Systèmes Quantiques Ouverts

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HAM-MARK

Spectral properties for a mathematical model of the weak interaction : the decay of the intermediate vector bosons $W^{\pm}$

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Joint work with Jean-Claude Guillot

References

• "Spectral theory for a mathematical model of the weak interaction : The decay of the intermediate vector bosons W+/-. I ", Preprint arXiv : 0904.3171, to appear in *Advances in Mathematical Physics*.
J.-M. B., J.-C. Guillot.

• Limiting absorption principle at low energies for a mathematical model of weak interactions : the decay of a boson., C.R. Acad. Sci. Paris, Ser. I, Vol 347 (2009).
J.-M. B., J.-C. Guillot.

• Work in progress with W. Aschbacher, J. Faupin, and J.-C. Guillot.

Weak Interaction

The weak interaction

It is one of the four fundamental interactions of nature. In the *Standard Model* of particle physics, it is due to the exchange of the heavy W and Z bosons. Its most familiar effects are

- The muon decay, via the weak interaction to an electron.
- The beta decay, which is one type of radioactivity. This is the decay of a neutron (in a nucleus) into a proton with emission of an electron (or a positron).

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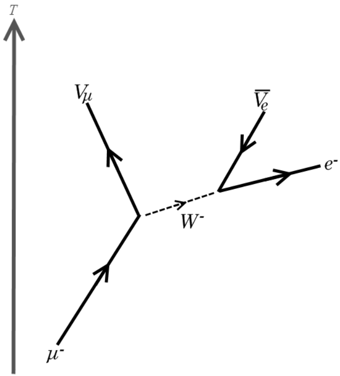
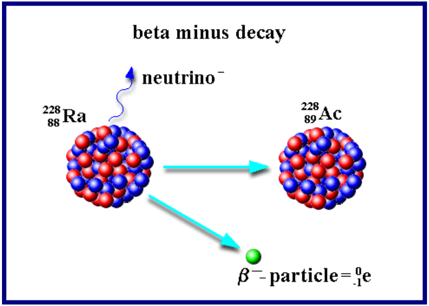
The field strength is about 10^{-11} times the strength of the electromagnetic force.

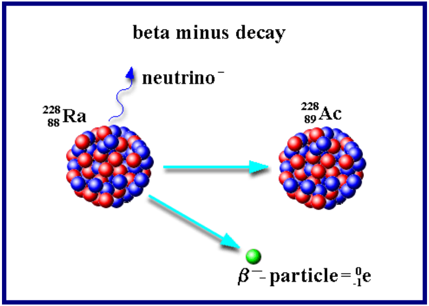


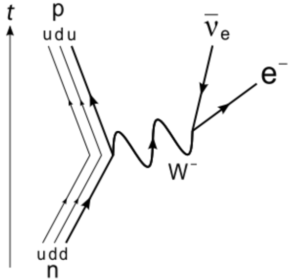
muon decay : $\mu^- \rightarrow e^- + \bar{\nu}_e + \nu_\mu$

β^- decay



β^- decay





beta-minus decay :
$$\begin{cases} \underbrace{n^0}_{udd} & \rightarrow & \underbrace{p^+}_{uud} + e^- + \bar{\nu}_e \\ d & \rightarrow & u + W^- \\ W^- & \rightarrow & e^- + \bar{\nu}_e. \end{cases}$$

$W^{\pm}$-decay

Weak decay of the boson $W^{\pm}$ involves the full family of leptons (e, μ, τ) (spin $\frac{1}{2}$ particles that interact only with the e.m. field), their associated neutrinos (first suggested by Pauli in 1930 to explain *continuous* energy spectrum of the emitted electron in β-decay) and the *massive* bosons $W^{\pm}$.

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- Leptons : ($\ell = 1, 2, 3$) corresponds to (e^-, μ^-, τ^-) and their antiparticles (e^+, μ^+, τ^+).
- Neutrinos : ($\ell = 1, 2, 3$) corresponds to $(\nu_e, \nu_\mu, \tau_\mu)$ and their antiparticles $(\bar{\nu_e}, \bar{\nu_\mu}, \bar{\tau_\mu})$
- Bosons : $W^{\pm}$.

