Classical pulsed and kicked rotors

S. De Bièvre and B. Aguer – Université de Lille

Université de Cergy – Rencontre HAM-MARK November 2009

The problem

Hamiltonian of a soft, inelastic, non-dissipative, time and space periodic Lorentz gas:

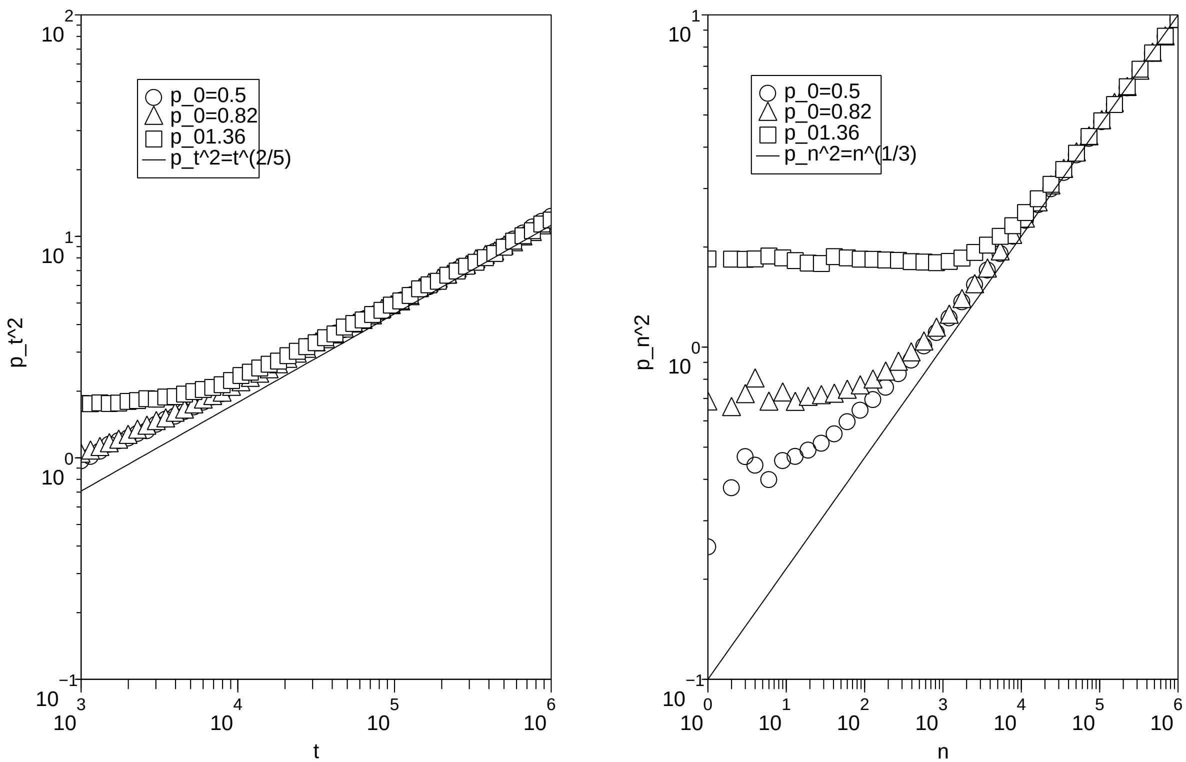
$$H(q,p,t) = \frac{p^2}{2} + \lambda V(q,t), V(q,t) = \sum_{m \in \mathbb{Z}^d} W(\|q - x_m\|, \omega t + \phi_0), \quad q, p \in \mathbb{R}^d,$$

with $x_m = \sum m_i \mathrm{e_i}$; $e_i, i = 1 \ldots d$ basis of $\mathbb{R}^d$, $\|e_i\| = 1$;
$W : \mathbb{R}^+ \times \mathbb{T}^m \to \mathbb{R}$, $W(\|q\|, \phi) = 0$ if $\|q\| > 1/2$;
$\omega \in \mathbb{R}^m$ a frequency vector.

QUESTION: What is the time-behaviour of the averaged kinetic energy $\langle v^2(t) \rangle$ and mean squared displacement $\langle q^2(t) \rangle$ of an ensemble of particles with fixed given initial energy $v_0^2 >> \lambda$ starting of in a random initial direction from a position close to the origin? How does it depend on d? And on the smoothness of W in t?

