Fluctuation et réponse dans un système markovien classique

HAM-MARK, Cergy Pontoise

Raphaël Chetrite

Physics of Complex Systems. Weizmann Institute of Science

23/08/2009

What is non equilibrium physics ?

”The study of non-linear physics is like the study of non-elephant biology.”
Stanislaw Ulam

What is non equilibrium physics ?

"The study of non-linear physics is like the study of non-elephant biology."
Stanislaw Ulam

Rough characterizations : Irreversibility

- Such phenomena , shooting and projected backward, appear unrealistic.
- Define an entropy production out of equilibrium.

What is non equilibrium physics ?

"The study of non-linear physics is like the study of non-elephant biology."
Stanislaw Ulam

Rough characterizations : Irreversibility

- Such phenomena , shooting and projected backward, appear unrealistic.
- Define an entropy production out of equilibrium.

By negation, what is Markovian Equilibrium systems ?

- Detailed balance : The invariant density is reversible :

$$\rho(x)P_s^t(x,y) = \rho(y)P_s^t(y,x)$$

- Fluctuation-Dissipation Theorem (FDT)

Fluctuation Dissipation Theorem

Equilibrium systems

Hamiltonian

$$H(x)$$

$$\langle A_t \rangle_h = \langle A_t \rangle_{h=0} + \beta \int_0^t ds\, h_{b,s} \partial_s \left\langle O_s^b A_t \right\rangle_{h=0} + O(h^2)$$

Einstein (1905)-Nyquist (1928)-Callen-Welton (1951)-Kubo (1966).

Weakly Non-Equilibrium systems

Hamiltonian

$$H'_t(x) = H(x) - h_{a,t} O^a(x)$$

Non Equilibrium systems

Equilibrium systems

Hamiltonien

$H(x)$

Far from Equilibrium systems

Non-stationnary Hamiltonian :

$H_t(x)$

Non Equilibrium systems

Equilibrium systems

Hamiltonien

$H(x)$

Far from Equilibrium systems

Non-stationnary Hamiltonian :

$H_t(x)$

Non-Equilibrium systems

Hamiltonian $H_t(x)$

+ external force $f_t(x)$

+ initial density

Non-Equilibrium systems

Hamiltonian

$$H'_t(x) = H_t(x) - h_{a,t} O^a(x)$$

+
external force $f_t(x)$

Plan

Part I : Kinematics of Markovian process

Generalization of Nelson derivative and general formulation of Fluctuation-Dissipation Theorem.

Plan

Part I : Kinematics of Markovian process

Generalization of Nelson derivative and general formulation of Fluctuation-Dissipation Theorem.

Part II : Lagrangian picture of a non equilibrium systems

Any diffusion process becomes reversible when viewed in a good Lagrangian frame.

Plan

Part I : Kinematics of Markovian process

Generalization of Nelson derivative and general formulation of Fluctuation-Dissipation Theorem.

Part II : Lagrangian picture of a non equilibrium systems

Any diffusion process becomes reversible when viewed in a good Lagrangian frame.

Part III : Fluctuation relation for a non equilibrium systems

Gallavotti-Cohen relation = Symmetry of the Level 1 of the Large Deviation theory for the entropy production.
We show the generalization of Gallavotti-Cohen at the level $2,5$ of the Large deviation Theory.

Kinematics of a Markovian process

We consider an inhomogeneous Markov semi-group with generator L_t. We note the instantaneous probability density $\rho_t(x) \equiv \langle \delta(X_t - x) \rangle$

Forward derivative $\frac{d_+}{dt} = \partial_t + L_t$

$$\frac{d_+ f}{dt}(x) \equiv \lim_{h \to 0} \frac{\left\langle \frac{f(t+h, X_{t+h}) - f(t, X_t)}{h} \delta(X_t - x) \right\rangle}{\langle \delta(X_t - x) \rangle}$$

$$=$$

Kinematics of a Markovian process

We consider an inhomogeneous Markov semi-group with generator L_t. We note the instantaneous probability density $\rho_t(x) \equiv \langle \delta(X_t - x) \rangle$

Forward derivative $\frac{d_+}{dt} = \partial_t + L_t$

$$\frac{d_+ f}{dt}(x) \equiv \lim_{h \to 0} \frac{\left\langle \frac{f(t+h, X_{t+h}) - f(t, X_t)}{h} \delta(X_t - x) \right\rangle}{\langle \delta(X_t - x) \rangle}$$

$$= \lim_{h \to 0} \frac{1}{h} \int dy P_t^{t+h}(x, y) \left(f(t+h, y) - f(t, x) \right)$$

$$=$$

Kinematics of a Markovian process

We consider an inhomogeneous Markov semi-group with generator L_t. We note the instantaneous probability density $\rho_t(x) \equiv \langle \delta(X_t - x) \rangle$

Forward derivative $\frac{d_+}{dt} = \partial_t + L_t$

$$\frac{d_+ f}{dt}(x) \equiv \lim_{h \to 0} \frac{\left\langle \frac{f(t+h, X_{t+h}) - f(t, X_t)}{h} \delta(X_t - x) \right\rangle}{\langle \delta(X_t - x) \rangle}$$

$$= \lim_{h \to 0} \frac{1}{h} \int dy P_t^{t+h}(x, y) \left(f(t+h, y) - f(t, x) \right)$$

$$= \lim_{h \to 0} \frac{1}{h} \left[\left(\int dy P_t^{t+h}(x, y) f(t+h, y) \right) - f(t, x) \right]$$

=

Kinematics of a Markovian process

We consider an inhomogeneous Markov semi-group with generator L_t. We note the instantaneous probability density $\rho_t(x) \equiv \langle \delta(X_t - x) \rangle$

Forward derivative $\frac{d_+}{dt} = \partial_t + L_t$

$$\frac{d_+ f}{dt}(x) \equiv \lim_{h\to 0} \frac{\left\langle \frac{f(t+h, X_{t+h}) - f(t, X_t)}{h} \delta(X_t - x) \right\rangle}{\langle \delta(X_t - x) \rangle}$$

$$= \lim_{h\to 0} \frac{1}{h} \int dy P_t^{t+h}(x,y) \left(f(t+h,y) - f(t,x) \right)$$

$$= \lim_{h\to 0} \frac{1}{h} \left[\left(\int dy P_t^{t+h}(x,y) f(t+h,y) \right) - f(t,x) \right]$$

$$= \lim_{h\to 0} \frac{1}{h} \left[\left(\int dy P_t^{t+h}(x,y) f(t+h,y) \right) - f(t,x) \right]$$

$$=$$

Kinematics of a Markovian process

We consider an inhomogeneous Markov semi-group with generator L_t. We note the instantaneous probability density $\rho_t(x) \equiv \langle \delta(X_t - x) \rangle$

Forward derivative $\frac{d_+}{dt} = \partial_t + L_t$

$$\frac{d_+ f}{dt}(x) \equiv \lim_{h \to 0} \frac{\left\langle \frac{f(t+h, X_{t+h}) - f(t, X_t)}{h} \delta(X_t - x) \right\rangle}{\langle \delta(X_t - x) \rangle}$$

$$= \lim_{h \to 0} \frac{1}{h} \int dy P_t^{t+h}(x,y) \left(f(t+h,y) - f(t,x) \right)$$

$$= \lim_{h \to 0} \frac{1}{h} \left[\left(\int dy P_t^{t+h}(x,y) f(t+h,y) \right) - f(t,x) \right]$$

$$= \lim_{h \to 0} \frac{1}{h} \left[\left(\int dy P_t^{t+h}(x,y) f(t+h,y) \right) - f(t,x) \right]$$

$$= \lim_{h \to 0} \frac{1}{h} \left[\left(\int dy P_t^{t+h}(x,y) f(t+h,y) \right) - f(t,x) \right]$$

$=$

Kinematics of a Markovian process

We consider an inhomogeneous Markov semi-group with generator L_t. We note the instantaneous probability density $\rho_t(x) \equiv \langle \delta(X_t - x) \rangle$

Forward derivative $\frac{d_+}{dt} = \partial_t + L_t$

$$\frac{d_+ f}{dt}(x) \equiv \lim_{h\to 0} \frac{\left\langle \frac{f(t+h,X_{t+h})-f(t,X_t)}{h} \delta(X_t - x) \right\rangle}{\langle \delta(X_t - x) \rangle}$$

$$= \lim_{h\to 0} \frac{1}{h} \int dy P_t^{t+h}(x,y) \left(f(t+h,y) - f(t,x) \right)$$

$$= \lim_{h\to 0} \frac{1}{h} \left[\left(\int dy P_t^{t+h}(x,y) f(t+h,y) \right) - f(t,x) \right]$$

$$= \lim_{h\to 0} \frac{1}{h} \left[\left(\int dy P_t^{t+h}(x,y) f(t+h,y) \right) - f(t,x) \right]$$

$$= \lim_{h\to 0} \frac{1}{h} \left[\left(\int dy P_t^{t+h}(x,y) f(t+h,y) \right) - f(t,x) \right]$$

$$= \lim_{h\to 0} \frac{1}{h} \left[\int dy h L(x,y) f(t,y) + h \partial_t f(t,x) + o(h) \right]$$

Introduce by Edward Nelson for diffusion process with additive noise.

Backward derivative $\frac{d_-}{dt} = \partial_t - \rho_t^{-1} \circ L_t^\dagger \circ \rho_t + \rho_t^{-1} L_t^\dagger[\rho_t] Id$

$$\frac{d_- f}{dt}(x) \equiv \lim_{h \to 0} \frac{\left\langle \frac{f(t,X_t) - f(t-h,X_{t-h})}{h} \delta(X_t - x) \right\rangle}{\langle \delta(X_t - x) \rangle}$$

=

Backward derivative $\frac{d_-}{dt} = \partial_t - \rho_t^{-1} \circ L_t^\dagger \circ \rho_t + \rho_t^{-1} L_t^\dagger[\rho_t] Id$

$$\frac{d_- f}{dt}(x) \equiv \lim_{h\to 0} \frac{\left\langle \frac{f(t,X_t)-f(t-h,X_{t-h})}{h} \delta(X_t - x) \right\rangle}{\langle \delta(X_t - x) \rangle}$$

$$= \lim_{h\to 0} \frac{1}{h} \frac{\int dy \rho_{t-h}(y) P^t_{t-h}(y,x)(f(t,x) - f(t-h,y))}{\rho_t(x)}$$

=

Backward derivative $\frac{d_-}{dt} = \partial_t - \rho_t^{-1} \circ L_t^\dagger \circ \rho_t + \rho_t^{-1} L_t^\dagger[\rho_t] Id$

$$\frac{d_- f}{dt}(x) \equiv \lim_{h \to 0} \frac{\left\langle \frac{f(t,X_t) - f(t-h,X_{t-h})}{h} \delta(X_t - x) \right\rangle}{\langle \delta(X_t - x) \rangle}$$

$$= \lim_{h \to 0} \frac{1}{h} \frac{\int dy \rho_{t-h}(y) P^t_{t-h}(y,x)(f(t,x) - f(t-h,y))}{\rho_t(x)}$$

$$= \lim_{h \to 0} \frac{1}{h} \frac{\rho_t(x) f(t,x) - \int dy \rho_{t-h}(y) P^t_{t-h}(y,x) f(t-h,y)}{\rho_t(x)}$$

=

Backward derivative $\frac{d_-}{dt} = \partial_t - \rho_t^{-1} \circ L_t^\dagger \circ \rho_t + \rho_t^{-1} L_t^\dagger[\rho_t] Id$

$$\frac{d_- f}{dt}(x) \equiv \lim_{h\to 0} \frac{\left\langle \frac{f(t,X_t)-f(t-h,X_{t-h})}{h} \delta(X_t - x) \right\rangle}{\langle \delta(X_t - x)\rangle}$$

$$= \lim_{h\to 0} \frac{1}{h} \frac{\int dy \rho_{t-h}(y) P_{t-h}^t(y,x)(f(t,x)-f(t-h,y))}{\rho_t(x)}$$

$$= \lim_{h\to 0} \frac{1}{h} \frac{\rho_t(x) f(t,x) - \int dy \rho_{t-h}(y) P_{t-h}^t(y,x) f(t-h,y)}{\rho_t(x)}$$

$$= \lim_{h\to 0} \frac{1}{h} \frac{\rho_t(x) f(t,x) - \int dy (\rho_t(y) - h\partial_t \rho_t(y) + o(h))(\delta(y,x) + hL_t(y,x) + o(h))(f(t,y) - h\partial_t f(t,x) + o(h))}{\rho_t(x)}$$

$$=$$

Backward derivative $\frac{d_-}{dt} = \partial_t - \rho_t^{-1} \circ L_t^\dagger \circ \rho_t + \rho_t^{-1} L_t^\dagger[\rho_t] Id$

$$\frac{d_- f}{dt}(x) \equiv \lim_{h\to 0} \frac{\left\langle \frac{f(t,X_t)-f(t-h,X_{t-h})}{h} \delta(X_t - x) \right\rangle}{\langle \delta(X_t - x) \rangle}$$

$$= \lim_{h\to 0} \frac{1}{h} \frac{\int dy \rho_{t-h}(y) P_{t-h}^t(y,x)(f(t,x)-f(t-h,y))}{\rho_t(x)}$$

$$= \lim_{h\to 0} \frac{1}{h} \frac{\rho_t(x) f(t,x) - \int dy \rho_{t-h}(y) P_{t-h}^t(y,x) f(t-h,y)}{\rho_t(x)}$$

$$= \lim_{h\to 0} \frac{1}{h} \frac{\rho_t(x) f(t,x) - \int dy (\rho_t(y) - h\partial_t \rho_t(y) + o(h))(\delta(y,x) + hL_t(y,x) + o(h))(f(t,y) - h\partial_t f(t,x) + o(h))}{\rho_t(x)}$$

$$= \partial_t f(t,x) - \rho_t^{-1}(x) L_t^\dagger [\rho_t f(t,x)] + \rho_t^{-1} L_t^\dagger [\rho_t] f(t,x)$$

