\Gamma(\tilde{\omega}_\sigma)+H_{I,\sigma}. \tag{5.2}$$

Then we have the following result:

Proposition 5.1. *For any $\sigma>0, H^\sigma$ has a ground state ψ_σ. We denote by E_σ its ground state energy.*

To prove this result we use the following lemma:

Lemma 5.2. *([G], Lemma 3.2) H^σ has a ground state if and only if $\tilde{H}^\sigma$ has one.*

Proof of Proposition 5.1 : According to the previous lemma, it suffices to show that $\tilde{H}^\sigma$ has a ground state. But $\tilde{H}^\sigma$ is a Hamiltonian of the form studied in Sect. 4.2, so, according to Theorem 4.2, it has a ground state. □

Proposition 5.2. *H^σ tends to H in the norm resolvent sense.*

Proof : We use Lemma A.2 of [G] which says that it suffices to show that Q^σ converges to Q in the topology of $\mathcal{D}(Q)$, where Q^σ and Q are the quadratic forms associated to H^σ and H. But, with a similar computation to the one of Lemma 3.1, one has

$$\begin{aligned} |Q(u,v)-Q^\sigma(u,v)| &\leq \left(\int_{\mathbb{R}^d}\mathrm{d}x\int_{\omega(k)\leq\sigma}\mathrm{d}k\,\frac{\rho_1(x-q)^2|\hat{\rho}_2(k)|^2}{2\omega^2(k)}\right)^{\frac{1}{2}}\\ &\qquad\times(Q(u,u)\|v\|+Q(v,v)\|u\|). \end{aligned}$$

Corollary 5.1. $\lim_{\sigma\to 0}E_\sigma=E_0$.

Remark 5.1. *As in the massive case, one has* $E_\sigma \leq E_p^0$ *for all* $\sigma > 0$.

Using Propositions 5.1 and 5.2 together with Corollary 5.1, one can see that the operators H^σ and H satisfy assumptions $(i)-(ii)-(iii)$ of Lemma 5.1. So, it remains to check condition (iv) and Theorem 3.1 will be proven.

5.2 *Uniform estimates*

Lemma 5.3. *There exists* $C_1 > 0$ *such that for all* $\sigma > 0$, $\langle \psi_\sigma ; H_0 \psi_\sigma \rangle \leq C_1$.

This inequality comes from the fact that $H_{I,\sigma}$ is relatively H_0 bounded with infinitesimal bound, uniformly with respect to $\sigma > 0$. We will also need an estimate on the number of soft bosons, estimate which uses the infrared condition (IR).

Lemma 5.4. *There exists* $C_2 > 0$ *such that for all* $\sigma > 0$, $\langle \psi_\sigma ; \mathbb{1} \otimes N \psi_\sigma \rangle \leq C_2$.

Proof : As in Lemma 4.2, one can show that

$$\|\mathbb{1} \otimes a(x,k)\psi_\sigma\| \leq \frac{1}{\omega(k)} \|\rho_1(x-Q)\frac{\hat{\rho}_2(k)}{\sqrt{2\omega_m(k)}}\chi_{\omega(k)\geq\sigma}(k) \otimes \mathbb{1}\psi_\sigma\|. \tag{5.3}$$

Thus,

$$\begin{aligned}
\langle \psi_\sigma ; \mathbb{1} \otimes N \psi_\sigma \rangle &= \int_{\mathbb{R}^d} \mathrm{d}x \int_{\mathbb{R}^n} \mathrm{d}k \, \|\mathbb{1} \otimes a(x,k)\psi_\sigma\|^2_{\mathcal{H}} \\
&\leq \int_{\mathbb{R}^d} \mathrm{d}q \int_{\mathbb{R}^d} \mathrm{d}x \int_{\omega(k)\geq\sigma} \mathrm{d}k \, \frac{|\hat{\rho}_2(k)|^2}{2\omega^3(k)} |\rho_1(x-q)|^2 \|\psi_\sigma(q)\|^2_{\mathcal{F}} \\
&\leq \|\rho_1\|_2^2 \left(\int_{\mathbb{R}^n} \mathrm{d}k \, \frac{|\hat{\rho}_2(k)|^2}{2\omega^3(k)} \right) \int_{\mathbb{R}^d} \mathrm{d}q \, \|\psi_\sigma(q)\|^2_{\mathcal{F}} \ \leq C_2.
\end{aligned}$$