SOME ANSWERS (NUMERICS):THE KINETIC ENERGY, $d \geq 2$

$$d = 2, W(q,t) = \chi_{[0,1/2]}(\|q\|)\cos t$$

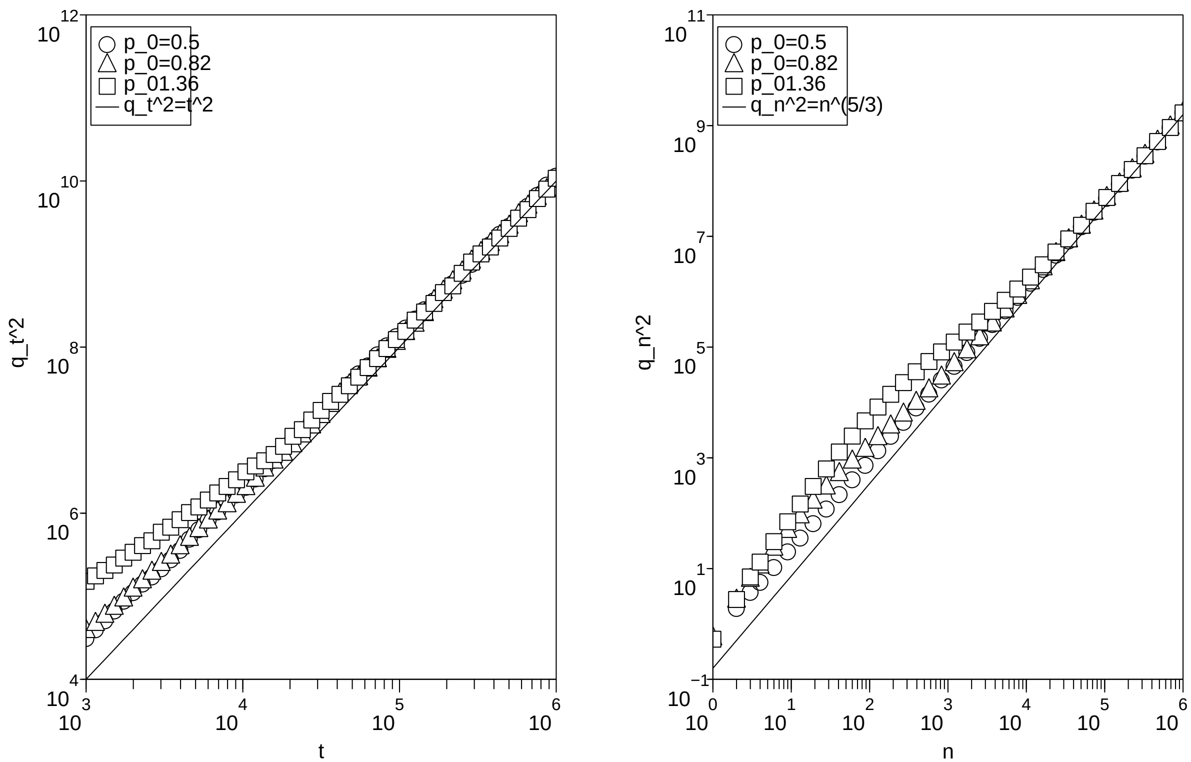


$$\langle v^2(t)\rangle \sim t^{2/5}, \langle v_n^2\rangle \sim n^{1/3}.$$

REMARKS: Powers identical for all dimensions $d \geq 2$. The same also for a random lattice, random coupling constants (Aguer-De Bièvre-Lafitte-Parris, JSP, to appear).

SOME ANSWERS (NUMERICS):THE MEAN SQUARED DISPLACEMENT, $d \geq 2$

$$d = 2, W(q,t) = \chi_{[0,1/2]}(\|q\|)\cos t$$



$$\langle q^2(t)\rangle \sim t^2, \langle q_n^2\rangle \sim n^{5/3}.$$

REMARKS: Powers identical for all dimensions $d \geq 2$. The same also for a random lattice, random coupling constants (Aguer-De Bièvre-Lafitte-Parris, JSP, to appear).