Backward derivative $\frac{d_-}{dt} = \partial_t - \rho_t^{-1} \circ L_t^\dagger \circ \rho_t + \rho_t^{-1} L_t^\dagger[\rho_t] Id$

$$\frac{d_- f}{dt}(x) \equiv \lim_{h\to 0} \frac{\left\langle \frac{f(t,X_t)-f(t-h,X_{t-h})}{h} \delta(X_t - x) \right\rangle}{\langle \delta(X_t - x) \rangle}$$

$$= \lim_{h\to 0} \frac{1}{h} \frac{\int dy \rho_{t-h}(y) P_{t-h}^t(y,x)(f(t,x)-f(t-h,y))}{\rho_t(x)}$$

$$= \lim_{h\to 0} \frac{1}{h} \frac{\rho_t(x) f(t,x) - \int dy \rho_{t-h}(y) P_{t-h}^t(y,x) f(t-h,y)}{\rho_t(x)}$$

$$= \lim_{h\to 0} \frac{1}{h} \frac{\rho_t(x) f(t,x) - \int dy (\rho_t(y) - h\partial_t \rho_t(y) + o(h))(\delta(y,x) + hL_t(y,x) + o(h))(f(t,y) - h\partial_t f(t,x) + o(h))}{\rho_t(x)}$$

$$= \partial_t f(t,x) - \rho_t^{-1}(x) L_t^\dagger [\rho_t f(t,x)] + \rho_t^{-1} L_t^\dagger [\rho_t] f(t,x)$$

Symmetric derivative : $\frac{d}{dt} = \frac{1}{2}\left(\frac{d_+}{dt} + \frac{d_-}{dt}\right)$

- $\frac{d}{dt} = \partial_t + \frac{L_t - \rho_t^{-1} \circ L_t^\dagger \circ \rho_t + \rho_t^{-1} L_t^\dagger[\rho_t] Id}{2}$

Backward derivative $\frac{d_-}{dt} = \partial_t - \rho_t^{-1} \circ L_t^\dagger \circ \rho_t + \rho_t^{-1} L_t^\dagger[\rho_t] Id$

$$\frac{d_- f}{dt}(x) \equiv \lim_{h\to 0} \frac{\left\langle \frac{f(t,X_t)-f(t-h,X_{t-h})}{h} \delta(X_t - x) \right\rangle}{\langle \delta(X_t - x)\rangle}$$

$$= \lim_{h\to 0} \frac{1}{h} \frac{\int dy \rho_{t-h}(y) P^t_{t-h}(y,x)(f(t,x)-f(t-h,y))}{\rho_t(x)}$$

$$= \lim_{h\to 0} \frac{1}{h} \frac{\rho_t(x) f(t,x) - \int dy \rho_{t-h}(y) P^t_{t-h}(y,x) f(t-h,y)}{\rho_t(x)}$$

$$= \lim_{h\to 0} \frac{1}{h} \frac{\rho_t(x) f(t,x) - \int dy(\rho_t(y) - h\partial_t \rho_t(y) + o(h))(\delta(y,x) + hL_t(y,x) + o(h))(f(t,y) - h\partial_t f(t,x) + o(h))}{\rho_t(x)}$$

$$= \partial_t f(t,x) - \rho_t^{-1}(x) L_t^\dagger [\rho_t f(t,x)] + \rho_t^{-1} L_t^\dagger [\rho_t] f(t,x)$$

Symmetric derivative : $\frac{d}{dt} = \frac{1}{2}\left(\frac{d_+}{dt} + \frac{d_-}{dt}\right)$

- $\frac{d}{dt} = \partial_t + \frac{L_t - \rho_t^{-1} \circ L_t^\dagger \circ \rho_t + \rho_t^{-1} L_t^\dagger [\rho_t] Id}{2}$
- System in invariant density (Steady state) :
  $L_t^\dagger[\rho] = 0 \Rightarrow \frac{d}{dt} = \partial_t + \frac{L_t - \rho^{-1} \circ L_t^\dagger \circ \rho}{2}$
- System in equilibrium (reversible) density : $\rho^{-1} \circ L_t^\dagger \circ \rho = L_t \Rightarrow \frac{d}{dt} = \partial_t$

Local Symmetric velocity $v_t(x) \equiv \frac{dX_t}{dt}(x)$

Local Symmetric velocity $v_t(x) = \frac{dX_t}{dt}(x)$

Time derivative of pair correlation between observable

$$\partial_t \langle U_s V_t \rangle = \left\langle U_s \frac{d_+ V_t}{dt} \right\rangle \text{ and } \partial_s \langle U_s V_t \rangle = \left\langle \frac{d_- U_s}{ds} V_t \right\rangle$$

Perturbation of a Markovian process

At $t = 0$, we kick the systems $\Rightarrow$ the perturbed generator becomes

$$L'_t = \exp\left(-\frac{\beta}{2} h_t O\right) \circ L_t \circ \exp\left(\frac{\beta}{2} h_t O\right) - \exp\left(-\frac{\beta}{2} h_t O\right) L_t \left[\exp\left(\frac{\beta}{2} h_t O\right)\right]$$

Comparaison of the process near $h = 0$: TFD

Perturbation of a Markovian process

At $t = 0$, we kick the systems $\Rightarrow$ the perturbed generator becomes

$$L'_t = \exp\left(-\frac{\beta}{2} h_t O\right) \circ L_t \circ \exp\left(\frac{\beta}{2} h_t O\right) - \exp\left(-\frac{\beta}{2} h_t O\right) L_t \left[\exp\left(\frac{\beta}{2} h_t O\right)\right]$$

Comparaison of the process near $h = 0$: TFD

- In 1^{st} order $L'_t = L_t + h_t N_t$ then $N = \frac{\beta}{2}\left(L \circ O - OL - L[O]\right)$
- 1^{st} order response of an observable A : $\left.\frac{\delta \langle A_t \rangle}{\delta h_s}\right|_{h=0} = \left\langle \left(\rho_s^{-1} N_s[\rho_s]\right) A_t \right\rangle_{h=0}$

Perturbation of a Markovian process

At $t = 0$, we kick the systems $\Rightarrow$ the perturbed generator becomes

$$L'_t = \exp\left(-\frac{\beta}{2}h_t O\right) \circ L_t \circ \exp\left(\frac{\beta}{2}h_t O\right) - \exp\left(-\frac{\beta}{2}h_t O\right) L_t \left[\exp\left(\frac{\beta}{2}h_t O\right)\right]$$

Comparaison of the process near $h = 0$: TFD

- In 1^{st} order $L'_t = L_t + h_t N_t$ then $N = \frac{\beta}{2}(L \circ O - OL - L[O])$
- 1^{st} order response of an observable A : $\left.\frac{\delta\langle A_t\rangle}{\delta h_s}\right|_{h=0} = \left\langle \left(\rho_s^{-1} N_s[\rho_s]\right) A_t \right\rangle_{h=0}$
- We have $\rho^{-1}N^{+}[\rho] = \frac{\beta}{2}\left(\frac{d_-}{dt} - \frac{d_+}{dt}\right)[O]$
- TFD First form :
  $\frac{2}{\beta}\left.\frac{\delta\langle A_t\rangle}{\delta h_s}\right|_{h=0} = \partial_s \langle O_s A_t\rangle - \left\langle \frac{d_+}{ds}[O] A_t \right\rangle =$

Perturbation of a Markovian process

At $t = 0$, we kick the systems $\Rightarrow$ the perturbed generator becomes

$$L'_t = \exp\left(-\frac{\beta}{2} h_t O\right) \circ L_t \circ \exp\left(\frac{\beta}{2} h_t O\right) - \exp\left(-\frac{\beta}{2} h_t O\right) L_t \left[\exp\left(\frac{\beta}{2} h_t O\right)\right]$$

Comparaison of the process near $h = 0$: TFD

- In 1^{st} order $L'_t = L_t + h_t N_t$ then $N = \frac{\beta}{2}(L \circ O - OL - L[O])$
- 1^{st} order response of an observable A : $\left.\frac{\delta\langle A_t\rangle}{\delta h_s}\right|_{h=0} = \left\langle \left(\rho_s^{-1} N_s[\rho_s]\right) A_t \right\rangle_{h=0}$
- We have $\rho^{-1} N^{+}[\rho] = \frac{\beta}{2}\left(\frac{d_-}{dt} - \frac{d_+}{dt}\right)[O]$
- TFD First form :
  $\frac{2}{\beta} \left.\frac{\delta\langle A_t\rangle}{\delta h_s}\right|_{h=0} = \partial_s \langle O_s A_t\rangle - \left\langle \frac{d_+}{ds}[O] A_t \right\rangle = \partial_s \langle O_s A_t\rangle - \langle \partial_s O_s A_t\rangle - \langle (LO)_s A_t\rangle$

Perturbation of a Markovian process

At $t = 0$, we kick the systems $\Rightarrow$ the perturbed generator becomes

$$L'_t = \exp\left(-\frac{\beta}{2} h_t O\right) \circ L_t \circ \exp\left(\frac{\beta}{2} h_t O\right) - \exp\left(-\frac{\beta}{2} h_t O\right) L_t \left[\exp\left(\frac{\beta}{2} h_t O\right)\right]$$

Comparaison of the process near $h = 0$: TFD

- In 1^{st} order $L'_t = L_t + h_t N_t$ then $N = \frac{\beta}{2}\left(L \circ O - OL - L[O]\right)$
- 1^{st} order response of an observable A : $\left.\frac{\delta\langle A_t\rangle}{\delta h_s}\right|_{h=0} = \left\langle \left(\rho_s^{-1} N_s[\rho_s]\right) A_t \right\rangle_{h=0}$
- We have $\rho^{-1} N^{+}[\rho] = \frac{\beta}{2}\left(\frac{d_-}{dt} - \frac{d_+}{dt}\right)[O]$
- TFD First form :
  $\frac{2}{\beta} \left.\frac{\delta\langle A_t\rangle}{\delta h_s}\right|_{h=0} = \partial_s \langle O_s A_t\rangle - \left\langle \frac{d_+}{ds}[O] A_t \right\rangle = \partial_s \langle O_s A_t\rangle - \langle \partial_s O_s A_t\rangle - \left\langle (LO)_s A_t \right\rangle$

  Kurchan...J.Phys 1994 for Homogeneous diffusion process with additive noise.

Perturbation of a Markovian process

At $t = 0$, we kick the systems $\Rightarrow$ the perturbed generator becomes

$$L'_t = \exp\left(-\frac{\beta}{2} h_t O\right) \circ L_t \circ \exp\left(\frac{\beta}{2} h_t O\right) - \exp\left(-\frac{\beta}{2} h_t O\right) L_t \left[\exp\left(\frac{\beta}{2} h_t O\right)\right]$$

Comparaison of the process near $h = 0$: TFD

- In 1^{st} order $L'_t = L_t + h_t N_t$ then $N = \frac{\beta}{2}\left(L \circ O - OL - L[O]\right)$
- 1^{st} order response of an observable A : $\left.\frac{\delta\langle A_t\rangle}{\delta h_s}\right|_{h=0} = \left\langle \left(\rho_s^{-1} N_s[\rho_s]\right) A_t \right\rangle_{h=0}$
- We have $\rho^{-1} N^+[\rho] = \frac{\beta}{2}\left(\frac{d_-}{dt} - \frac{d_+}{dt}\right)[O]$
- TFD First form :
  $\frac{2}{\beta}\left.\frac{\delta\langle A_t\rangle}{\delta h_s}\right|_{h=0} = \partial_s \langle O_s A_t\rangle - \left\langle \frac{d_+}{ds}[O] A_t \right\rangle = \partial_s \langle O_s A_t\rangle - \langle \partial_s O_s A_t\rangle - \langle (LO)_s A_t\rangle$

  Kurchan...J.Phys 1994 for Homogeneous diffusion process with additive noise.
- TFD Second form : $\frac{d_+}{dt} = 2\frac{d}{dt} - \frac{d_-}{dt}$ and then

  $$\frac{1}{\beta}\left.\frac{\delta\langle A_t\rangle}{\delta h_s}\right|_{h=0} = \partial_s \langle O_s A_t\rangle - \left\langle \frac{d}{ds} O_s A_t \right\rangle$$

  Chetrite-Gawedzki-Falkovich (J.Stat Mech 2008, J. stat Phys 2009)for diffusion process

  Gomez-Solano...P.R.L (2009) for experimental verification.

Pure jump process : $L_t(f)(x) = \int dy W_t(x,y)\left(f(y) - f(x)\right)$

- Mean current between x and y : $j_t(x,y) = \rho_t(x) W_t(x,y) - \rho_t(y) W_t(y,x)$
- Invariant density (Steady state) $\Leftrightarrow \int dy j_t(x,y) = 0$.
- Equilibrium density $\Leftrightarrow j_t(x,y) = 0$.

Pure jump process : $L_t(f)(x) = \int dy W_t(x,y)\,(f(y) - f(x))$

- Mean current between x and y : $j_t(x,y) = \rho_t(x) W_t(x,y) - \rho_t(y) W_t(y,x)$
- Invariant density (Steady state) $\Leftrightarrow \int dy j_t(x,y) = 0$.
- Equilibrium density $\Leftrightarrow j_t(x,y) = 0$.
- The local Symmetric velocity is here : $v_t(x) = \frac{1}{2}\rho_t^{-1}(x) \int dy j_t(x,y)(y - x)$

Pure jump process : $L_t(f)(x) = \int dy W_t(x,y)\,(f(y) - f(x))$

- Mean current between x and y : $j_t(x,y) = \rho_t(x) W_t(x,y) - \rho_t(y) W_t(y,x)$
- Invariant density (Steady state) $\Leftrightarrow \int dy j_t(x,y) = 0$.
- Equilibrium density $\Leftrightarrow j_t(x,y) = 0$.
- The local Symmetric velocity is here : $v_t(x) = \frac{1}{2}\rho_t^{-1}(x) \int dy j_t(x,y)(y-x)$
- Equilibrium = Usual TFD $\Rightarrow$ Vanishing $v_t(x)$.

Pure jump process : $L_t(f)(x) = \int dy W_t(x,y)\,(f(y) - f(x))$

- Mean current between x and y : $j_t(x,y) = \rho_t(x) W_t(x,y) - \rho_t(y) W_t(y,x)$
- Invariant density (Steady state) $\Leftrightarrow \int dy j_t(x,y) = 0$.
- Equilibrium density $\Leftrightarrow j_t(x,y) = 0$.
- The local Symmetric velocity is here : $v_t(x) = \frac{1}{2}\rho_t^{-1}(x) \int dy j_t(x,y)(y-x)$
- Equilibrium = Usual TFD $\Rightarrow$ Vanishing $v_t(x)$.

Diffusion Process : $L_t = \widehat{u}_t.\nabla + \nabla.\frac{d_t}{2}.\nabla$

- Current of the Focker-Planck equation : $j_t = \widehat{u}_t \rho_t - \frac{d_t}{2} \nabla \rho_t$
- Invariant density (Steady state) $\Leftrightarrow \nabla.j_t = 0$.
- Equilibrium density $\Leftrightarrow j_t = 0$.