We have obtained a control on the total number of bosons. However, we will also need some control (uniform with respect to σ) on the number of "far away bosons", that is on the following quantities: $\langle \psi_\sigma ; N(|x| > R)\psi_\sigma \rangle$, $\langle \psi_\sigma ; N(|y| > S)\psi_\sigma \rangle$ and $\langle \psi_\sigma ; N(|p| > P)\psi_\sigma \rangle$ where $N(|x| > R)$ was defined in (4.5) and

$$N(|y| > S) = \int_{\mathbb{R}^d} \mathrm{d}x \int_{|y|>S} \mathrm{d}y \, \tilde{a}^*(x,y)\tilde{a}(x,y),$$

$$N(|p| > P) = \int_{|p|>P} \mathrm{d}p \int_{\mathbb{R}^n} \mathrm{d}k \, \hat{a}^*(p,k)\hat{a}(p,k).$$

The operators $\tilde{a}$ and $\tilde{a}^*$ come from a and a^* via a partial Fourier transform in the k variable, and the operators $\hat{a}$ and $\hat{a}^*$ via a partial Fourier transform in the x variable. We then prove a result similar to Proposition 4.8.

Lemma 5.5. *For any* $\alpha > 0$, *there exists* $C(\alpha) > 0$ *such that*

$$\langle \psi_\sigma ; \mathbb{1} \otimes N(|x| > R)\psi_\sigma \rangle \leq C(\alpha)e^{-\alpha R}.$$

The proof of this lemma is exactly the same as the one of Proposition 4.8. This lemma gives us a control on the number of “far away” bosons in the x direction. Similarly one can control the number of bosons whose momentum in the x direction is large:

Lemma 5.6. *For any $s > 0$, there exists $C(s) > 0$ such that*

$$\langle \psi_\sigma; \mathbb{1} \otimes N(|p| > P)\psi_\sigma \rangle \leq \frac{C(s)}{1 + P^s}.$$

Finally, to control $N(|y| > S)$, we use the following result noting that $\mathrm{d}\Gamma(1 - F_S(y)) \leq N(|y| > \frac{S}{2})$.

Lemma 5.7. *Let $F \in C_0^\infty(\mathbb{R}^n)$ such that*

$$0 \leq F(y) \leq 1, \quad F(y) = 1 \text{ for } |y| \leq 1/2, \quad \text{and} \quad F(y) = 0 \text{ for } |y| \geq 1.$$

Let $F_S(y) = F(\frac{|y|}{S})$. Then

$$\lim_{\sigma \to 0, S \to +\infty} \langle \psi_\sigma; \mathrm{d}\Gamma(1 - F_S(y))\psi_\sigma \rangle = 0.$$

Proof : There is a similar result in [G] (Lemma 4.5), and we essentially follow its proof. The main difference is that the norm of $\rho_1(x - Q)$ as an operator on $L^2(\mathbb{R}^d)$ does not depend on x and is therefore not square integrable with respect to this variable. As in Sect. 4.2.2, to control this problem, we will use the exponential decay of the spectral projectors in the Q variable (Proposition 4.7). First, one easily sees that

$$\mathrm{d}\Gamma(1 - F_S(y)) = \int \mathrm{d}x\, \mathrm{d}k\, a^*(x,k)(1 - F(\frac{|D_k|}{S}))a(x,k). \tag{5.4}$$

Then, one can prove ([G], Prop 4.4) that

$$\lim_{\sigma \to 0} a(x,k)\psi_\sigma - (E_0 - H - \omega(k))^{-1}\frac{\rho_1(x - Q)\hat{\rho}_2(k)}{\sqrt{2\omega(k)}}\psi_\sigma = 0$$

in $L^2(\mathbb{R}^{d+n}, dx\, dk; \mathcal{H})$. Using this together with (5.4), we then have

$$\begin{aligned}
&\langle \psi_\sigma; \mathrm{d}\Gamma(1 - F_S(y))\psi_\sigma \rangle \\
&\quad \leq \|(E_0 - H - \omega(k))^{-1}\tfrac{\rho_1(x-Q)\hat{\rho}_2(k)}{\sqrt{2\omega(k)}}\psi_\sigma\|_{L^2(\mathbb{R}^{d+n};\mathcal{H})} \\
&\qquad \times \|(1 - F(\tfrac{|D_k|}{S}))(E_0 - H - \omega(k))^{-1}\tfrac{\rho_1(x-Q)\hat{\rho}_2(k)}{\sqrt{2\omega(k)}}\psi_\sigma\|_{L^2(\mathbb{R}^{d+n};\mathcal{H})} + o(\sigma^0) \\
&\quad \leq \|(E_0 - H - \omega(k))^{-1}\tfrac{\rho_1(x-Q)e^{-\alpha|Q|}\hat{\rho}_2(k)}{\sqrt{2\omega(k)}}\|_{L^2(\mathbb{R}^{d+n};\mathcal{B}(\mathcal{H}))} \times \|e^{\alpha|Q|}\psi_\sigma\|_{\mathcal{H}} \\
&\qquad \times \|(1 - F(\tfrac{|D_k|}{S}))(E_0 - H - \omega(k))^{-1}\tfrac{\rho_1(x-Q)e^{-\alpha|Q|}\hat{\rho}_2(k)}{\sqrt{2\omega(k)}}\|_{L^2(\mathbb{R}^{d+n};\mathcal{B}(\mathcal{H}))} \\
&\qquad\quad \times \|e^{\alpha|Q|}\psi_\sigma\|_{\mathcal{H}} + o(\sigma^0).
\end{aligned}$$