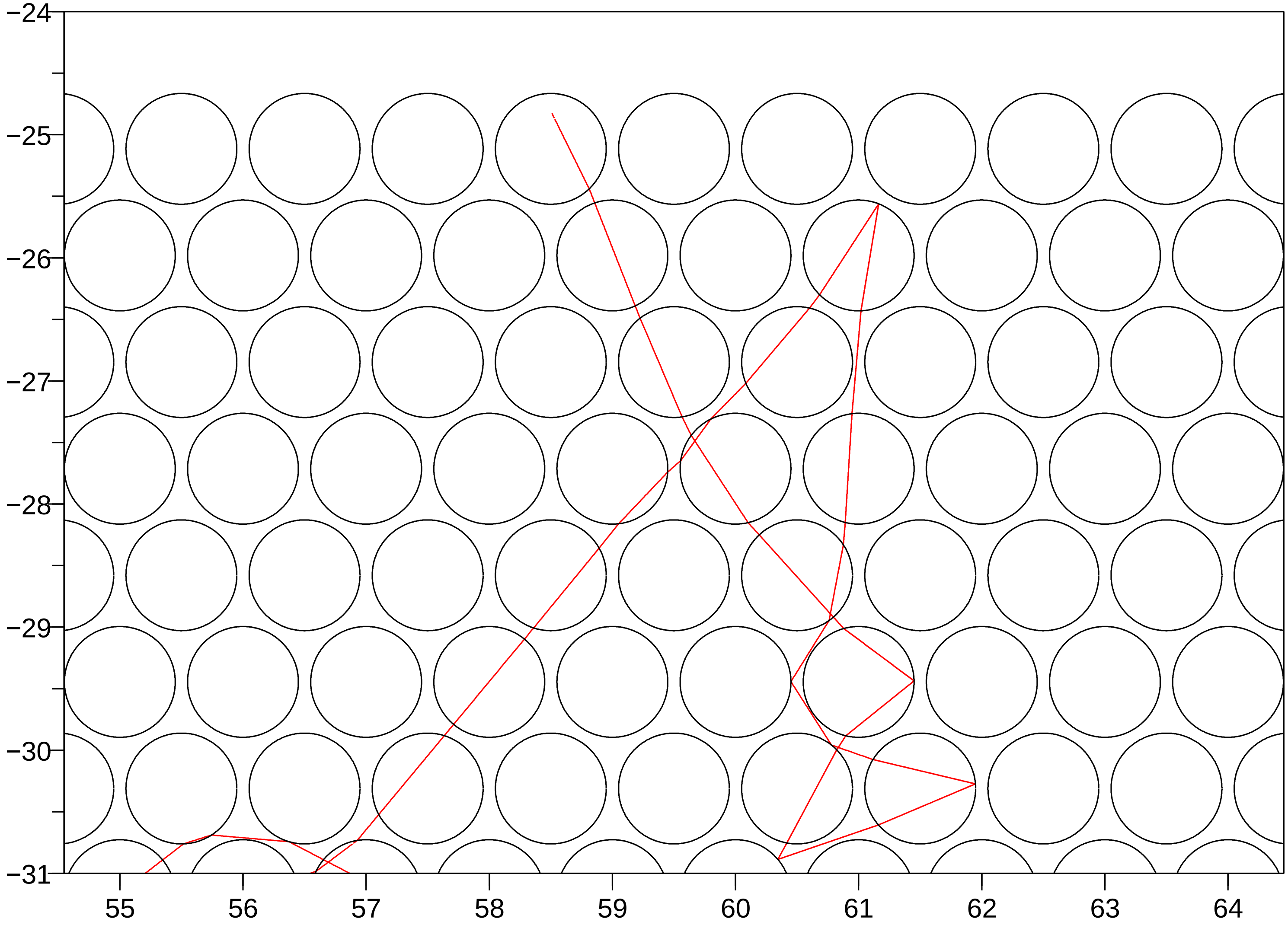
QUESTION: The particles speed up on average. But $\langle q^2(t)\rangle$ grows only quadratically with time, so in this sense the motion is ballistic. How is this possible?

ANSWER: The particle turns while traveling. The increase in its average speed is compensated exactly by these detours, leading to a ballistic growth of $\langle q(t)^2\rangle$. Mote on this later.

A PULSED ROTOR: The energy growth of those systems when $d \geq 2$ shows they are unstable in the following sense. The Hamiltonian is invariant under the lattice translation group $\mathbb{Z}e_1 + \cdots + \mathbb{Z}e_d$ and therefore generates a Hamiltonian dynamics on the torus, with the same momentum behaviour. It describes a particle moving on a flat torus with a scatterer described by the time (quasi-)periodic potential W: a pulsed rotor. As will become clearer below, the potential constitutes an increasingly small perturbation of the free particle, as the particle momentum gets larger. The unbounded growth of the energy observed here therefore suggests that either all invariant tori of this completely integrable system are destroyed by the perturbation (unlikely), or (more likely) that the momentum variable diffuses between the remaining tori, slowly increasing with time.