Pure jump process : $L_t(f)(x) = \int dy W_t(x,y)\,(f(y) - f(x))$

- Mean current between x and y : $j_t(x,y) = \rho_t(x)W_t(x,y) - \rho_t(y)W_t(y,x)$
- Invariant density (Steady state) $\Leftrightarrow \int dy j_t(x,y) = 0$.
- Equilibrium density $\Leftrightarrow j_t(x,y) = 0$.
- The local Symmetric velocity is here : $v_t(x) = \frac{1}{2}\rho_t^{-1}(x)\int dy j_t(x,y)(y-x)$
- Equilibrium = Usual TFD $\Rightarrow$ Vanishing $v_t(x)$.

Diffusion Process : $L_t = \widehat{u}_t.\nabla + \nabla.\frac{d_t}{2}.\nabla$

- Current of the Focker-Planck equation : $j_t = \widehat{u}_t\rho_t - \frac{d_t}{2}\nabla\rho_t$
- Invariant density (Steady state) $\Leftrightarrow \nabla.j_t = 0$.
- Equilibrium density $\Leftrightarrow j_t = 0$.
- The local symmetric velocity is $v_t(x) = \rho_t^{-1}(x)j_t(x)$ and $\frac{d}{dt} = \partial_t + v_t.\nabla$

Pure jump process : $L_t(f)(x) = \int dy W_t(x,y)\left(f(y) - f(x)\right)$

- Mean current between x and y : $j_t(x,y) = \rho_t(x) W_t(x,y) - \rho_t(y) W_t(y,x)$
- Invariant density (Steady state) $\Leftrightarrow \int dy j_t(x,y) = 0$.
- Equilibrium density $\Leftrightarrow j_t(x,y) = 0$.
- The local Symmetric velocity is here : $v_t(x) = \frac{1}{2}\rho_t^{-1}(x) \int dy j_t(x,y)(y-x)$
- Equilibrium = Usual TFD $\Rightarrow$ Vanishing $v_t(x)$.

Diffusion Process : $L_t = \widehat{u}_t.\nabla + \nabla.\frac{d_t}{2}.\nabla$

- Current of the Focker-Planck equation : $j_t = \widehat{u}_t \rho_t - \frac{d_t}{2} \nabla \rho_t$
- Invariant density (Steady state) $\Leftrightarrow \nabla.j_t = 0$.
- Equilibrium density $\Leftrightarrow j_t = 0$.
- The local symmetric velocity is $v_t(x) = \rho_t^{-1}(x) j_t(x)$ and $\frac{d}{dt} = \partial_t + v_t.\nabla$
- Equilibrium = Usual TFD $\Leftrightarrow$ Vanishing $v_t(x) \Leftrightarrow \widehat{u}_t = \frac{d_t}{2} \nabla ln(\rho)$.

Same type of result : Darses-Nourdin Elec. Comm. in Prob. (2007)

Pure jump process : $L_t(f)(x) = \int dy W_t(x,y)\,(f(y) - f(x))$

- Mean current between x and y : $j_t(x,y) = \rho_t(x) W_t(x,y) - \rho_t(y) W_t(y,x)$
- Invariant density (Steady state) $\Leftrightarrow \int dy j_t(x,y) = 0$.
- Equilibrium density $\Leftrightarrow j_t(x,y) = 0$.
- The local Symmetric velocity is here : $v_t(x) = \frac{1}{2}\rho_t^{-1}(x) \int dy j_t(x,y)(y-x)$
- Equilibrium = Usual TFD $\Rightarrow$ Vanishing $v_t(x)$.

Diffusion Process : $L_t = \widehat{u}_t.\nabla + \nabla.\frac{d_t}{2}.\nabla$

- Current of the Focker-Planck equation : $j_t = \widehat{u}_t \rho_t - \frac{d_t}{2} \nabla \rho_t$
- Invariant density (Steady state) $\Leftrightarrow \nabla.j_t = 0$.
- Equilibrium density $\Leftrightarrow j_t = 0$.
- The local symmetric velocity is $v_t(x) = \rho_t^{-1}(x) j_t(x)$ and $\frac{d}{dt} = \partial_t + v_t.\nabla$
- Equilibrium = Usual TFD $\Leftrightarrow$ Vanishing $v_t(x) \Leftrightarrow \widehat{u}_t = \frac{d_t}{2} \nabla ln(\rho)$.

Same type of result : Darses-Nourdin Elec. Comm. in Prob. (2007) for Langevin process with additive noise : $d_+ d_+(X_t) = d_- d_-(X_t) \Leftrightarrow$ Drift gradiant : $u_t = -\nabla H_t$.

Lagrangian picture of diffusion process

Lagrangian frame of mean local velocity

- Let $x \to \Phi_t(x)$ be the Lagrangian flow of v :

$$\partial_t \Phi_t(x) = v_t(\Phi_t(x)) \text{ and } \Phi_{t_0}(x) = x$$

Lagrangian picture of diffusion process

Lagrangian frame of mean local velocity

- Let $x \to \Phi_t(x)$ be the Lagrangian flow of v :

$$\partial_t \Phi_t(x) = v_t(\Phi_t(x)) \text{ and } \Phi_{t_0}(x) = x$$

- In the Lagrangian frame of v the process x_t becomes

$$\widetilde{x}_t = \Phi_t^{-1}(x_t)$$

- Chetrite-Gawedzki J. Stat Phys 2009.

Lagrangian picture of diffusion process

Lagrangian frame of mean local velocity

- Let $x \to \Phi_t(x)$ be the Lagrangian flow of v :

$$\partial_t \Phi_t(x) = v_t(\Phi_t(x)) \text{ and } \Phi_{t_0}(x) = x$$

- In the Lagrangian frame of v the process x_t becomes

$$\widetilde{x}_t = \Phi_t^{-1}(x_t)$$

- Chetrite-Gawedzki J. Stat Phys 2009.

Step 1 : Freeze of the instantaneous density in the Lagrangian frame to the initial time value of the Eulerian frame density

- $\widetilde{\rho}_t(x) = \left\langle \delta\left(\Phi_t^{-1}(x_t) - x\right)\right\rangle$

Lagrangian picture of diffusion process

Lagrangian frame of mean local velocity

- Let $x \to \Phi_t(x)$ be the Lagrangian flow of v :

$$\partial_t \Phi_t(x) = v_t(\Phi_t(x)) \text{ and } \Phi_{t_0}(x) = x$$

- In the Lagrangian frame of v the process x_t becomes

$$\widetilde{x}_t = \Phi_t^{-1}(x_t)$$

- Chetrite-Gawedzki J. Stat Phys 2009.

Step 1 : Freeze of the instantaneous density in the Lagrangian frame to the initial time value of the Eulerian frame density

- $\widetilde{\rho}_t(x) = \left\langle \delta\left(\Phi_t^{-1}(x_t) - x\right)\right\rangle = \langle \delta\left(x_t - \Phi_t(x)\right)\rangle \det\left(\nabla \Phi_t(x)\right)$

Lagrangian picture of diffusion process

Lagrangian frame of mean local velocity

- Let $x \to \Phi_t(x)$ be the Lagrangian flow of v :

$$\partial_t \Phi_t(x) = v_t(\Phi_t(x)) \text{ and } \Phi_{t_0}(x) = x$$

- In the Lagrangian frame of v the process x_t becomes

$$\widetilde{x}_t = \Phi_t^{-1}(x_t)$$

- Chetrite-Gawedzki J. Stat Phys 2009.

Step 1 : Freeze of the instantaneous density in the Lagrangian frame to the initial time value of the Eulerian frame density

- $\widetilde{\rho}_t(x) = \left\langle \delta\left(\Phi_t^{-1}(x_t) - x\right)\right\rangle = \langle \delta\left(x_t - \Phi_t(x)\right)\rangle \det\left(\nabla \Phi_t(x)\right) = \rho_t\left(\Phi_t(x)\right) \det\left(\nabla \Phi_t(x)\right)$

Lagrangian picture of diffusion process

Lagrangian frame of mean local velocity

- Let $x \to \Phi_t(x)$ be the Lagrangian flow of v :

$$\partial_t \Phi_t(x) = v_t(\Phi_t(x)) \text{ and } \Phi_{t_0}(x) = x$$

- In the Lagrangian frame of v the process x_t becomes

$$\widetilde{x}_t = \Phi_t^{-1}(x_t)$$

- Chetrite-Gawedzki J. Stat Phys 2009.

Step 1 : Freeze of the instantaneous density in the Lagrangian frame to the initial time value of the Eulerian frame density

- $\widetilde{\rho}_t(x) = \left\langle \delta\left(\Phi_t^{-1}(x_t) - x\right)\right\rangle = \langle \delta\left(x_t - \Phi_t(x)\right)\rangle \det\left(\nabla \Phi_t(x)\right) = \rho_t\left(\Phi_t(x)\right) \det\left(\nabla \Phi_t(x)\right)$
- The solution of the Cauchy problem for the Focker-Planck equation : $\rho_t(x) = \int dy \delta\left(x - \Phi_t(y)\right) \rho_{t_0}(y)$

Lagrangian picture of diffusion process

Lagrangian frame of mean local velocity

- Let $x \to \Phi_t(x)$ be the Lagrangian flow of v :

$$\partial_t \Phi_t(x) = v_t(\Phi_t(x)) \text{ and } \Phi_{t_0}(x) = x$$

- In the Lagrangian frame of v the process x_t becomes

$$\widetilde{x}_t = \Phi_t^{-1}(x_t)$$

- Chetrite-Gawedzki J. Stat Phys 2009.

Step 1 : Freeze of the instantaneous density in the Lagrangian frame to the initial time value of the Eulerian frame density

- $\widetilde{\rho}_t(x) = \left\langle \delta\left(\Phi_t^{-1}(x_t) - x\right)\right\rangle = \langle \delta\left(x_t - \Phi_t(x)\right)\rangle \det\left(\nabla \Phi_t(x)\right) = \rho_t\left(\Phi_t(x)\right) \det\left(\nabla \Phi_t(x)\right)$
- The solution of the Cauchy problem for the Focker-Planck equation : $\rho_t(x) = \int dy \delta\left(x - \Phi_t(y)\right) \rho_{t_0}(y) = \det^{-1}\left(\nabla \Phi_t\left(\Phi_t^{-1}(x)\right)\right) \rho_{t_0}\left(\Phi_t^{-1}(x)\right) \Rightarrow \widetilde{\rho}_t(x) = \rho_{t_0}(x)$

Step 2 : Equilibrium form for the diffusion process in the Lagrangian frame ; Long but trivial calculus

$$\dot{\widetilde{x}} = \widetilde{d}_t(\widetilde{x})\nabla \ln(\rho_{t_0}) + \widetilde{r}_t(\widetilde{x}) + \widetilde{\eta}_t(\widetilde{x})$$

with :

- $\widetilde{\eta}_t^i(\widetilde{x}) = \partial_k \left(\Phi_t^{-1}\right)^i (x)\eta_t^k(x)$
- $\widetilde{D}_t^{ij}(\widetilde{x},\widetilde{y}) = \partial_k \left(\Phi_t^{-1}\right)^i (x) D_t^{kl}(x,y)\partial_l \left(\Phi_t^{-1}\right)^j (y)$

Step 2 : Equilibrium form for the diffusion process in the Lagrangian frame ; Long but trivial calculus

$$\dot{\widetilde{x}} = \widetilde{d}_t(\widetilde{x})\nabla \ln(\rho_{t_0}) + \widetilde{r}_t(\widetilde{x}) + \widetilde{\eta}_t(\widetilde{x})$$

with :

- $\widetilde{\eta}^i_t(\widetilde{x}) = \partial_k \left(\Phi_t^{-1}\right)^i (x)\eta^k_t(x)$
- $\widetilde{D}^{ij}_t(\widetilde{x},\widetilde{y}) = \partial_k \left(\Phi_t^{-1}\right)^i (x) D^{kl}_t(x,y)\partial_l \left(\Phi_t^{-1}\right)^j (y)$

Conclusions of the two first part

- Equilibrium and non-equilibrium are closer than usually perceived.
- This mapping transform by example a stationary non equilibrium systems (NESS) in an equilibrium but non stationary systems.

Step 2 : Equilibrium form for the diffusion process in the Lagrangian frame ; Long but trivial calculus

$$\dot{\tilde{x}} = \tilde{d}_t(\tilde{x})\nabla \ln(\rho_{t_0}) + \tilde{r}_t(\tilde{x}) + \tilde{\eta}_t(\tilde{x})$$

with :

- $\tilde{\eta}_t^i(\tilde{x}) = \partial_k \left(\Phi_t^{-1}\right)^i (x)\eta_t^k(x)$
- $\tilde{D}_t^{ij}(\tilde{x},\tilde{y}) = \partial_k \left(\Phi_t^{-1}\right)^i (x) D_t^{kl}(x,y)\partial_l \left(\Phi_t^{-1}\right)^j (y)$

Conclusions of the two first part

- Equilibrium and non-equilibrium are closer than usually perceived.
- This mapping transform by example a stationary non equilibrium systems (NESS) in an equilibrium but non stationary systems.
- This Lagrangian picture is broken when Φ_t^{-1} don't exist. It's seems the case for the EDP stochastic.
- Practical interest ?

Physics is like ♡ : sure, it may give practical results, but that not why we do it. Richard Feynman

Entropy Production, Entropy creation, Work

Space of trajectories on $[0, T]$

- Measure $M_{T,\mu_0}[dx]$ define by $\int F[x] M_{T,\mu_0}[dx] = \int dx \mu_0(x) E_x \left(F[x]\right)$ for initial density μ_0 at $t = 0$.

Entropy Production, Entropy creation, Work

Space of trajectories on $[0, T]$

- Measure $M_{T,\mu_0}[dx]$ define by $\int F[x] M_{T,\mu_0}[dx] = \int dx \mu_0(x) E_x(F[x])$ for initial density μ_0 at $t = 0$.
- Backward trajectories $[\widetilde{x}]$ such that $\widetilde{x}_t = x^*_{T-t}$ with $(x^*)^* = x$.

Entropy Production, Entropy creation, Work

Space of trajectories on $[0, T]$

- Measure $M_{T,\mu_0}[dx]$ define by $\int F[x] M_{T,\mu_0}[dx] = \int dx \mu_0(x) E_x(F[x])$ for initial density μ_0 at $t = 0$.
- Backward trajectories $[\widetilde{x}]$ such that $\widetilde{x}_t = x^*_{T-t}$ with $(x^*)^* = x$.
- $M_{T,\mu_0}[d\widetilde{x}]$: Measure of $[\widetilde{x}]$ for a process which begin with μ_0.