The interaction

In the Schrödinger representation, the interaction for $e^\pm$, ν_e, $W^\pm$ is formally

$$\int \mathrm{d}^3x\ \overline{\Psi_e}(x)\gamma^\alpha(1-\gamma_5)\Psi_{\nu_e}(x)W_\alpha(x)$$
$$+\int \mathrm{d}^3x\ \overline{\Psi_{\nu_e}}(x)\gamma^\alpha(1-\gamma_5)\Psi_e(x)W_\alpha(x)^*\ ,$$

γ^α, $\alpha = 0,1,2,3$ and γ^5 are the Dirac matrices.
$\Psi_.(x)$ and $\overline{\Psi_.}(x)$ are the Dirac fields for e^-, e^+, ν_e and $\bar{\nu}_e$.

- The electron/positron fields are

$$\Psi_e(x) =$$
$$\frac{1}{(2\pi)^{\frac{3}{2}}}\sum_{s=\pm\frac{1}{2}}\int \mathrm{d}^3p\left(b_{e,+}(p,s)\frac{u(p,s)}{\sqrt{p_0}}\mathrm{e}^{ip.x}+b^*_{e,-}(p,s)\frac{v(p,s)}{\sqrt{p_0}}\mathrm{e}^{-ip.x}\right),$$

$u(p,s)$, $v(p,s)$ are normalized solutions of the Dirac equation, $p_0 = (|p|^2+m_e^2)^{\frac{1}{2}}$.

• The neutrino fields ψ_{ν_e} are defined similarly, with $c^*_{\nu_e,\pm}(p,s)$ and $c_{\nu_e,\pm}(p,s)$ creation and annihilation operators instead. However, since the mass of the neutrino is zero, $m_{\nu_e} = 0$, the term $1/\sqrt{p_0} = 1/\sqrt{|p|}$ is singular.

• The W_α fields are given by

$$W_\alpha(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \sum_{\lambda=-1,0,1} \int \frac{\mathrm{d}^3 k}{\sqrt{2k_0}} \left(\epsilon_\alpha(k,\lambda) a_+(k,\lambda) \mathrm{e}^{ik.x} + \epsilon^*_\alpha(k,\lambda) a^*_-(k,\lambda) \mathrm{e}^{-ik.x} \right),$$

with

$$k_0 = (|k|^2 + m_W^2), \quad m_W > 0$$

The interaction

In order to get a well defined operator for the interaction :

1) Adapt Glimm-Jaffe work for Yukawa Hamiltonian :
$L^1 \cap L^2$-spatial cutoff $g(x)$ + momentum cutoffs $\chi(p)$ and $\rho(k)$.

$$\text{Kernel : } \underbrace{\chi(p_1)}_{\text{massive lepton}} \underbrace{\chi(p_2)}_{\text{massless neutrino}} \rho(k)\hat{g}(p_1 + p_2 - k)$$

2) Present work :
More general L^2-Kernels (no L^1 assumption required)

Hilbert spaces

Massive leptons (fermions) :
$e^{\pm}$, $\mu^{\pm}$, $\tau^{\pm}$ ($\ell = 1, 2, 3$).
quantum variables :
$$\xi_1 = (p_1, s_1) \in \Sigma_1 = \underbrace{\mathbb{R}^3}_{\text{momentum}} \times \underbrace{\{-1/2, +1/2\}}_{\text{spin}}.$$
Fock space : $\mathfrak{F}_{a,\ell} = \bigoplus_{n=0}^{\infty} \bigotimes_{\text{antisym.}}^{n} L^2(\Sigma_1)$