We check that $(E_0 - H - \omega(k))^{-1} \frac{\rho_1(x-Q)e^{-\alpha|Q|}\hat{\rho}_2(k)}{\sqrt{2\omega(k)}}$ belongs to $L^2(\mathbb{R}^{d+n}; \mathcal{B}(\mathcal{H}))$, using the fact that $\|(E_0 - H - \omega(k))^{-1}\| \leq \omega(k)^{-1}$ and condition (IR). Thus

$$\lim_{S\to+\infty} \|(1 - F(\frac{|D_k|}{S}))(E_0 - H - \omega(k))^{-1} \frac{\rho_1(x-Q)e^{-\alpha|Q|}\hat{\rho}_2(k)}{\sqrt{2\omega(k)}}\|_{L^2(\mathbb{R}^{d+n};\mathcal{B}(\mathcal{H}))} = 0.$$

Moreover $\|e^{\alpha|Q|}\psi_\sigma\|_{\mathcal{H}}$ is uniformly bounded (w.r.t σ), which can be proven as for $\|e^{\alpha|Q|}\psi_m(L)\|_{\mathcal{H}}$ (see Sect. 4.2), and the result follows. □

5.3 *Proof of Theorem 3.1*

We have seen that the only thing we had to check was condition (iv) of Lemma 5.1. The unit ball of $\mathcal{H}$ is weakly compact, so there exists a sequence $\sigma_n \to 0$ and $\psi \in \mathcal{H}$ such that ψ_{σ_n} converges weakly to ψ. It then suffices to prove that $\psi \neq 0$. The idea is to find a compact operator K such that for n large enough one has such an estimate:

$$\|K\psi_{\sigma_n}\| \geq \delta > 0. \tag{5.5}$$

This will ensure that ψ is non zero. Indeed, K is compact, so $K\psi_{\sigma_n}$ tends strongly to $K\psi$. If ψ was zero then $\|K\psi_{\sigma_n}\|$ would go to zero, which contradicts (5.5).

Let us then take $F \in C_0^\infty(\mathbb{R}^n)$ and $G \in C_0^\infty(\mathbb{R}^d)$ satisfying the conditions of Lemma 5.7. Remembering that p is the variable conjugate to x, *i.e.* $p = -i\nabla_x$ on $L^2(\mathbb{R}^{d+n}, dx\, dk)$, one has the following inequalities:

$$(1 - \Gamma(F_S(y)))^2 \leq (1 - \Gamma(F_S(y))) \leq \mathrm{d}\Gamma(1 - F_S(y)), \tag{5.6}$$

$$(1 - \Gamma(G_R(x)))^2 \leq N(|x| > \frac{R}{2}), \quad \text{and} \quad (1 - \Gamma(G_P(p)))^2 \leq N(|p| > \frac{P}{2}). \tag{5.7}$$

Finally, let $\chi(s \leq s_0)$ be a function with support in $\{|s| \leq s_0\}$ and equal to 1 in $\{|s| \leq \frac{s_0}{2}\}$. For any non negative θ, P, R and S, we define

$$K(\theta, P, R, S) := \chi(N \leq \theta)\chi(H_0 \leq \theta)\Gamma(F_S(y))\Gamma(G_R(x))\Gamma(G_P(p)). \tag{5.8}$$

The assumptions on F, G, χ and ω ensure that $K(\theta, P, R, S)$ is compact for any θ, P, R and S.