Entropy Production, Entropy creation, Work

Space of trajectories on $[0, T]$

- Measure $M_{T,\mu_0}[dx]$ define by $\int F[x] M_{T,\mu_0}[dx] = \int dx \mu_0(x) E_x (F[x])$ for initial density μ_0 at $t = 0$.
- Backward trajectories $[\widetilde{x}]$ such that $\widetilde{x}_t = x^*_{T-t}$ with $(x^*)^* = x$.
- $M_{T,\mu_0}[d\widetilde{x}]$: Measure of $[\widetilde{x}]$ for a process which begin with μ_0.

Backward process : Markovian generator L^r

Measure M^r_T on the space of trajectories.

Entropy Production, Entropy creation, Work

Space of trajectories on $[0, T]$

- Measure $M_{T,\mu_0}[dx]$ define by $\int F[x] M_{T,\mu_0}[dx] = \int dx \mu_0(x) E_x(F[x])$ for initial density μ_0 at $t = 0$.
- Backward trajectories $[\widetilde{x}]$ such that $\widetilde{x}_t = x^*_{T-t}$ with $(x^*)^* = x$.
- $M_{T,\mu_0}[d\widetilde{x}]$: Measure of $[\widetilde{x}]$ for a process which begin with μ_0.

Backward process : Markovian generator L^r

Measure M^r_T on the space of trajectories.

Action functional W_T

- $\exp(-W_T[x])$: the Radon-Nykodym derivative

$$\exp(-W_T[x]) = \frac{M^r_{T,\mu^r_0}[d\widetilde{x}]}{M_{T,\mu_0}[dx]}$$

Entropy Production, Entropy creation, Work

Space of trajectories on $[0, T]$

- Measure $M_{T,\mu_0}[dx]$ define by $\int F[x] M_{T,\mu_0}[dx] = \int dx \mu_0(x) E_x(F[x])$ for initial density μ_0 at $t = 0$.
- Backward trajectories $[\widetilde{x}]$ such that $\widetilde{x}_t = x^*_{T-t}$ with $(x^*)^* = x$.
- $M_{T,\mu_0}[d\widetilde{x}]$: Measure of $[\widetilde{x}]$ for a process which begin with μ_0.

Backward process : Markovian generator L^r

Measure M^r_T on the space of trajectories.

Action functional W_T

- $\exp(-W_T[x])$: the Radon-Nykodym derivative

$$\exp(-W_T[x]) = \frac{M^r_{T,\mu^r_0}[d\widetilde{x}]}{M_{T,\mu_0}[dx]}$$

- $\Rightarrow$ Entropy creation σ_T if $\mu^r_0(dx^*) = \mu_T(dx)$

Entropy Production, Entropy creation, Work

Space of trajectories on $[0, T]$

- Measure $M_{T,\mu_0}[dx]$ define by $\int F[x] M_{T,\mu_0}[dx] = \int dx \mu_0(x) E_x\left(F[x]\right)$ for initial density μ_0 at $t = 0$.
- Backward trajectories $\left[\widetilde{x}\right]$ such that $\widetilde{x}_t = x^*_{T-t}$ with $(x^*)^* = x$.
- $M_{T,\mu_0}[d\widetilde{x}]$: Measure of $\left[\widetilde{x}\right]$ for a process which begin with μ_0.

Backward process : Markovian generator L^r

Measure M^r_T on the space of trajectories.

Action functional W_T

- $\exp(-W_T\,[x])$: the Radon-Nykodym derivative

$$\exp(-W_T\,[x]) = \frac{M^r_{T,\mu^r_0}[d\widetilde{x}]}{M_{T,\mu_0}[dx]}$$

- $\Rightarrow$ Entropy creation σ_T if $\mu^r_0(dx^*) = \mu_T\,(dx)$
- $\Rightarrow$ Entropy Production J_t if $\mu_0(dx) = dx = \mu^r_0(dx)$.

Expression obtained with Girsanov and Feymann-Kac relation

- Maes J.Stat.Phys 99, Lebowitz-Spohn J.Stat.Phys 99,...Chetrite-Gawedzki C.M.P 08.
- Ex : Pure jump process

$$J_t\left[x\right],\left[t\right],n] = \int_0^t du\left(\lambda^r_{T-u}(x^*_u) - \lambda_u(x_u)\right) + \sum_{i=1}^{i=N_t} \ln \frac{W_{T_i}(x^-_{T_i}, x^+_{T_i})}{W^r_{T-T_i}(\left(x^+_{T_i}\right)^*, \left(x^-_{T_i}\right)^*)}$$

Expression obtained with Girsanov and Feymann-Kac relation

- Maes J.Stat.Phys 99, Lebowitz-Spohn J.Stat.Phys 99,...Chetrite-Gawedzki C.M.P 08.
- Ex : Pure jump process

$$J_t\left[x\right],\left[t\right],n] = \int_0^t du \left(\lambda^r_{T-u}(x^*_u) - \lambda_u(x_u)\right) + \sum_{i=1}^{i=N_t} \ln \frac{W_{T_i}(x^-_{T_i}, x^+_{T_i})}{W^r_{T-T_i}(\left(x^+_{T_i}\right)^*, \left(x^-_{T_i}\right)^*)}$$

Central fluctuation relation : $\langle \exp\left(-W_T\left[x\right]\right) F\left[x\right]\rangle_{\mu_0} = \left\langle F\left[\widetilde{x}\right]\right\rangle^r_{\mu^r_0}$

- $\langle \exp\left(-W_T\left[x\right]\right) F\left[x\right]\rangle_{\mu_0} = \int M_{T,\mu_0}[dx] \exp\left(-W_T\right) F[x]$

Expression obtained with Girsanov and Feymann-Kac relation

- Maes J.Stat.Phys 99, Lebowitz-Spohn J.Stat.Phys 99,...Chetrite-Gawedzki C.M.P 08.
- Ex : Pure jump process

$$J_t\left[x\right],\left[t\right],n] = \int_0^t du\left(\lambda^r_{T-u}(x_u^*) - \lambda_u(x_u)\right) + \sum_{i=1}^{i=N_t} \ln \frac{W_{T_i}(x_{T_i}^-, x_{T_i}^+)}{W^r_{T-T_i}(\left(x_{T_i}^+\right)^*, \left(x_{T_i}^-\right)^*)}$$

Central fluctuation relation : $\left\langle \exp\left(-W_T\left[x\right]\right) F\left[x\right]\right\rangle_{\mu_0} = \left\langle F\left[\widetilde{x}\right]\right\rangle^r_{\mu_0^r}$

- $\left\langle \exp\left(-W_T\left[x\right]\right) F\left[x\right]\right\rangle_{\mu_0} = \int M_{T,\mu_0}[dx] \exp\left(-W_T\right) F[x] = \int M^r_{T,\mu_0^r}[d\widetilde{x}] F[x] = \int M^r_{T,\mu_0^r}[dx] F[\widetilde{x}]$

Expression obtained with Girsanov and Feymann-Kac relation

- Maes J.Stat.Phys 99, Lebowitz-Spohn J.Stat.Phys 99,...Chetrite-Gawedzki C.M.P 08.
- Ex : Pure jump process

$$J_t\left[x\right],\left[t\right],n] = \int_0^t du\left(\lambda^r_{T-u}(x^*_u) - \lambda_u(x_u)\right) + \sum_{i=1}^{i=N_t} \ln \frac{W_{T_i}(x^-_{T_i}, x^+_{T_i})}{W^r_{T-T_i}(\left(x^+_{T_i}\right)^*, \left(x^-_{T_i}\right)^*)}$$

Central fluctuation relation : $\langle \exp\left(-W_T\left[x\right]\right) F\left[x\right]\rangle_{\mu_0} = \left\langle F\left[\widetilde{x}\right]\right\rangle^r_{\mu^r_0}$

- $\langle \exp\left(-W_T\left[x\right]\right) F\left[x\right]\rangle_{\mu_0} = \int M_{T,\mu_0}[dx] \exp\left(-W_T\right) F[x] = \int M^r_{T,\mu^r_0}[d\widetilde{x}]F[x] = \int M^r_{T,\mu^r_0}[dx]F[\widetilde{x}]$
- $F = 1 \Rightarrow$ Jarzynski relation : $\langle \exp(-W_T)\rangle_{\mu_0} = 1$
- Jenssen inegality $\Rightarrow \langle W_T\rangle_{\mu_0} \geq 0$

Expression obtained with Girsanov and Feymann-Kac relation

- Maes J.Stat.Phys 99, Lebowitz-Spohn J.Stat.Phys 99,...Chetrite-Gawedzki C.M.P 08.
- Ex : Pure jump process

$$J_t\left[x\right],\left[t\right],n] = \int_0^t du\left(\lambda^r_{T-u}(x^*_u) - \lambda_u(x_u)\right) + \sum_{i=1}^{i=N_t} \ln \frac{W_{T_i}(x^-_{T_i}, x^+_{T_i})}{W^r_{T-T_i}(\left(x^+_{T_i}\right)^*, \left(x^-_{T_i}\right)^*)}$$

Central fluctuation relation : $\left\langle \exp\left(-W_T\left[x\right]\right) F\left[x\right]\right\rangle_{\mu_0} = \left\langle F\left[\widetilde{x}\right]\right\rangle^r_{\mu^r_0}$

- $\left\langle \exp\left(-W_T\left[x\right]\right) F\left[x\right]\right\rangle_{\mu_0} = \int M_{T,\mu_0}[dx] \exp\left(-W_T\right) F[x] = \int M^r_{T,\mu^r_0}[d\widetilde{x}] F[x] = \int M^r_{T,\mu^r_0}[dx] F[\widetilde{x}]$
- $F = 1 \Rightarrow$ Jarzynski relation : $\left\langle \exp(-W_T)\right\rangle_{\mu_0} = 1$
- Jenssen inegality $\Rightarrow \left\langle W_T\right\rangle_{\mu_0} \geq 0$
- Backward functional action : $\exp(-W^r_T\left[x\right]) = \frac{M_{T,\mu_0}[d\widetilde{x}]}{M^r_{T,\mu^r_0}[dx]} \Rightarrow W^r = -\widetilde{W}$

Expression obtained with Girsanov and Feymann-Kac relation

- Maes J.Stat.Phys 99, Lebowitz-Spohn J.Stat.Phys 99,...Chetrite-Gawedzki C.M.P 08.
- Ex : Pure jump process

$$J_t\left[x\right],\left[t\right],n] = \int_0^t du\left(\lambda^r_{T-u}(x^*_u) - \lambda_u(x_u)\right) + \sum_{i=1}^{i=N_t} \ln \frac{W_{T_i}(x^-_{T_i}, x^+_{T_i})}{W^r_{T-T_i}(\left(x^+_{T_i}\right)^*, \left(x^-_{T_i}\right)^*)}$$

Central fluctuation relation : $\left\langle \exp\left(-W_T\left[x\right]\right) F\left[x\right]\right\rangle_{\mu_0} = \left\langle F\left[\widetilde{x}\right]\right\rangle^r_{\mu^r_0}$

- $\left\langle \exp\left(-W_T\left[x\right]\right) F\left[x\right]\right\rangle_{\mu_0} = \int M_{T,\mu_0}[dx] \exp\left(-W_T\right) F[x] = \int M^r_{T,\mu^r_0}[d\widetilde{x}] F[x] = \int M^r_{T,\mu^r_0}[dx] F[\widetilde{x}]$
- $F = 1 \Rightarrow$ Jarzynski relation : $\left\langle \exp(-W_T)\right\rangle_{\mu_0} = 1$
- Jenssen inegality $\Rightarrow \left\langle W_T\right\rangle_{\mu_0} \geq 0$
- Backward functional action : $\exp(-W^r_T\left[x\right]) = \frac{M_{T,\mu_0}[d\widetilde{x}]}{M^r_{T,\mu^r_0}[dx]} \Rightarrow W^r = -\widetilde{W}$
- By taking $F[x] = \delta(W_T[x] - W)\delta(x_0 - x)\delta(x_T - y) \Rightarrow$

$$\rho_{0,T}(x, y; W) = \rho^r_{0,T}(y^*, x^*; -W)\exp(W)$$

Detailed Fluctuation Theorem (Generalization of detailed balance).

Homogeneous case : Large deviation symmetry

Gallavotti-Cohen relation

- For large times, if we admit the large deviation form
$\rho_{0,T}(x, y; Tw) \asymp \exp(-TH(w))$ and $\rho^r_{0,T}(y^*, x^*; Tw) \asymp \exp(-TH^r(w))$

$$H(w) + w = H^r(-w)$$

Homogeneous case : Large deviation symmetry

Gallavotti-Cohen relation

- For large times, if we admit the large deviation form $\rho_{0,T}(x,y;Tw) \asymp \exp(-TH(w))$ and $\rho^r_{0,T}(y^*,x^*;Tw) \asymp \exp(-TH^r(w))$

$$H(w) + w = H^r(-w)$$

- If backward process = forward process $\Rightarrow H(w) + w = H(-w)$

Homogeneous case : Large deviation symmetry

Gallavotti-Cohen relation

- For large times, if we admit the large deviation form $\rho_{0,T}(x, y; Tw) \asymp \exp(-TH(w))$ and $\rho^r_{0,T}(y^*, x^*; Tw) \asymp \exp(-TH^r(w))$
$$H(w) + w = H^r(-w)$$
- If backward process = forward process $\Rightarrow H(w) + w = H(-w)$

Large deviation Level 2, level 2,5

- empirical density : $\rho^e_{1,T}(x) = \frac{1}{T}\int_0^T \delta(X_t - x)\, dt$
- Level 2 : $\left\langle \delta\left(\rho^e_{1,T} - \rho\right)\right\rangle \asymp \exp(-TI_2[\rho])$ with

Homogeneous case : Large deviation symmetry

Gallavotti-Cohen relation

- For large times, if we admit the large deviation form $\rho_{0,T}(x, y; Tw) \asymp \exp(-TH(w))$ and $\rho^r_{0,T}(y^*, x^*; Tw) \asymp \exp(-TH^r(w))$

$$H(w) + w = H^r(-w)$$

- If backward process = forward process $\Rightarrow H(w) + w = H(-w)$

Large deviation Level 2, level 2,5

- empirical density : $\rho^e_{1,T}(x) = \frac{1}{T}\int_0^T \delta(X_t - x)\,dt$
- Level 2 : $\left\langle \delta\left(\rho^e_{1,T} - \rho\right)\right\rangle \asymp \exp(-TI_2[\rho])$ with $I_2[\rho] = \inf_{[f]}\left(\int dx\,\rho(x)\,(\exp(-f)\,L\exp f)(x)\right) \Rightarrow$ Not explicit form.