Massless neutrinos :
ν_e, ν_μ, ν_τ ($\ell = 1, 2, 3$), and antineutrinos $\bar{\nu}_e$, $\bar{\nu}_\mu$, $\bar{\nu}_\tau$.
quantum variables : $\xi_2 = (p_2, s_2) \in \Sigma_2 = \mathbb{R}^3 \times \{-1/2, +1/2\}$.
Fock space : $\mathfrak{F}_{a,\ell} = \bigoplus_{n=0}^{\infty} \bigotimes_{\text{antisym.}}^{n} L^2(\Sigma_2)$

Massive bosons :
$W^{\pm}$
quantum variables : $\xi_3 = (k, \lambda) \in \Sigma_3 = \underbrace{\mathbb{R}^3}_{\text{momentum}} \times \underbrace{\{-1, 0, 1\}}_{\text{helicity}}$.
Fock space : $\mathfrak{F}_s = \bigoplus_{n=0}^{\infty} \bigotimes_{\text{sym.}}^{n} L^2(\Sigma_3)$

Hilbert spaces

Total Fock space :

$$\mathfrak{F} = \text{total fermionic Fock space} \bigotimes \text{total bosonic Fock space}$$

$$= \left(\bigotimes_{\ell=1,2,3} \underbrace{\mathfrak{F}_{a,\ell}}_{\text{fermionic particle } \ell} \otimes \underbrace{\mathfrak{F}_{a,\ell}}_{\text{neutrino of particle } \ell} \otimes \underbrace{\mathfrak{F}_{a,\ell}}_{\text{fermionic antiparticle } \ell} \otimes \underbrace{\mathfrak{F}_{a,\ell}}_{\text{neutrino of antiparticle } \ell} \right)$$

$$\bigotimes \left(\underbrace{\mathfrak{F}_s}_{\text{boson } W^-} \otimes \underbrace{\mathfrak{F}_s}_{\text{boson } W^+} \right)$$

fermionic/bosonic creation/annihilation operators

Massive leptons : $\ell = 1$: electron ; $\ell = 2$: muon ; $\ell = 3$: tauon.
$b^*_{\ell,\epsilon}(\xi_1)$, $b_{\ell,\epsilon}(\xi_1)$: fermionic creation/annihilation operators for the massive particle ℓ, when $\epsilon = +$ and for the massive antiparticle ℓ when $\epsilon = -$.

Massless neutrinos :
$c^*_{\ell,\epsilon}(\xi_2)$, $c_{\ell,\epsilon}(\xi_2)$: fermionic creation/annihilation operators for the massless neutrino ℓ when $\epsilon = +$, and the massless antineutrino when $\epsilon = -$.

Massive bosons :
$a^*_{\epsilon}(\xi_3)$, $a_{\epsilon}(\xi_3)$: bosonic creation/annihilation operators for the massive boson W^- when $\epsilon = +$, and for W^+ when $\epsilon = -$.

- $a^\sharp(x_1)$ commute with $b^\sharp(\xi_2)$ and $c^\sharp(\xi_3)$.
- Additional requirement (see e.g. S. Weinberg) : $b^\sharp_{\ell,\epsilon}(\xi_1)$ and $c^\sharp_{\ell',\epsilon'}(\xi_2)$ always anticommute.