Using Lemmas 5.3 and 5.4, there exists $\theta_0 > 0$ such that, for all n, one has:

$$\|(1 - \chi(N \leq \theta))\psi_{\sigma_n}\| \leq \frac{1}{10}, \|(1 - \chi(H_0 \leq \theta))\psi_{\sigma_n}\| \leq \frac{1}{10}. \tag{5.9}$$

Likewise, using Lemmas 5.5 and 5.6 together with (5.7), there exist $R_0, P_0 > 0$ such that, for all n, one has:

$$\|(1 - \Gamma(G_R(x)))\psi_{\sigma_n}\| \leq \frac{1}{10}, \|(1 - \Gamma(G_P(p)))\psi_{\sigma_n}\| \leq \frac{1}{10}. \tag{5.10}$$

Finally, using Lemma 5.7 and (5.6), there exist $S_0 > 0$ and n_0 such that, for all $n \geq n_0$, one has:

$$\|(1 - \Gamma(F_S(y)))\psi_{\sigma_n}\| \leq \frac{1}{10}. \tag{5.11}$$

Then, for any $n \geq n_0$, using the last three estimates, we have

$$\|\psi_{\sigma_n}\| \leq \frac{1}{2} + \|K(\theta_0, P_0, R_0, S_0)\psi_{\sigma_n}\|.$$

But $\|\psi_{\sigma_n}\| = 1$ for all n, thus $\|K(\theta_0, P_0, R_0, S_0)\psi_{\sigma_n}\| \geq \frac{1}{2}$, for any $n \geq n_0$, which is an estimate of the form (5.5). □

5.4 *Proof of Proposition 3.2*

The idea of the proof is adapted from [DG2]. Once again, one of the main tools is the pullthrough formula, which comes from the commutator between H and annihilation operators

$$[H, \mathbb{1} \otimes a(x,k)] = -\omega(k)\mathbb{1} \otimes a(x,k) - \rho_1(x - Q)\frac{\hat{\rho}_2(k)}{\sqrt{2\omega(k)}} \otimes \mathbb{1}. \tag{5.12}$$

In order to get our result we will need to use this formula taking into account the membranes alltogether, which, on a formal level, means that we will integrate the previous formula over the "x-space". It is therefore more convenient to look at the Hamiltonian not in the (x, k) variables but in the (p, k) variables, where p is the variable conjugate to x *via* Fourier transform, and then consider the value $p = 0$. In such variables, the pullthrough formula just becomes

$$[H, \mathbb{1} \otimes \hat{a}(p,k)] = -\omega(k)\mathbb{1} \otimes \hat{a}(p,k) - \hat{\rho}_1(p)e^{-ipQ}\frac{\hat{\rho}_2(k)}{\sqrt{2\omega(k)}} \otimes \mathbb{1}. \tag{5.13}$$

Suppose now that $\Psi \in \mathcal{H}$ satisfies $H\Psi = E_0\Psi$, where E_0 is the ground state energy of H. We will show that $\Psi = 0$. We apply equation (5.13) on such a vector. One then gets the following equality

$$\mathbb{1} \otimes \hat{a}(p,k)\ \Psi = -(H + \omega(k) - E_0)^{-1}\left(\frac{\hat{\rho}_1(p)e^{-ipQ}\hat{\rho}_2(k)}{\sqrt{2\omega(k)}} \otimes \mathbb{1}\right)\Psi.$$

We denote with an exponent (m) the component of a vector in the m-particle sector. We have, for any m,

$$(\mathbb{1} \otimes \hat{a}(p,k)\ \Psi)^{(m)}(p_1, k_1, \ldots, p_m, k_m) = \sqrt{m+1}\Psi^{(m+1)}(p, k, p_1, k_1, \ldots, p_m, k_m)$$

and the righthand side is square integrable with respect to all its arguments because $\Psi \in \mathcal{H}$. Therefore, for all m,

$$\Phi^{(m)}(p,k) := \left(-(H + \omega(k) - E_0)^{-1}\frac{\hat{\rho}_1(p)e^{-ipQ}\hat{\rho}_2(k)}{\sqrt{2\omega(k)}} \otimes \mathbb{1}\ \Psi\right)^{(m)}$$

is square integrable with respect to (p,k). On the other hand, it is a continuous function on $\mathbb{R}^d \times (\mathbb{R}^n - \{0\})$. Then, for any $p_0 \in \mathbb{R}^d$, $\Phi^{(m)}(p_0,k)$ is a well defined function of k and it is square integrable. As we have said previously, we consider the value $p_0 = 0$. But

$$\Phi^{(m)}(0,k) = \frac{\hat{\rho}_1(0)\hat{\rho}_2(k)}{\sqrt{2}\omega(k)^{\frac{3}{2}}}\Psi^{(m)},$$

which is not square integrable if the infrared condition is violated, unless $\hat{\rho}_1(0)\Psi^{(m)} = 0$. By assumption, $\hat{\rho}_1(0) \neq 0$, so $\Psi^{(m)} = 0$ for all m which means that $\Psi = 0$. $\square$

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