Homogeneous case : Large deviation symmetry

Gallavotti-Cohen relation

- For large times, if we admit the large deviation form $\rho_{0,T}(x,y;Tw) \asymp \exp\left(-TH(w)\right)$ and $\rho^r_{0,T}(y^*,x^*;Tw) \asymp \exp\left(-TH^r(w)\right)$

$$H(w) + w = H^r(-w)$$

- If backward process = forward process $\Rightarrow H(w) + w = H(-w)$

Large deviation Level 2, level 2,5

- empirical density : $\rho^e_{1,T}(x) = \frac{1}{T}\int_0^T \delta\left(X_t - x\right) dt$
- Level 2 : $\left\langle \delta\left(\rho^e_{1,T} - \rho\right)\right\rangle \asymp \exp\left(-TI_2\left[\rho\right]\right)$ with $I_2\left[\rho\right] = \inf_{[f]}\left(\int dx \rho(x)\left(\exp\left(-f\right) L \exp f\right)(x)\right) \Rightarrow$ Not explicit form.
- Ergodic Markov Chains : The Large deviation becomes explicit for the pair empirical measure (Level 2,5).

Homogeneous case : Large deviation symmetry

Gallavotti-Cohen relation

- For large times, if we admit the large deviation form
$\rho_{0,T}(x,y;Tw) \asymp \exp\left(-TH(w)\right)$ and $\rho^r_{0,T}(y^*,x^*;Tw) \asymp \exp\left(-TH^r(w)\right)$

$$H(w) + w = H^r(-w)$$

- If backward process = forward process $\Rightarrow H(w) + w = H(-w)$

Large deviation Level 2, level 2,5

- empirical density : $\rho^e_{1,T}(x) = \frac{1}{T}\int_0^T \delta\left(X_t - x\right) dt$
- Level 2 : $\left\langle \delta\left(\rho^e_{1,T} - \rho\right)\right\rangle \asymp \exp\left(-TI_2\left[\rho\right]\right)$ with
$I_2\left[\rho\right] = \inf_{[f]}\left(\int dx \rho(x)\left(\exp\left(-f\right) L \exp f\right)(x)\right) \Rightarrow$ Not explicit form.
- Ergodic Markov Chains : The Large deviation becomes explicit for the pair empirical measure (Level 2,5).
- Ergodic pure jump process : LD explicit for joint empirical density and "empirical jump distribution" : $\rho^e_{2,T}(x,y) = \frac{1}{T}\sum_{i=1}^{N_T}\delta\left(X^-_{t_i} - x\right)\delta\left(X^+_{t_i} - y\right)$
Level 2,5 : $\left\langle \delta\left(\rho^e_{1,T} - \rho\right)\delta\left(\rho^e_{2,T} - C\right)\right\rangle \asymp \exp\left(-TI_{2,5}\left[\rho, C\right]\right)$

$$I_{2,5}[\rho, C] = \begin{cases} \int dxdy \left(-C(x,y) + \rho(x)W(x,y) + C(x,y) \ln \frac{C(x,y)}{\rho(x)W(x,y)} \right) \text{ si } \quad C^1 = C^2 \\ \infty \text{ sinon} \end{cases}$$

Baldi-Piccioni. Stat Prob Lett 1999

$$I_{2,5}[\rho, C] = \begin{cases} \int dxdy \left(-C(x,y) + \rho(x) W(x,y) + C(x,y) \ln \frac{C(x,y)}{\rho(x) W(x,y)} \right) & \text{si} \quad C^1 = C^2 \\ \infty \ \text{sinon} \end{cases}$$

Baldi-Piccioni. Stat Prob Lett 1999

- Ergodic diffusion process : LD explicit for joint empirical density and empirical current $j_T^e(x) = \frac{1}{T} \int_{0\ STR}^{T} \delta(X_t - x) \dot{X}_t dt$

$$I_{2,5}[\rho, C] = \begin{cases} \int dxdy \left(-C(x,y) + \rho(x)W(x,y) + C(x,y)\ln \frac{C(x,y)}{\rho(x)W(x,y)}\right) & \text{si} \quad C^1 = C^2 \\ \infty \text{ sinon} \end{cases}$$

Baldi-Piccioni. Stat Prob Lett 1999

- Ergodic diffusion process : LD explicit for joint empirical density and empirical current $j_T^e(x) = \frac{1}{T}\int_{0\ STR}^{T} \delta\left(X_t - x\right)\dot{X}_t dt$

$$I_{2,5}[\rho, j] = \begin{cases} \frac{1}{2}\int dx\,(\rho d)^{-1}\,(j - J_\rho)\,(j - J_\rho) & \text{si } \nabla.j = 0 \\ \infty \quad \text{sinon} \end{cases}$$

Reference : Math Literature : ?,Physics literature : Maes-Netocny-Wynants Physica A 2008

Prove "a la" theoretician physicist by tilting method

- We perturb the forward diffusion process by replace the drift by another $a_{\rho,j}$ such that $\rho_i = \rho$ and $j_i \equiv \widehat{a(\rho,j)}\rho - \frac{d}{2}\nabla\rho = j$. We note $M^{\rho,j}$ his trajectorial measure.

$$I_{2,5}[\rho, C] = \begin{cases} \int dxdy \left(-C(x,y) + \rho(x)W(x,y) + C(x,y)\ln\frac{C(x,y)}{\rho(x)W(x,y)}\right) & \text{si } C^1 = C^2 \\ \infty & \text{sinon} \end{cases}$$

Baldi-Piccioni. Stat Prob Lett 1999

- Ergodic diffusion process : LD explicit for joint empirical density and empirical current $j^e_T(x) = \frac{1}{T}\int_{0\ STR}^{T} \delta(X_t - x)\,\dot{X}_t dt$

$$I_{2,5}[\rho, j] = \begin{cases} \frac{1}{2}\int dx\,(\rho d)^{-1}(j - J_\rho)(j - J_\rho) & \text{si } \nabla . j = 0 \\ \infty & \text{sinon} \end{cases}$$

Reference : Math Literature : ?,Physics literature : Maes-Netocny-Wynants Physica A 2008

Prove "a la" theoretician physicist by tilting method

- We perturb the forward diffusion process by replace the drift by another $a_{\rho,j}$ such that $\rho_i = \rho$ and $j_i \equiv \widehat{a(\rho, j)}\rho - \frac{d}{2}\nabla\rho = j$. We note $M^{\rho,j}$ his trajectorial measure.
- $\exp(-TI[\rho, j]) \asymp \langle \delta([\rho^e_T] - [\rho])\,\delta([j^e_T] - [j])\rangle$

$$= \int \frac{M_{\mu_0,T}}{M^{\rho,j}_{\mu_0,T}}[x] M^{\rho,j}_{\mu_0,T}[dx]\,\delta([\rho^e_T] - [\rho])\,\delta([j^e_T] - [j])$$

$$I_{2,5}[\rho, C] = \begin{cases} \int dxdy \left(-C(x,y) + \rho(x)W(x,y) + C(x,y)\ln \frac{C(x,y)}{\rho(x)W(x,y)}\right) & \text{si} \quad C^1 = C^2 \\ \infty \text{ sinon} \end{cases}$$

Baldi-Piccioni. Stat Prob Lett 1999

- Ergodic diffusion process : LD explicit for joint empirical density and empirical current $j_T^e(x) = \frac{1}{T}\int_{0\,STR}^{T} \delta(X_t - x)\,\dot{X}_t dt$

$$I_{2,5}[\rho, j] = \begin{cases} \frac{1}{2}\int dx\,(\rho d)^{-1}(j - J_\rho)(j - J_\rho) & \text{si } \nabla . j = 0 \\ \infty \quad \text{sinon} \end{cases}$$

Reference : Math Literature : ?,Physics literature : Maes-Netocny-Wynants Physica A 2008

Prove "a la" theoretician physicist by tilting method

- We perturb the forward diffusion process by replace the drift by another $a_{\rho,j}$ such that $\rho_i = \rho$ and $j_i \equiv \widehat{a(\rho,j)}\rho - \frac{d}{2}\nabla\rho = j$. We note $M^{\rho,j}$ his trajectorial measure.
- $\exp\left(-TI[\rho,j]\right) \asymp \left\langle \delta\left(\left[\rho_T^e\right] - [\rho]\right)\delta\left(\left[j_T^e\right] - [j]\right)\right\rangle$

$$= \int \frac{M_{\mu_0,T}}{M^{\rho,j}_{\mu_0,T}}[x] M^{\rho,j}_{\mu_0,T}[dx]\delta\left(\left[\rho_T^e\right] - [\rho]\right)\delta\left(\left[j_T^e\right] - [j]\right)$$

$$= \int \frac{M_{\mu_0,T}}{M^{\rho,j}_{\mu_0,T}}\left[\rho_T^e, j_T^e\right] M^{\rho,j}_{\mu_0,T}[dx]\delta\left(\left[\rho_T^e\right] - [\rho]\right)\delta\left(\left[j_T^e\right] - [j]\right)$$

$$I_{2,5}[\rho, C] = \begin{cases} \int dxdy \left(-C(x,y) + \rho(x)W(x,y) + C(x,y)\ln \frac{C(x,y)}{\rho(x)W(x,y)}\right) & \text{si} \quad C^1 = C^2 \\ \infty & \text{sinon} \end{cases}$$

Baldi-Piccioni. Stat Prob Lett 1999

- Ergodic diffusion process : LD explicit for joint empirical density and empirical current $j_T^e(x) = \frac{1}{T}\int_{0\ STR}^{T} \delta(X_t - x)\dot{X}_t dt$

$$I_{2,5}[\rho, j] = \begin{cases} \frac{1}{2}\int dx\,(\rho d)^{-1}(j - J_\rho)(j - J_\rho) & \text{si } \nabla.j = 0 \\ \infty & \text{sinon} \end{cases}$$

Reference : Math Literature : ?,Physics literature : Maes-Netocny-Wynants Physica A 2008

Prove "a la" theoretician physicist by tilting method

- We perturb the forward diffusion process by replace the drift by another $a_{\rho,j}$ such that $\rho_i = \rho$ and $j_i \equiv \widehat{a(\rho,j)}\rho - \frac{d}{2}\nabla\rho = j$. We note $M^{\rho,j}$ his trajectorial measure.
- $\exp(-TI[\rho,j]) \asymp \langle \delta([\rho_T^e] - [\rho])\,\delta([j_T^e] - [j])\rangle$

$$= \int \frac{M_{\mu_0,T}}{M^{\rho,j}_{\mu_0,T}}[x] M^{\rho,j}_{\mu_0,T}[dx]\delta([\rho_T^e] - [\rho])\,\delta([j_T^e] - [j])$$

$$= \int \frac{M_{\mu_0,T}}{M^{\rho,j}_{\mu_0,T}}[\rho_T^e, j_T^e]\, M^{\rho,j}_{\mu_0,T}[dx]\delta([\rho_T^e] - [\rho])\,\delta([j_T^e] - [j])$$

$$= \frac{M_{\mu_0,T}}{M^{\rho,j}_{\mu_0,T}}[\rho, j] \int M^{\rho,j}_{\mu_0,T}[dx]\delta([\rho_T^e] - [\rho])\,\delta([j_T^e] - [j]) \asymp$$

$$I_{2,5}[\rho, C] = \begin{cases} \int dxdy \left(-C(x,y) + \rho(x)W(x,y) + C(x,y)\ln \frac{C(x,y)}{\rho(x)W(x,y)}\right) & \text{si } C^1 = C^2 \\ \infty \text{ sinon} \end{cases}$$

Baldi-Piccioni. Stat Prob Lett 1999

- Ergodic diffusion process : LD explicit for joint empirical density and empirical current $j_T^e(x) = \frac{1}{T}\int_{0\ STR}^{T} \delta(X_t - x)\dot{X}_t dt$

$$I_{2,5}[\rho, j] = \begin{cases} \frac{1}{2}\int dx\, (\rho d)^{-1}(j - J_\rho)(j - J_\rho) & \text{si } \nabla.j = 0 \\ \infty \quad \text{sinon} \end{cases}$$

Reference : Math Literature : ?,Physics literature : Maes-Netocny-Wynants Physica A 2008

Prove "a la" theoretician physicist by tilting method

- We perturb the forward diffusion process by replace the drift by another $a_{\rho,j}$ such that $\rho_i = \rho$ and $j_i \equiv \widehat{a(\rho,j)}\rho - \frac{d}{2}\nabla\rho = j$. We note $M^{\rho,j}$ his trajectorial measure.
- $\exp(-TI[\rho,j]) \asymp \langle \delta([\rho_T^e] - [\rho])\,\delta([j_T^e] - [j])\rangle$

$$= \int \frac{M_{\mu_0,T}}{M^{\rho,j}_{\mu_0,T}}[x] M^{\rho,j}_{\mu_0,T}[dx]\delta([\rho_T^e] - [\rho])\,\delta([j_T^e] - [j])$$

$$= \int \frac{M_{\mu_0,T}}{M^{\rho,j}_{\mu_0,T}}[\rho_T^e, j_T^e]\, M^{\rho,j}_{\mu_0,T}[dx]\delta([\rho_T^e] - [\rho])\,\delta([j_T^e] - [j])$$

$$= \frac{M_{\mu_0,T}}{M^{\rho,j}_{\mu_0,T}}[\rho,j]\int M^{\rho,j}_{\mu_0,T}[dx]\delta([\rho_T^e] - [\rho])\,\delta([j_T^e] - [j]) \asymp \frac{M_{\mu_0,T}}{M^{\rho,j}_{\mu_0,T}}[\rho,j].1 \asymp \frac{M_{\mu_0,T}}{M^{\rho,j}_{\mu_0,T}}[\rho,j]$$

Gallavotti-Cohen in the 2,5 Large deviation theory

For Diffusion process, case $x^* = x$ for simplify calculation

- $\exp\left(-\sigma_T[x]\right) \equiv \frac{M^r_{\mu_T}[d\widetilde{x}]}{M_{\mu_0}[dx]}$

Gallavotti-Cohen in the 2,5 Large deviation theory

For Diffusion process, case $x^* = x$ for simplify calculation

- $\exp\left(-\sigma_T[x]\right) \equiv \frac{M^r_{\mu_T}[d\widetilde{x}]}{M_{\mu_0}[dx]} = \frac{M^{\rho,j}_{\mu_0,T}[dx]}{M_{\mu_0}[dx]} \cdot \frac{M^r_{\mu_T}[d\widetilde{x}]}{M^{r,\rho,-j}_{\mu_T}[d\widetilde{x}]} \cdot \frac{M^{r,\rho,-j}_{\mu_T}[d\widetilde{x}]}{M^{\rho,j}_{\mu_0,T}[dx]}$