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$$\{b_{\ell,\epsilon}(\xi_1), b^*_{\ell',\epsilon'}(\xi_1')\} = \delta_{\ell\ell'}\delta_{\epsilon\epsilon'}\delta(\xi_1 - \xi_1') \,,$$
$$\{c_{\ell,\epsilon}(\xi_2), c^*_{\ell',\epsilon'}(\xi_2')\} = \delta_{\ell\ell'}\delta_{\epsilon\epsilon'}\delta(\xi_2 - \xi_2') \,,$$
$$[a_\epsilon(\xi_3), a^*_{\epsilon'}(\xi_3')] = \delta_{\epsilon\epsilon'}\delta(\xi_3 - \xi_3') \,,$$
$$\{b_{\ell,\epsilon}(\xi_1), b_{\ell',\epsilon'}(\xi_1')\} = \{c_{\ell,\epsilon}(\xi_2), c_{\ell',\epsilon'}(\xi_2')\} = 0 \,,$$
$$[a_\epsilon(\xi_3), a_{\epsilon'}(\xi_3')] = 0 \,,$$
$$\{b_{\ell,\epsilon}(\xi_1), c_{\ell',\epsilon'}(\xi_2)\} = \{b_{\ell,\epsilon}(\xi_1), c^*_{\ell',\epsilon'}(\xi_2)\} = 0 \,,$$
$$[b_{\ell,\epsilon}(\xi_1), a_{\epsilon'}(\xi_3)] = [b_{\ell,\epsilon}(\xi_1), a^*_{\epsilon'}(\xi_3)] = [c_{\ell,\epsilon}(\xi_2), a_{\epsilon'}(\xi_3)] = [c_{\ell,\epsilon}(\xi_2), a^*_{\epsilon'}(\xi_3)] = 0 \,,$$

The Hamiltonian - free part H_0

$$H_0 = \sum_{\ell=1}^{3} \sum_{\epsilon=\pm} \underbrace{\int w_\ell^{(1)}(\xi_1)\, b_{\ell,\epsilon}^*(\xi_1) b_{\ell,\epsilon}(\xi_1) \mathrm{d}\xi_1}_{\text{electron, muon, tauon}}$$

$$+ \sum_{\ell=1}^{3} \sum_{\epsilon=\pm} \underbrace{\int w_\ell^{(2)}(\xi_2)\, c_{\ell,\epsilon}^*(\xi_2) c_{\ell,\epsilon}(\xi_2) \mathrm{d}\xi_2}_{\text{associated neutrinos}}$$

$$+ \sum_{\epsilon=\pm} \underbrace{\int w^{(3)}(\xi_3)\, a_\epsilon^*(\xi_3) a_\epsilon(\xi_3) \mathrm{d}\xi_3}_{\text{boson } W^\pm} \ .$$

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The free energy of leptons, neutrinos, and bosons are

$$w_\ell^{(1)}(\xi_1) = (|p_1|^2 + m_\ell^2)^{\frac{1}{2}},\ w_\ell^{(2)}(\xi_2) = |p_2|\ , w^{(3)}(\xi_3) = (|k|^2 + m_W^2)^{\frac{1}{2}}\ .$$

m_ℓ = mass of the lepton ℓ, m_W = the mass of the boson,
and $0 < m_1 < m_2 < m_3 < m_W$.

The Hamiltonian - interaction part H_I

$H_I^{(1)}$ describes the decay of the bosons $W^\pm$ into leptons :

$$H_I^{(1)} = \sum_{\ell=1}^{3} \sum_{\epsilon\neq\epsilon'} \int G^{(1)}_{\ell,\epsilon,\epsilon'}(\xi_1,\xi_2,\xi_3)\, \underbrace{b^*_{\ell,\epsilon}(\xi_1)}_{e^\pm,\,\mu^\pm,\,\tau^\pm}\, \underbrace{c^*_{\ell,\epsilon'}(\xi_2)}_{\text{neutrino } \ell}\, \underbrace{a_\epsilon(\xi_3)}_{\text{boson } W} \mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}\xi_3$$

$$+ \sum_{\ell=1}^{3} \sum_{\epsilon\neq\epsilon'} \int \overline{G^{(1)}_{\ell,\epsilon,\epsilon'}(\xi_1,\xi_2,\xi_3)} a^*_\epsilon(\xi_3) c_{\ell,\epsilon'}(\xi_2) b_{\ell,\epsilon}(\xi_1) \mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}\xi_3 \ ,$$