Gallavotti-Cohen in the 2,5 Large deviation theory

For Diffusion process, case $x^* = x$ for simplify calculation

- $\exp\left(-\sigma_T[x]\right) \equiv \frac{M^r_{\mu_T}[d\widetilde{x}]}{M_{\mu_0}[dx]} = \frac{M^{\rho,j}_{\mu_0,T}[dx]}{M_{\mu_0}[dx]} \cdot \frac{M^r_{\mu_T}[d\widetilde{x}]}{M^{r,\rho,-j}_{\mu_T}[d\widetilde{x}]} \cdot \frac{M^{r,\rho,-j}_{\mu_T}[d\widetilde{x}]}{M^{\rho,j}_{\mu_0,T}[dx]}$

$= \frac{M^{\rho,j}_{\mu_0,T}}{M_{\mu_0,T}}\left[\rho^e_T, j^e_T\right] \cdot \frac{M^r_{\mu_T}}{M^{r,\rho,-j}_{\mu_T}}\left[\rho^e_T, -j^e_T\right] \cdot \frac{M^{\rho,-j}_{\mu_T}[d\widetilde{x}]}{M^{\rho,j}_{\mu_0,T}[dx]}$

Gallavotti-Cohen in the 2,5 Large deviation theory

For Diffusion process, case $x^* = x$ for simplify calculation

- $\exp\left(-\sigma_T[x]\right) \equiv \frac{M^r_{\mu_T}[d\widetilde{x}]}{M_{\mu_0}[dx]} = \frac{M^{\rho,j}_{\mu_0,T}[dx]}{M_{\mu_0}[dx]} \cdot \frac{M^r_{\mu_T}[d\widetilde{x}]}{M^{r,\rho,-j}_{\mu_T}[d\widetilde{x}]} \cdot \frac{M^{r,\rho,-j}_{\mu_T}[d\widetilde{x}]}{M^{\rho,j}_{\mu_0,T}[dx]}$
  $= \frac{M^{\rho,j}_{\mu_0,T}}{M_{\mu_0,T}}\left[\rho^e_T, j^e_T\right] \cdot \frac{M^r_{\mu_T}}{M^{r,\rho,-j}_{\mu_T}}\left[\rho^e_T, -j^e_T\right] \cdot \frac{M^{\rho,-j}_{\mu_T}[d\widetilde{x}]}{M^{\rho,j}_{\mu_0,T}[dx]} = \frac{M^{\rho,j}_{\mu_0,T}}{M_{\mu_0,T}}\left[\rho^e_T, j^e_T\right] \cdot \frac{M^r_{\mu_T}}{M^{r,\rho,-j}_{\mu_T}}\left[\rho^e_T, -j^e_T\right].1$

Gallavotti-Cohen in the 2,5 Large deviation theory

For Diffusion process, case $x^* = x$ for simplify calculation

- $\exp(-\sigma_T[x]) \equiv \frac{M^r_{\mu_T}[d\widetilde{x}]}{M_{\mu_0}[dx]} = \frac{M^{\rho,j}_{\mu_0,T}[dx]}{M_{\mu_0}[dx]} \cdot \frac{M^r_{\mu_T}[d\widetilde{x}]}{M^{r,\rho,-j}_{\mu_T}[d\widetilde{x}]} \cdot \frac{M^{r,\rho,-j}_{\mu_T}[d\widetilde{x}]}{M^{\rho,j}_{\mu_0,T}[dx]}$

$$= \frac{M^{\rho,j}_{\mu_0,T}}{M_{\mu_0,T}}\left[\rho^e_T, j^e_T\right] \cdot \frac{M^r_{\mu_T}}{M^{r,\rho,-j}_{\mu_T}}\left[\rho^e_T, -j^e_T\right] \cdot \frac{M^{\rho,-j}_{\mu_T}[d\widetilde{x}]}{M^{\rho,j}_{\mu_0,T}[dx]} = \frac{M^{\rho,j}_{\mu_0,T}}{M_{\mu_0,T}}\left[\rho^e_T, j^e_T\right] \cdot \frac{M^r_{\mu_T}}{M^{r,\rho,-j}_{\mu_T}}\left[\rho^e_T, -j^e_T\right].1$$

- We note $\sigma_T[x] = Tw\left[\rho^e_T, j^e_T\right]$. and we obtain the relation :

$$w[\rho, j] = I^r[\rho, -j] - I[\rho, j]$$

Gallavotti-Cohen in the 2,5 Large deviation theory

For Diffusion process, case $x^* = x$ for simplify calculation

- $\exp\left(-\sigma_T[x]\right) \equiv \frac{M^r_{\mu_T}[d\widetilde{x}]}{M_{\mu_0}[dx]} = \frac{M^{\rho,j}_{\mu_0,T}[dx]}{M_{\mu_0}[dx]} \cdot \frac{M^r_{\mu_T}[d\widetilde{x}]}{M^{r,\rho,-j}_{\mu_T}[d\widetilde{x}]} \cdot \frac{M^{r,\rho,-j}_{\mu_T}[d\widetilde{x}]}{M^{\rho,j}_{\mu_0,T}[dx]}$

$$= \frac{M^{\rho,j}_{\mu_0,T}}{M_{\mu_0,T}}\left[\rho^e_T, j^e_T\right] \cdot \frac{M^r_{\mu_T}}{M^{r,\rho,-j}_{\mu_T}}\left[\rho^e_T, -j^e_T\right] \cdot \frac{M^{\rho,-j}_{\mu_T}[d\widetilde{x}]}{M^{\rho,j}_{\mu_0,T}[dx]} = \frac{M^{\rho,j}_{\mu_0,T}}{M_{\mu_0,T}}\left[\rho^e_T, j^e_T\right] \cdot \frac{M^r_{\mu_T}}{M^{r,\rho,-j}_{\mu_T}}\left[\rho^e_T, -j^e_T\right].1$$

- We note $\sigma_T[x] = Tw\left[\rho^e_T, j^e_T\right]$. and we obtain the relation :

$$w\left[\rho, j\right] = I^r\left[\rho, -j\right] - I\left[\rho, j\right]$$

Level 1 by contraction : $H(w) = \min_{w[j,\rho]=w} I\left[\rho, j\right]$

Gallavotti-Cohen in the 2,5 Large deviation theory

For Diffusion process, case $x^* = x$ for simplify calculation

- $\exp\left(-\sigma_T[x]\right) \equiv \frac{M^r_{\mu_T}[d\widetilde{x}]}{M_{\mu_0}[dx]} = \frac{M^{\rho,j}_{\mu_0,T}[dx]}{M_{\mu_0}[dx]} \cdot \frac{M^r_{\mu_T}[d\widetilde{x}]}{M^{r,\rho,-j}_{\mu_T}[d\widetilde{x}]} \cdot \frac{M^{r,\rho,-j}_{\mu_T}[d\widetilde{x}]}{M^{\rho,j}_{\mu_0,T}[dx]}$
  $= \frac{M^{\rho,j}_{\mu_0,T}}{M_{\mu_0,T}}\left[\rho^e_T, j^e_T\right] \cdot \frac{M^r_{\mu_T}}{M^{r,\rho,-j}_{\mu_T}}\left[\rho^e_T, -j^e_T\right] \cdot \frac{M^{\rho,-j}_{\mu_T}[d\widetilde{x}]}{M^{\rho,j}_{\mu_0,T}[dx]} = \frac{M^{\rho,j}_{\mu_0,T}}{M_{\mu_0,T}}\left[\rho^e_T, j^e_T\right] \cdot \frac{M^r_{\mu_T}}{M^{r,\rho,-j}_{\mu_T}}\left[\rho^e_T, -j^e_T\right] .1$
- We note $\sigma_T[x] = Tw\left[\rho^e_T, j^e_T\right]$. and we obtain the relation :

$$w\left[\rho, j\right] = I^r\left[\rho, -j\right] - I\left[\rho, j\right]$$

Level 1 by contraction : $H(w) = \min_{w[j,\rho]=w} I\left[\rho, j\right]$

- $\min_{w[j,\rho]=w}\left(I\left[\rho, j\right] + w\left[\rho, j\right]\right) = H(x) + x$

Gallavotti-Cohen in the 2,5 Large deviation theory

For Diffusion process, case $x^* = x$ for simplify calculation

- $\exp\left(-\sigma_T[x]\right) \equiv \frac{M^r_{\mu_T}[d\widetilde{x}]}{M_{\mu_0}[dx]} = \frac{M^{\rho,j}_{\mu_0,T}[dx]}{M_{\mu_0}[dx]} \cdot \frac{M^r_{\mu_T}[d\widetilde{x}]}{M^{r,\rho,-j}_{\mu_T}[d\widetilde{x}]} \cdot \frac{M^{r,\rho,-j}_{\mu_T}[d\widetilde{x}]}{M^{\rho,j}_{\mu_0,T}[dx]}$
  $= \frac{M^{\rho,j}_{\mu_0,T}}{M_{\mu_0,T}}\left[\rho^e_T, j^e_T\right] \cdot \frac{M^r_{\mu_T}}{M^{r,\rho,-j}_{\mu_T}}\left[\rho^e_T, -j^e_T\right] \cdot \frac{M^{\rho,-j}_{\mu_T}[d\widetilde{x}]}{M^{\rho,j}_{\mu_0,T}[dx]} = \frac{M^{\rho,j}_{\mu_0,T}}{M_{\mu_0,T}}\left[\rho^e_T, j^e_T\right] \cdot \frac{M^r_{\mu_T}}{M^{r,\rho,-j}_{\mu_T}}\left[\rho^e_T, -j^e_T\right].1$
- We note $\sigma_T[x] = Tw\left[\rho^e_T, j^e_T\right]$. and we obtain the relation :

$$w[\rho, j] = I^r[\rho, -j] - I[\rho, j]$$

Level 1 by contraction : $H(w) = \min_{w[j,\rho]=w} I[\rho, j]$

- $\min_{w[j,\rho]=w}\left(I[\rho, j] + w[\rho, j]\right) = H(x) + x = \min_{w[j,\rho]=w} I^r[\rho, -j]$

Gallavotti-Cohen in the 2,5 Large deviation theory

For Diffusion process, case $x^* = x$ for simplify calculation

- $\exp\left(-\sigma_T[x]\right) \equiv \frac{M^r_{\mu T}[d\widetilde{x}]}{M_{\mu_0}[dx]} = \frac{M^{\rho,j}_{\mu_0,T}[dx]}{M_{\mu_0}[dx]} \cdot \frac{M^r_{\mu T}[d\widetilde{x}]}{M^{r,\rho,-j}_{\mu T}[d\widetilde{x}]} \cdot \frac{M^{r,\rho,-j}_{\mu T}[d\widetilde{x}]}{M^{\rho,j}_{\mu_0,T}[dx]}$
  $= \frac{M^{\rho,j}_{\mu_0,T}}{M_{\mu_0,T}}\left[\rho^e_T, j^e_T\right] \cdot \frac{M^r_{\mu T}}{M^{r,\rho,-j}_{\mu T}}\left[\rho^e_T, -j^e_T\right] \cdot \frac{M^{\rho,-j}_{\mu T}[d\widetilde{x}]}{M^{\rho,j}_{\mu_0,T}[dx]} = \frac{M^{\rho,j}_{\mu_0,T}}{M_{\mu_0,T}}\left[\rho^e_T, j^e_T\right] \cdot \frac{M^r_{\mu T}}{M^{r,\rho,-j}_{\mu T}}\left[\rho^e_T, -j^e_T\right].1$
- We note $\sigma_T[x] = Tw\left[\rho^e_T, j^e_T\right]$. and we obtain the relation :

$$w\left[\rho, j\right] = I^r\left[\rho, -j\right] - I\left[\rho, j\right]$$

Level 1 by contraction : $H(w) = \min_{w[j,\rho]=w} I\left[\rho, j\right]$

- $\min_{w[j,\rho]=w}\left(I\left[\rho, j\right] + w\left[\rho, j\right]\right) = H(x) + x = \min_{w[j,\rho]=w} I^r\left[\rho, -j\right]$
$= \min_{w[-j,\rho]=x} I^r\left[\rho, j\right]$

Gallavotti-Cohen in the 2,5 Large deviation theory

For Diffusion process, case $x^* = x$ for simplify calculation

- $\exp(-\sigma_T[x]) \equiv \frac{M^r_{\mu_T}[d\widetilde{x}]}{M_{\mu_0}[dx]} = \frac{M^{\rho,j}_{\mu_0,T}[dx]}{M_{\mu_0}[dx]} \cdot \frac{M^r_{\mu_T}[d\widetilde{x}]}{M^{r,\rho,-j}_{\mu_T}[d\widetilde{x}]} \cdot \frac{M^{r,\rho,-j}_{\mu_T}[d\widetilde{x}]}{M^{\rho,j}_{\mu_0,T}[dx]}$

$= \frac{M^{\rho,j}_{\mu_0,T}}{M_{\mu_0,T}}[\rho^e_T, j^e_T] \cdot \frac{M^r_{\mu_T}}{M^{r,\rho,-j}_{\mu_T}}[\rho^e_T, -j^e_T] \cdot \frac{M^{\rho,-j}_{\mu_T}[d\widetilde{x}]}{M^{\rho,j}_{\mu_0,T}[dx]} = \frac{M^{\rho,j}_{\mu_0,T}}{M_{\mu_0,T}}[\rho^e_T, j^e_T] \cdot \frac{M^r_{\mu_T}}{M^{r,\rho,-j}_{\mu_T}}[\rho^e_T, -j^e_T] .1$

- We note $\sigma_T[x] = Tw[\rho^e_T, j^e_T]$. and we obtain the relation :

$$w[\rho, j] = I^r[\rho, -j] - I[\rho, j]$$

Level 1 by contraction : $H(w) = \min_{w[j,\rho]=w} I[\rho, j]$

- $\min_{w[j,\rho]=w}(I[\rho, j] + w[\rho, j]) = H(x) + x = \min_{w[j,\rho]=w} I^r[\rho, -j]$

$=\min_{w[-j,\rho]=x} I^r[\rho, j] = \min_{w^r[j,\rho]=-x} I^r[\rho, j]$

Gallavotti-Cohen in the 2,5 Large deviation theory

For Diffusion process, case $x^* = x$ for simplify calculation

- $\exp\left(-\sigma_T[x]\right) \equiv \frac{M^r_{\mu_T}[d\widetilde{x}]}{M_{\mu_0}[dx]} = \frac{M^{\rho,j}_{\mu_0,T}[dx]}{M_{\mu_0}[dx]} \cdot \frac{M^r_{\mu_T}[d\widetilde{x}]}{M^{r,\rho,-j}_{\mu_T}[d\widetilde{x}]} \cdot \frac{M^{r,\rho,-j}_{\mu_T}[d\widetilde{x}]}{M^{\rho,j}_{\mu_0,T}[dx]}$

$$= \frac{M^{\rho,j}_{\mu_0,T}}{M_{\mu_0,T}}\left[\rho^e_T, j^e_T\right] \cdot \frac{M^r_{\mu_T}}{M^{r,\rho,-j}_{\mu_T}}\left[\rho^e_T, -j^e_T\right] \cdot \frac{M^{\rho,-j}_{\mu_T}[d\widetilde{x}]}{M^{\rho,j}_{\mu_0,T}[dx]} = \frac{M^{\rho,j}_{\mu_0,T}}{M_{\mu_0,T}}\left[\rho^e_T, j^e_T\right] \cdot \frac{M^r_{\mu_T}}{M^{r,\rho,-j}_{\mu_T}}\left[\rho^e_T, -j^e_T\right].1$$

- We note $\sigma_T[x] = Tw\left[\rho^e_T, j^e_T\right]$. and we obtain the relation :

$$w\left[\rho, j\right] = I^r\left[\rho, -j\right] - I\left[\rho, j\right]$$

Level 1 by contraction : $H(w) = \min_{w[j,\rho]=w} I\left[\rho, j\right]$

- $\min_{w[j,\rho]=w}\left(I\left[\rho, j\right] + w\left[\rho, j\right]\right) = H(x) + x = \min_{w[j,\rho]=w} I^r\left[\rho, -j\right]$

$= \min_{w[-j,\rho]=x} I^r\left[\rho, j\right] = \min_{w^r[j,\rho]=-x} I^r\left[\rho, j\right] = H^r(-x)$

Gallavotti-Cohen in the 2,5 Large deviation theory

For Diffusion process, case $x^* = x$ for simplify calculation

- $\exp\left(-\sigma_T[x]\right) \equiv \frac{M^r_{\mu_T}[d\widetilde{x}]}{M_{\mu_0}[dx]} = \frac{M^{\rho,j}_{\mu_0,T}[dx]}{M_{\mu_0}[dx]} \cdot \frac{M^r_{\mu_T}[d\widetilde{x}]}{M^{r,\rho,-j}_{\mu_T}[d\widetilde{x}]} \cdot \frac{M^{r,\rho,-j}_{\mu_T}[d\widetilde{x}]}{M^{\rho,j}_{\mu_0,T}[dx]}$

$= \frac{M^{\rho,j}_{\mu_0,T}}{M_{\mu_0,T}}\left[\rho^e_T, j^e_T\right] \cdot \frac{M^r_{\mu_T}}{M^{r,\rho,-j}_{\mu_T}}\left[\rho^e_T, -j^e_T\right] \cdot \frac{M^{\rho,-j}_{\mu_T}[d\widetilde{x}]}{M^{\rho,j}_{\mu_0,T}[dx]} = \frac{M^{\rho,j}_{\mu_0,T}}{M_{\mu_0,T}}\left[\rho^e_T, j^e_T\right] \cdot \frac{M^r_{\mu_T}}{M^{r,\rho,-j}_{\mu_T}}\left[\rho^e_T, -j^e_T\right] .1$

- We note $\sigma_T[x] = Tw\left[\rho^e_T, j^e_T\right]$. and we obtain the relation :

$$w\left[\rho, j\right] = I^r\left[\rho, -j\right] - I\left[\rho, j\right]$$

Level 1 by contraction : $H(w) = \min_{w[j,\rho]=w} I\left[\rho, j\right]$

- $\min_{w[j,\rho]=w}\left(I\left[\rho, j\right] + w\left[\rho, j\right]\right) = H(x) + x = \min_{w[j,\rho]=w} I^r\left[\rho, -j\right]$

$=\min_{w[-j,\rho]=x} I^r\left[\rho, j\right] = \min_{w^r[j,\rho]=-x} I^r\left[\rho, j\right] = H^r(-x)$

For pure jump process : $w[\rho, C] = I^r\left[\rho, C^T\right] - I\left[\rho, C\right]$

Gallavotti-Cohen in the 2,5 Large deviation theory

For Diffusion process, case $x^* = x$ for simplify calculation

- $\exp\left(-\sigma_T[x]\right) \equiv \frac{M^r_{\mu_T}[d\widetilde{x}]}{M_{\mu_0}[dx]} = \frac{M^{\rho,j}_{\mu_0,T}[dx]}{M_{\mu_0}[dx]} \cdot \frac{M^r_{\mu_T}[d\widetilde{x}]}{M^{r,\rho,-j}_{\mu_T}[d\widetilde{x}]} \cdot \frac{M^{r,\rho,-j}_{\mu_T}[d\widetilde{x}]}{M^{\rho,j}_{\mu_0,T}[dx]}$

$= \frac{M^{\rho,j}_{\mu_0,T}}{M_{\mu_0,T}}\left[\rho^e_T, j^e_T\right] \cdot \frac{M^r_{\mu_T}}{M^{r,\rho,-j}_{\mu_T}}\left[\rho^e_T, -j^e_T\right] \cdot \frac{M^{\rho,-j}_{\mu_T}[d\widetilde{x}]}{M^{\rho,j}_{\mu_0,T}[dx]} = \frac{M^{\rho,j}_{\mu_0,T}}{M_{\mu_0,T}}\left[\rho^e_T, j^e_T\right] \cdot \frac{M^r_{\mu_T}}{M^{r,\rho,-j}_{\mu_T}}\left[\rho^e_T, -j^e_T\right].1$

- We note $\sigma_T[x] = Tw\left[\rho^e_T, j^e_T\right]$. and we obtain the relation :

$$w\left[\rho, j\right] = I^r\left[\rho, -j\right] - I\left[\rho, j\right]$$

Level 1 by contraction : $H(w) = \min_{w[j,\rho]=w} I\left[\rho, j\right]$

- $\min_{w[j,\rho]=w}\left(I\left[\rho, j\right] + w\left[\rho, j\right]\right) = H(x) + x = \min_{w[j,\rho]=w} I^r\left[\rho, -j\right]$

$= \min_{w[-j,\rho]=x} I^r\left[\rho, j\right] = \min_{w^r[j,\rho]=-x} I^r\left[\rho, j\right] = H^r(-x)$

For pure jump process : $w[\rho, C] = I^r\left[\rho, C^T\right] - I\left[\rho, C\right]$

- Case backward systems = forward systems : $w[C] = I\left[\rho, C^T\right] - I\left[\rho, C\right]$ and $w[C] = w(J = C - C^T)$.

Gallavotti-Cohen in the 2,5 Large deviation theory

For Diffusion process, case $x^* = x$ for simplify calculation

- $\exp\left(-\sigma_T[x]\right) \equiv \frac{M^r_{\mu_T}[d\widetilde{x}]}{M_{\mu_0}[dx]} = \frac{M^{\rho,j}_{\mu_0,T}[dx]}{M_{\mu_0}[dx]} \cdot \frac{M^r_{\mu_T}[d\widetilde{x}]}{M^{r,\rho,-j}_{\mu_T}[d\widetilde{x}]} \cdot \frac{M^{r,\rho,-j}_{\mu_T}[d\widetilde{x}]}{M^{\rho,j}_{\mu_0,T}[dx]}$

$$= \frac{M^{\rho,j}_{\mu_0,T}}{M_{\mu_0,T}}\left[\rho^e_T, j^e_T\right] \cdot \frac{M^r_{\mu_T}}{M^{r,\rho,-j}_{\mu_T}}\left[\rho^e_T, -j^e_T\right] \cdot \frac{M^{\rho,-j}_{\mu_T}[d\widetilde{x}]}{M^{\rho,j}_{\mu_0,T}[dx]} = \frac{M^{\rho,j}_{\mu_0,T}}{M_{\mu_0,T}}\left[\rho^e_T, j^e_T\right] \cdot \frac{M^r_{\mu_T}}{M^{r,\rho,-j}_{\mu_T}}\left[\rho^e_T, -j^e_T\right].1$$

- We note $\sigma_T[x] = Tw\left[\rho^e_T, j^e_T\right]$. and we obtain the relation :

$$w\left[\rho, j\right] = I^r\left[\rho, -j\right] - I\left[\rho, j\right]$$

Level 1 by contraction : $H(w) = \min_{w[j,\rho]=w} I\left[\rho, j\right]$

- $\min_{w[j,\rho]=w}\left(I\left[\rho, j\right] + w\left[\rho, j\right]\right) = H(x) + x = \min_{w[j,\rho]=w} I^r\left[\rho, -j\right]$

$= \min_{w[-j,\rho]=x} I^r\left[\rho, j\right] = \min_{w^r[j,\rho]=-x} I^r\left[\rho, j\right] = H^r(-x)$

For pure jump process : $w[\rho, C] = I^r\left[\rho, C^T\right] - I\left[\rho, C\right]$

- Case backward systems = forward systems : $w[C] = I\left[\rho, C^T\right] - I\left[\rho, C\right]$ and $w[C] = w(J = C - C^T)$.
- Contraction $\Rightarrow$ $w[J] = I\left[-J\right] - I\left[J\right]$

Remarques-Conclusion

- Tous les courants réels J^r (Combinaison linéaire des éléments de matrice du courant empirique) ne vérifient pas une relation de Gallavotti-Cohen.

Remarques-Conclusion

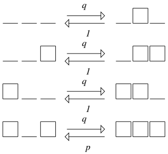
- Tous les courants réels J^r (Combinaison linéaire des éléments de matrice du courant empirique) ne vérifient pas une relation de Gallavotti-Cohen.
- Condition suffisante : $w[J] \sim J^r$ dans le limite des grandes déviations.

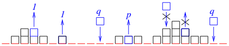
Remarques-Conclusion

- Tous les courants réels J^r (Combinaison linéaire des éléments de matrice du courant empirique) ne vérifient pas une relation de Gallavotti-Cohen.
- Condition suffisante : $w[J] \sim J^r$ dans le limite des grandes déviations.
- Exemple : SSEP avec 2 reservoirs : Résultat de Derrida-Bodineau C.R Phys 2007 pour les grandes déviations du nombre de particule algébrique allant d'un reservoir vers le systeme.

Remarques-Conclusion

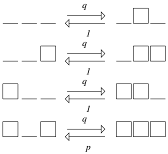
- Tous les courants réels J^r (Combinaison linéaire des éléments de matrice du courant empirique) ne vérifient pas une relation de Gallavotti-Cohen.
- Condition suffisante : $w[J] \sim J^r$ dans le limite des grandes déviations.
- Exemple : SSEP avec 2 reservoirs : Résultat de Derrida-Bodineau C.R Phys 2007 pour les grandes déviations du nombre de particule algébrique allant d'un reservoir vers le systeme.
- Contre Exemple : RSOS model : modèle discret de croissance d'interfaces. $w[J] \nsim$ Hauteur moyenne.

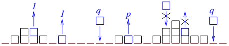




Remarques-Conclusion

- Tous les courants réels J^r (Combinaison linéaire des éléments de matrice du courant empirique) ne vérifient pas une relation de Gallavotti-Cohen.
- Condition suffisante : $w[J] \sim J^r$ dans le limite des grandes déviations.
- Exemple : SSEP avec 2 reservoirs : Résultat de Derrida-Bodineau C.R Phys 2007 pour les grandes déviations du nombre de particule algébrique allant d'un reservoir vers le systeme.
- Contre Exemple : RSOS model : modèle discret de croissance d'interfaces. $w[J] \nsim$ Hauteur moyenne.





On a une symétrie seulement pour les grandes déviations jointe de la hauteur moyenne et d'un autre courant Barato-Chetrite-Hinrichsen-Mukamel.

- Maintenant, les systèmes quantiques...

One-dimensional Kardar-Parisi-Zhang equation

$$\partial_t h_t(x) = \nu \nabla^2 h_t(x) + \frac{\lambda}{2} \left(\nabla h_t(x)\right)^2 + \eta_t(x) \quad \text{with}$$
$$\langle \eta_t(x) \eta_s(y) \rangle = 2D\delta(t-s)\delta(x-y)$$

Non Equilibrium diffusion without Lagrangian pictures

One-dimensional Kardar-Parisi-Zhang equation

$$\partial_t h_t(x) = \nu \nabla^2 h_t(x) + \frac{\lambda}{2} \left(\nabla h_t(x)\right)^2 + \eta_t(x) \quad \text{with}$$
$$\langle \eta_t(x) \eta_s(y) \rangle = 2D\delta(t-s)\delta(x-y)$$

- Invariant density : $\rho = \frac{1}{Z} \exp\left(-\frac{\nu}{2D} \int \left(\nabla h(x)\right)^2 dx\right)$
- Mean local velocity : $v[h](x) = \frac{\lambda}{2} \left(\nabla h(x)\right)^2$

Non Equilibrium diffusion without Lagrangian pictures

One-dimensional Kardar-Parisi-Zhang equation

$$\partial_t h_t(x) = \nu \nabla^2 h_t(x) + \frac{\lambda}{2} \left(\nabla h_t(x)\right)^2 + \eta_t(x) \quad \text{with}$$
$$\langle \eta_t(x) \eta_s(y) \rangle = 2D\delta(t-s)\delta(x-y)$$

- Invariant density : $\rho = \frac{1}{Z} \exp\left(-\frac{\nu}{2D} \int \left(\nabla h(x)\right)^2 dx \right)$
- Mean local velocity : $v[h](x) = \frac{\lambda}{2} \left(\nabla h(x)\right)^2$
- The Lagrangian flow is solution of : $\partial_t h_t = \frac{\lambda}{2} \left(\nabla h_t(x)\right)^2$ that becomes for $u_t(x) = -\lambda \nabla h_t(x)$ the inviscid Burgers equation :

$$\partial_t u_t(x) + u_t(x) \nabla u_t(x) = 0$$

Non Equilibrium diffusion without Lagrangian pictures

One-dimensional Kardar-Parisi-Zhang equation

$$\partial_t h_t(x) = \nu \nabla^2 h_t(x) + \frac{\lambda}{2} \left(\nabla h_t(x)\right)^2 + \eta_t(x) \quad \text{with}$$
$$\langle \eta_t(x) \eta_s(y) \rangle = 2D\delta(t-s)\delta(x-y)$$

- Invariant density : $\rho = \frac{1}{Z} \exp\left(-\frac{\nu}{2D} \int \left(\nabla h(x)\right)^2 dx\right)$
- Mean local velocity : $v[h](x) = \frac{\lambda}{2} \left(\nabla h(x)\right)^2$
- The Lagrangian flow is solution of : $\partial_t h_t = \frac{\lambda}{2} \left(\nabla h_t(x)\right)^2$ that becomes for $u_t(x) = -\lambda \nabla h_t(x)$ the inviscid Burgers equation :

$$\partial_t u_t(x) + u_t(x) \nabla u_t(x) = 0$$

$\Rightarrow$ Shocks $\Rightarrow$ No unique global invertible Lagrangian flow $\Rightarrow$ No global Lagrangian frame picture.