$H_I^{(2)}$ is the term for the “vacuum polarization” :

$$H_I^{(2)} = \sum_{\ell=1}^{3} \sum_{\epsilon\neq\epsilon'} \int G^{(2)}_{\ell,\epsilon,\epsilon'}(\xi_1,\xi_2,\xi_3) b^*_{\ell,\epsilon}(\xi_1) c^*_{\ell,\epsilon'}(\xi_2) a^*_\epsilon(\xi_3) \mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}\xi_3$$

$$+ \sum_{\ell=1}^{3} \sum_{\epsilon\neq\epsilon'} \int \overline{G^{(2)}_{\ell,\epsilon,\epsilon'}(\xi_1,\xi_2,\xi_3)} a_\epsilon(\xi_3) c_{\ell,\epsilon'}(\xi_2) b_{\ell,\epsilon}(\xi_1) \mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}\xi_3 \ .$$

Total interaction

$$H_I = H_I^{(1)} + H_I^{(2)} \ .$$

The Hamiltonian

Total interaction

$$H_I = H_I^{(1)} + H_I^{(2)} \, .$$

Total Hamiltonian

$$H = H_0 + g(H_I^{(1)} + H_I^{(2)}) = H_0 + gH_I \, .$$

Difficulties

- The spectral structure of the free Hamiltonian : one bound state corresponding to the vacuum located at the bottom of the a.c. spectrum ;
  Set of thresholds $\{p\, m_1 + q\, m2 + r\, m_3 + s\, m_W\}$.
- Obtain a *well-defined* interaction term H_I.
- Infrared divergence problems for the (fermionic) massless neutrinos.
- Complexity of the model that involves several kind of particles : undefinite numbers of fermions and bosons.

Spectral theory and techniques for similar models

Relative boundedness of the interaction :

- [J. Glimm, A. Jaffe '68]. ($L^1 \cap L^2$ assumption + cutoff)
- [J.-M. Barbaroux, M. Dimassi, J.-C. Guillot '04] : (QED)

Existence of a ground state :

- [V. Bach, J. Fröhlich, I.M. Sigal '99] : (Pauli-Fierz operator : nonrelativistic QED)
- [J.-M. Barbaroux, M. Dimassi, J.-C. Guillot '04] (QED)

Existence of asymptotic fields. Location of a.c. spectrum :

- [Hiroshima '01] (nonrelativistic QED)

Conjugate operator method and a.c. spectrum :

- [E. Mourre, '80/81] (positive commutator theory for unbounded operators)
- [W.O. Amrein, A. Boutet de Monvel, V. Georgescu '96] and [J. Sahbani 97]
- [V. Bach, J. Fröhlich, I.M. Sigal, A.Soffer '99] a.c. spectrum and LAP for nonrelativistic QED - Pauli-Fierz model.
- [V. Georgescu, C. Gérard, J.S. Møller JFA'04, CMP'04] and [J. Fröhlich, M. Griesemer, I.M. Sigal '08] (a.c. spectrum down to G.S. energy).
- [L. Amour, B. Grébert, J.-C. Guillot '07] (Fermi weak interactions)

Well defined Hamiltonian

Self-adjointness

Assume $G \in L^2$. Let $g_1 > 0$ be such that

$$\frac{3g_1^2}{m_W}\left(\frac{1}{m_1^2}+1\right)\sum_{\alpha=1,2}\sum_{\ell=1}^{3}\sum_{\epsilon\neq\epsilon'}\|G^{(\alpha)}_{\ell,\epsilon,\epsilon'}\|^2_{L^2(\Sigma_1\times\Sigma_1\times\Sigma_2)} < 1 .$$

Then for every g satisfying $g \leq g_1$, H is a self-adjoint operator in $\mathfrak{F}$ with domain $\mathcal{D}(H) = \mathcal{D}(H_0)$.

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Then for every g satisfying $g \leq g_1$, H is a self-adjoint operator in $\mathfrak{F}$ with domain $\mathcal{D}(H) = \mathcal{D}(H_0)$.

Proof.