Diffusive hydrodynamical limits of lattice particle systems

$$\partial_t n_t + \nabla j_t = 0 \text{ with } j_t(x) = -D(n_t(x))\nabla n_t(x) + \eta_t(x, n_t)$$
$$\text{and } \langle \eta_t(x,n)\eta_s(y,n)\rangle = \epsilon\chi(n)\delta(t-s)\delta(x-y)$$

Spohn, H. : Large scale dynamics of Interacting Particles. 1991

- Invariant density : $\rho[n] = \exp\left(-\frac{1}{\epsilon}S[n]\right)$

Diffusive hydrodynamical limits of lattice particle systems

$$\partial_t n_t + \nabla j_t = 0 \text{ with } j_t(x) = -D\left(n_t(x)\right)\nabla n_t(x) + \eta_t(x, n_t)$$
$$\text{and } \langle \eta_t(x,n)\eta_s(y,n)\rangle = \epsilon\chi(n)\delta(t-s)\delta(x-y)$$

Spohn, H. : Large scale dynamics of Interacting Particles. 1991

- Invariant density : $\rho[n] = \exp\left(-\frac{1}{\epsilon}S[n]\right)$

zero range processes :

$D^{ij}\left(n(x)\right) = \varphi'(n(x))\delta^{ij}$ and $\chi^{ij}\left(n(x)\right) = \varphi(n(x))\delta^{ij}$ for an increasing function φ. And $\frac{\delta S}{\delta n(x)} = \ln\frac{\varphi(n(x))}{\lambda(x)}$ with $\nabla^2\lambda(x) = 0$

Kipnis, C. , Landim, C. : Scaling limits of interacting Particle Systems. 1999

Diffusive hydrodynamical limits of lattice particle systems

$$\partial_t n_t + \nabla j_t = 0 \text{ with } j_t(x) = -D\left(n_t(x)\right)\nabla n_t(x) + \eta_t(x, n_t)$$
$$\text{and } \langle \eta_t(x,n)\eta_s(y,n)\rangle = \epsilon\chi(n)\delta(t-s)\delta(x-y)$$

Spohn, H. : Large scale dynamics of Interacting Particles. 1991

- Invariant density : $\rho[n] = \exp\left(-\frac{1}{\epsilon}S[n]\right)$

zero range processes :

$D^{ij}\left(n(x)\right) = \varphi'(n(x))\delta^{ij}$ and $\chi^{ij}\left(n(x)\right) = \varphi(n(x))\delta^{ij}$ for an increasing function φ. And $\frac{\delta S}{\delta n(x)} = \ln\frac{\varphi(n(x))}{\lambda(x)}$ with $\nabla^2\lambda(x) = 0$

Kipnis, C. , Landim, C. : Scaling limits of interacting Particle Systems. 1999

- Mean local velocity : $v[n](x) = \frac{1}{2}\nabla.\left(\varphi(n(x))\nabla\ln\lambda(x)\right)$

Diffusive hydrodynamical limits of lattice particle systems

$$\partial_t n_t + \nabla j_t = 0 \text{ with } j_t(x) = -D\left(n_t(x)\right)\nabla n_t(x) + \eta_t(x, n_t)$$
$$\text{and } \langle \eta_t(x,n)\eta_s(y,n)\rangle = \epsilon\chi(n)\delta(t-s)\delta(x-y)$$

Spohn, H. : Large scale dynamics of Interacting Particles. 1991

- Invariant density : $\rho[n] = \exp\left(-\frac{1}{\epsilon}S[n]\right)$

zero range processes :

$D^{ij}\left(n(x)\right) = \varphi'(n(x))\delta^{ij}$ and $\chi^{ij}\left(n(x)\right) = \varphi(n(x))\delta^{ij}$ for an increasing function φ. And $\frac{\delta S}{\delta n(x)} = \ln\frac{\varphi(n(x))}{\lambda(x)}$ with $\nabla^2\lambda(x) = 0$

Kipnis, C. , Landim, C. : Scaling limits of interacting Particle Systems. 1999

- Mean local velocity : $v[n](x) = \frac{1}{2}\nabla.\left(\varphi(n(x))\nabla \ln\lambda(x)\right)$
- The Lagrangian flow is a quasi-linear first order PDE whose global solution is again obstructed by caustics.

Diffusive hydrodynamical limits of lattice particle systems

$$\partial_t n_t + \nabla j_t = 0 \text{ with } j_t(x) = -D\left(n_t(x)\right)\nabla n_t(x) + \eta_t(x, n_t)$$
$$\text{and } \langle \eta_t(x,n)\eta_s(y,n)\rangle = \epsilon\chi(n)\delta(t-s)\delta(x-y)$$

Spohn, H. : Large scale dynamics of Interacting Particles. 1991

- Invariant density : $\rho[n] = \exp\left(-\frac{1}{\epsilon}S[n]\right)$

zero range processes :

$D^{ij}\left(n(x)\right) = \varphi'(n(x))\delta^{ij}$ and $\chi^{ij}\left(n(x)\right) = \varphi(n(x))\delta^{ij}$ for an increasing function φ. And $\frac{\delta S}{\delta n(x)} = \ln\frac{\varphi(n(x))}{\lambda(x)}$ with $\nabla^2\lambda(x) = 0$

Kipnis, C. , Landim, C. : Scaling limits of interacting Particle Systems. 1999

- Mean local velocity : $v[n](x) = \frac{1}{2}\nabla.\left(\varphi(n(x))\nabla\ln\lambda(x)\right)$
- The Lagrangian flow is a quasi-linear first order PDE whose global solution is again obstructed by caustics.

SSEP : ? ? ? ?

Backward process

$\frac{dx}{dt} = (u_{t^*,+}(x) - u_{t^*,-}(x) - v_{t^*}(x))$
Non trivial for the markovian generator :
$L^r_{t^*} = \left(L_t - 2\widehat{u}_{t,+}.\nabla - \nabla.\widehat{u}_{t,+} + \nabla.u_{t,-}\right)$

Example of time inversion

Natural inversion

Choice : $u_{t,+} = 0 \leftrightarrow u_{t,-} = u$.

Example of time inversion

Natural inversion

Choice : $u_{t,+} = 0 \leftrightarrow u_{t,-} = u$.

Deterministic case : $W_T^{nat} = -ln(\rho_0^r)(x_T) + ln(\rho_0)(x_0) - \int_0^T \nabla \cdot u_t(x_t)\, dt$,

$\Longrightarrow$ Phase space contraction, called "dissipation function" by Evans-Searles.

Example of time inversion

Natural inversion

Choice : $u_{t,+} = 0 \leftrightarrow u_{t,-} = u$.

Deterministic case : $W_T^{nat} = -ln(\rho_0^r)(x_T) + ln(\rho_0)(x_0) - \int_0^T \nabla \cdot u_t(x_t)\, dt$,

$\Longrightarrow$ Phase space contraction, called "dissipation function" by Evans-Searles.

Reversed protocol

Choice : $u_{t,+} = u \leftrightarrow u_{t,-} = 0$

$\Longrightarrow$ $W_T^{tot} = -ln(\rho_0^r)(x_T) + ln(\rho_0)(x_0) + \int_0^T \left(2\widehat{u}_t \cdot d_t^{-1}(x_t) \cdot \dot{x}_t - \nabla \cdot u_t(x_t) \right) dt$

Lebowitz-Spohn : *A Gallavotti-Cohen type symmetry in the large deviation functional for stochastic dynamics*. J. Stat. Phys.(1999)

Example of time inversion

Natural inversion

Choice : $u_{t,+} = 0 \leftrightarrow u_{t,-} = u$.

Deterministic case : $W_T^{nat} = -ln(\rho_0^r)(x_T) + ln(\rho_0)(x_0) - \int_0^T \nabla \cdot u_t(x_t)\, dt$,

$\Longrightarrow$ Phase space contraction, called "dissipation function" by Evans-Searles.

Reversed protocol

Choice : $u_{t,+} = u \leftrightarrow u_{t,-} = 0$

$$\Longrightarrow \; W_T^{tot} = -ln(\rho_0^r)(x_T) + ln(\rho_0)(x_0) + \int_0^T \Big(2\widehat{u}_t \cdot d_t^{-1}(x_t) \cdot \dot{x}_t - \nabla \cdot u_t(x_t) \Big)\, dt$$

Lebowitz-Spohn : *A Gallavotti-Cohen type symmetry in the large deviation functional for stochastic dynamics*. J. Stat. Phys.(1999)

Markov Chain $W_N^{tot}[x] = \ln\left[\frac{\mu_0(dx_0)dx_N}{dx_0 \mu_0^r(dx_N)}\right] + \ln\left[\frac{M_0(x_0,x_1)}{M_0(x_1,x_0)} \cdot \frac{M_1(x_1,x_2)}{M_1(x_2,x_1)} \cdots \frac{M_{N-1}(x_{N-1},x_N)}{M_{N-1}(x_N,x_{N-1})}\right]$

Example of time inversion

Natural inversion

Choice : $u_{t,+} = 0 \leftrightarrow u_{t,-} = u$.

Deterministic case : $W_T^{nat} = -ln(\rho_0^r)(x_T) + ln(\rho_0)(x_0) - \int_0^T \nabla \cdot u_t(x_t)\, dt,$

$\Longrightarrow$ Phase space contraction, called "dissipation function" by Evans-Searles.

Reversed protocol

Choice : $u_{t,+} = u \leftrightarrow u_{t,-} = 0$

$$\Longrightarrow \quad W_T^{tot} = -ln(\rho_0^r)(x_T) + ln(\rho_0)(x_0) + \int_0^T \Big(2\widehat{u}_t \cdot d_t^{-1}(x_t) \cdot \dot{x}_t - \nabla \cdot u_t(x_t) \Big)\, dt$$

Lebowitz-Spohn : *A Gallavotti-Cohen type symmetry in the large deviation functional for stochastic dynamics*. J. Stat. Phys.(1999)

Markov Chain $w_N^{tot}[x] = \ln \left[\frac{\mu_0(dx_0) dx_N}{dx_0 \mu_0^r(dx_N)} \right] + \ln \left[\frac{M_0(x_0, x_1)}{M_0(x_1, x_0)} \cdot \frac{M_1(x_1, x_2)}{M_1(x_2, x_1)} \cdots \frac{M_{N-1}(x_{N-1}, x_N)}{M_{N-1}(x_N, x_{N-1})} \right]$

Current inversion

We introduce the density "locally invariant" $\rho_t^{li} = \exp(-\varphi_t^{li})$ with : $div(j_t^{li}) = div(\rho_t^{li} v_t^{li}) = 0$.

Example of time inversion

Natural inversion

Choice : $u_{t,+} = 0 \leftrightarrow u_{t,-} = u$.

Deterministic case : $W_T^{nat} = -ln(\rho_0^r)(x_T) + ln(\rho_0)(x_0) - \int_0^T \nabla \cdot u_t(x_t)\, dt,$

$\Longrightarrow$ Phase space contraction, called "dissipation function" by Evans-Searles.

Reversed protocol

Choice : $u_{t,+} = u \leftrightarrow u_{t,-} = 0$

$$\Longrightarrow \quad W_T^{tot} = -ln(\rho_0^r)(x_T) + ln(\rho_0)(x_0) + \int_0^T \left(2\widehat{u}_t \cdot d_t^{-1}(x_t) \cdot \dot{x}_t - \nabla \cdot u_t(x_t)\right)\, dt$$

Lebowitz-Spohn : *A Gallavotti-Cohen type symmetry in the large deviation functional for stochastic dynamics*. J. Stat. Phys.(1999)

Markov Chain $w_N^{tot}[x] = \ln\left[\frac{\mu_0(dx_0)dx_N}{dx_0\,\mu_0^r(dx_N)}\right] + \ln\left[\frac{M_0(x_0,x_1)}{M_0(x_1,x_0)} \cdot \frac{M_1(x_1,x_2)}{M_1(x_2,x_1)} \cdots \frac{M_{N-1}(x_{N-1},x_N)}{M_{N-1}(x_N,x_{N-1})}\right]$

Current inversion

We introduce the density "locally invariant" $\rho_t^{li} = \exp(-\varphi_t^{li})$ with : $div(j_t^{li}) = div(\rho_t^{li} v_t^{li}) = 0$.

With the choice : $u_{t,+} = \frac{d_t}{2} \cdot \rho_t^{li} \leftrightarrow u_{t,-} = v_t^{li}$.

The backward process is associated with opposite probability current.

Example of time inversion

Natural inversion

Choice : $u_{t,+} = 0 \leftrightarrow u_{t,-} = u$.

Deterministic case : $W_T^{nat} = -ln(\rho_0^r)(x_T) + ln(\rho_0)(x_0) - \int_0^T \nabla \cdot u_t(x_t)\, dt,$

$\Longrightarrow$ Phase space contraction, called "dissipation function" by Evans-Searles.

Reversed protocol

Choice : $u_{t,+} = u \leftrightarrow u_{t,-} = 0$

$\Longrightarrow \quad W_T^{tot} = -ln(\rho_0^r)(x_T) + ln(\rho_0)(x_0) + \int_0^T \Big(2\widehat{u}_t \cdot d_t^{-1}(x_t) \cdot \dot{x}_t - \nabla \cdot u_t(x_t) \Big)\, dt$

Lebowitz-Spohn : *A Gallavotti-Cohen type symmetry in the large deviation functional for stochastic dynamics*. J. Stat. Phys.(1999)

Markov Chain $\; w_N^{tot}[x] = \ln \left[\frac{\mu_0(dx_0)dx_N}{dx_0 \mu_0^r(dx_N)} \right] + \ln \left[\frac{M_0(x_0,x_1)}{M_0(x_1,x_0)} \cdot \frac{M_1(x_1,x_2)}{M_1(x_2,x_1)} \cdots \frac{M_{N-1}(x_{N-1},x_N)}{M_{N-1}(x_N,x_{N-1})} \right]$

Current inversion

We introduce the density "locally invariant" $\rho_t^{li} = \exp(-\varphi_t^{li})$ with : $div(j_t^{li}) = div(\rho_t^{li} v_t^{li}) = 0$.

With the choice : $u_{t,+} = \frac{d_t}{2} \cdot \rho_t^{li} \leftrightarrow u_{t,-} = v_t^{li}$.

The backward process is associated with opposite probability current.

If we choose : $\rho_0(x) = \rho_0^{li}(x)$ et $\rho_0(x) = \rho_N^{li}(x)$. $\Longrightarrow\; w_T^{ex} = \int_0^T \left(\partial_t \varphi_t^{li} \right)(x_t) dt$.