- $B^{(\alpha)}_{\ell,\epsilon,\epsilon'} := \int G(\xi_1,\xi_2) b_{\ell,\epsilon}(\xi_1) c_{\ell,\epsilon'}(\xi_2) \mathrm{d}\xi_1 \mathrm{d}\xi_2$

Operator of massive lepton number : $N_\ell = \sum_\epsilon \int b^*_{\ell,\epsilon}(\xi_1) b_{\ell,\epsilon}(\xi_1)\mathrm{d}\xi_1$.

$$\|B^{(\alpha)}_{\ell,\epsilon,\epsilon'}\Psi\| \leq \|G\|_{L^2}\|N_\ell\Psi\| \ .$$

- $\|H_I\Psi\| \leq c\sum_\alpha\sum_\ell\sum_{\epsilon,\epsilon'}(\int|G|^2/w^{(3)})\|(N_\ell+1)^{\frac12}\otimes(\underbrace{H_0^{(3)}}_{\text{boson free ham.}})^{\frac12}\Psi\|$

Hypotheses

(0) Basic assumption for self-adjointness : $G^{(\alpha)}_{\ell,\epsilon,\epsilon'} \in L^2$.

(i) $\int_{\Sigma_1\times\Sigma_1\times\Sigma_2} \frac{|G^{(\alpha)}_{\ell,\epsilon,\epsilon'}(\xi_1,\xi_2,\xi_3)|^2}{|p_2|^2} \mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\xi_3 < \infty,$

(ii) Infrared regularization

$$\left(\int_{\Sigma_1\times\{|p_2|\le\sigma\}\times\Sigma_2} |G^{(\alpha)}_{\ell,\epsilon,\epsilon'}(\xi_1,\xi_2,\xi_3)|^2 \mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\xi_3\right)^{\frac{1}{2}} \le C\sigma^2.$$

(iii) (a) $\int_{\Sigma_1\times\Sigma_1\times\Sigma_2} \left|[(p_2\cdot\nabla_{p_2})G^{(\alpha)}_{\ell,\epsilon,\epsilon'}](\xi_1,\xi_2,\xi_3)\right|^2 \mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\xi_3 < \infty$,

(b) $\int_{\Sigma_1\times\Sigma_1\times\Sigma_2} p_{2,i}^2\, p_{2,j}^2 \left|\frac{\partial^2 G^{(\alpha)}_{\ell,\epsilon,\epsilon'}}{\partial p_{2,i}\partial p_{2,j}}(\xi_1,\xi_2,\xi_3)\right|^2 \mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\xi_3 < \infty$.

(iv) Ultraviolet cutoff
$G^{(\alpha)}_{\ell,\epsilon,\epsilon'}(\xi_1,\xi_2,\xi_3) = 0$ if $|p_2| \ge \Lambda$.

Ground state

Theorem : Existence of a dressed vacuum

Suppose the kernels $G^{(\alpha)}_{\ell,\epsilon,\epsilon'}$ satisfy (0) and (i), then there exists $0 < g_2 \leq g_1$ such that H has a unique ground state for $g \leq g_2$.

$$E_0 := \inf \sigma(H) \text{ is an eigenvalue .}$$

Proof. The scheme is well-known for Pauli-Fierz operator (atoms with fixed number of particles + quantized radiation field).

- [BFS'99] *"Spectral Analysis for systems of atoms and molecules coupled to the quantized radiation fields"*
- [GLL'01] *"Ground states in nonrelativistic QED"*
- [Hiroshima'01] *"Ground states and spectrum of QED of nonrelativistic particle"*

Ground state

- $H^{\sigma_n} := (H_0^{(1)} + H_0^{(2)\sigma_n} + H_0^{(3)}) + gH_{I,\sigma_n}$ with $H_0^{(2)\sigma_n}$ and and H_{I,σ_n} acting on $\{|p_2| \geq \sigma_n\}$: infrared cutoff.

$$H\phi_n = E^{\sigma_n}\phi_n, \quad E^{\sigma_n} := \inf \sigma(H^{\sigma_n})$$

  Moreover $(E^{\sigma_n}, E^{\sigma_n} + c\sigma_n) \cap \sigma(H^{\sigma_n}) = \emptyset$.

- $\tilde{\phi}_n := \phi_n \otimes \Omega_{\leq \sigma_n, neutrino}$ is a ground state for $(H^{\sigma_n} \otimes \mathbf{1}) + (\mathbf{1} \otimes \underbrace{H_{0,\sigma_n}^{(2)}}_{\text{small energies } |p_2| < \sigma_n})$

- $\tilde{\phi}_n \rightharpoonup \tilde{\phi}$ (weak convergence) and $\tilde{\phi} \neq 0$:
  - "Neutrino number" for $\tilde{\phi}_n$ is uniformly $\mathcal{O}(g^2)$: proved with pullthrough formula.
  - Control the number of neutrinos with small energies.

- $\lim_{\sigma_n \to 0} \left(H^{\sigma_n} \otimes \mathbf{1} + \mathbf{1} \otimes H_{0,\sigma_n}^{(2)}\right) \tilde{\phi}_n = \lim_{\sigma_n \to 0} \left(H^{\sigma_n} \otimes \mathbf{1} + \mathbf{1} \otimes H_{0,\sigma_n}^{(2)}\right) \tilde{\phi} = H\tilde{\phi}$

Limiting absorption principle

Based on the standard conjugate operator method, with self-adjoint conj. op.

- [Mourre '80/81],
- [Amrein, BoutetdeMonvel, Georgescu '96],
- [Sahbani '97]

We follow the techniques applied to models with indefinite number of particle :

- [BFSS, CMP'99] *"Positive commutators and the spectrum of Pauli-Fierz Hamiltonian of atoms and molecules"* : using the second quantized dilation operator to prove ac spectrum in Pauli-Fierz models.
- [FGS, CMP'08], *"Spectral theory for the Standard Model of nonrelativistic QED* : Proof down to the ground state energy, without using non self-adjoint conj. op. techniques as developed in [GeorGéMø, CMP'04] and ref. therein ([HüSp], [Skib], [DeJa]).
- [AGG, Cubo '07] *"A math. model of the Fermi weak interactions"* : Model with several kind of particles with undefinite particle number.

Limiting absorption principle

Conjugate operator : A is the second quantized dilation generator for the neutrinos :

$$a = \frac{1}{2}(p_2 \cdot i\nabla_{p_2} + i\nabla_{p_2} \cdot p_2) \text{ on } L^2(\Sigma_2) \quad \text{one neutrino conj. op.}$$

$$\mathrm{d}\Gamma(a) = \text{ second quantization of dilation operator}$$

$$A_\ell = \underbrace{\mathbf{1}}_{\text{part. } \ell} \otimes \underbrace{\mathbf{1}}_{\text{antipart. } \ell} \otimes \underbrace{\mathrm{d}\Gamma(a)}_{\text{neutrino}} \otimes \underbrace{\mathbf{1}}_{\text{antineutrino}} + \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1} \otimes \mathrm{d}\Gamma(a)$$

$$A = A_1 \otimes \mathbf{1}_2 \otimes \mathbf{1}_3 + \mathbf{1}_1 \otimes A_2 \otimes \mathbf{1}_3 + \mathbf{1}_1 \otimes \mathbf{1}_2 \otimes A_3 \ .$$

$$\langle A \rangle = (1 + A^2)^{\frac{1}{2}} \ .$$

Limiting absorption principle

Theorem : Limiting absorption principle

Assume $G^{(\alpha)}_{\ell,\epsilon,\epsilon'}$ satisfy (0)-(iv). For any $\delta > 0$, $0 < \delta < m_1$, there exists $0 < g_\delta \leq g_2$ such that, for $0 < g \leq g_\delta$,
(i) Limiting absorption principle.
For every $s > 1/2$ and φ, ψ in $\mathfrak{F}$, the limits

$$\lim_{\varepsilon \to 0} (\varphi,\ \langle A \rangle^{-s} (H - \lambda \pm i\varepsilon) \langle A \rangle^{-s} \psi)$$

exist uniformly for λ in any compact subset of $(\inf \sigma(H),\ m_1 - \delta)$.
(ii) $\sigma(H) \cap (\inf \sigma(H),\ m_1 - \delta)$ is purely absolutely continuous.
(iii) Pointwise decay in time.
If $\frac{1}{2} < s < 1$ and $f \in C_0^\infty(\mathbb{R})$ with $\mathrm{supp} f \subset (\inf \sigma(H),\ m_1 - \delta)$, then

$$\| \langle A \rangle^{-s} \mathrm{e}^{-itH} f(H) \langle A \rangle^{-s} \| = \mathcal{O}(t^{\frac{1}{2} - s}) \ , \quad \text{as } t \to +\infty$$

Limiting absorption principle

Proof.

- Regularity : H is locally ($C^1(A)$ and) $C^2(A)$-regular : for $f \in C_0^\infty((-\infty, m_1 - \delta))$, :

$$\mathrm{e}^{-isA} f(H) \mathrm{e}^{isA} \Psi \quad \text{is twice continuously differentiable}$$

- Positive commutator estimate (Mourre estimate)

$$E_{\Delta_{\sigma_n}}(H - E)\,[H,\, iA]\, E_{\Delta_{\sigma_n}}(H - E) \geq c\,\sigma_n E_{\Delta_{\sigma_n}}(H - E)\ ,$$

  where $(\Delta_n)_{n\in\mathbb{N}}$ is a covering of $(\inf \sigma(H), m_1 - \delta)$ by intervals of smaller and smaller sizes and closer and closer to $E := \inf \sigma(H)$. Achieved via auxiliary operators (infrared cutoff for neutrinos free energy $H_0^{(2)}$, and infrared cutoff for neutrinos in the interaction H_I) : H^{σ_n} and $H_{\sigma_n} := H^{\sigma_n} \otimes \mathbf{1} + \mathbf{1} \otimes H^{(2)}_{0,\sigma_n}$
  - Gap estimate for H^{σ_n} : $(E_{\sigma_n}, E_{\sigma_n} + c\sigma_n) \cap \sigma(H^{\sigma_n}) = \emptyset$, to get rid of small energies in Mourre estimate (loc. reg. + virial thm).
  - Mourre estimate for H and H_{σ_n} : $E_{\Delta_n}(H_{\sigma_n} - E_{\sigma_n})[H,\, iA_n]E_{\Delta_n}(H_{\sigma_n} - E_{\sigma_n}) \geq c\sigma_n E_{\Delta_n}(H_{\sigma_n} - E_{\sigma_n})$
  - Control $\|E_{\Delta_n}(H_{\sigma_n} - E_{\sigma_n}) - E_{\Delta_n}(H - E)\| \leq c\,\sigma_n\, g$.

Theorem : Location of the absolutely continuous spectrum

Suppose the kernels $G^{(\alpha)}_{\ell,\epsilon,\epsilon'}$ satisfy (0) and (i), then there exists $0 < g_2 \leq g_1$ such that

$$\sigma(H) = \sigma_{\mathrm{ac}}(H) = [\inf \sigma(H), \infty) \ ,$$

with $\inf \sigma(H) \leq 0$.

Proof. Based on the proof of existence of asymptotic Fock representations for the CAR associated with $c^\sharp_{\ell,\epsilon}(f)$: For

$$c^{\sharp\, t}_{\ell,\epsilon} := \mathrm{e}^{itH}\mathrm{e}^{-itH_0} c^\sharp_{\ell,\epsilon}(f)\mathrm{e}^{itH_0}\mathrm{e}^{-itH} \ ,$$

the following are well defined

$$\lim_{t\to\pm\infty} c^{\sharp\, t}_{\ell,\epsilon}(f)\psi =: c^{\sharp\, \pm}_{\ell,\epsilon}(f)\psi$$

and for $\tilde{\phi}$ ground state of H,

$$c^{\sharp\, \pm}_{\ell,\epsilon}(f)\tilde{\phi} = 0 \ .$$