Explaining the power laws: a random walk description of the motion

CLAIM: The statistical properties of the motion are well described by a random walk, as in the case where the phases and coupling constants of the scatterers are random. This is, as we will see, a result of the fact that the geometry of the periodic lattice together with the instabilities in the dynamics of the individual scattering events undergone by the particle suffice to effectively randomize the motion, even if the potential itself is deterministic. As a result, the correct exponents can be obtained relatively simply from physically straightforward arguments.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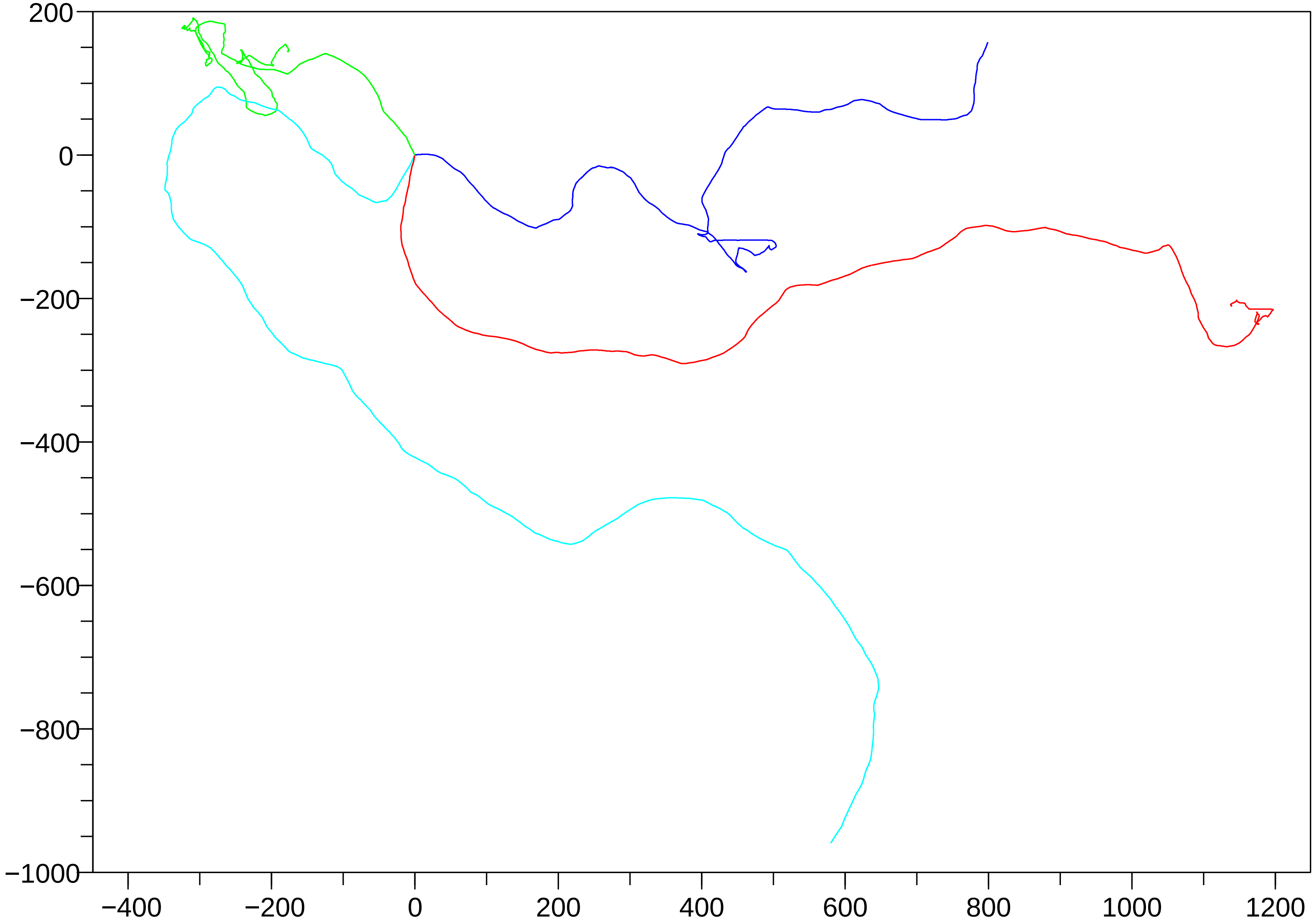

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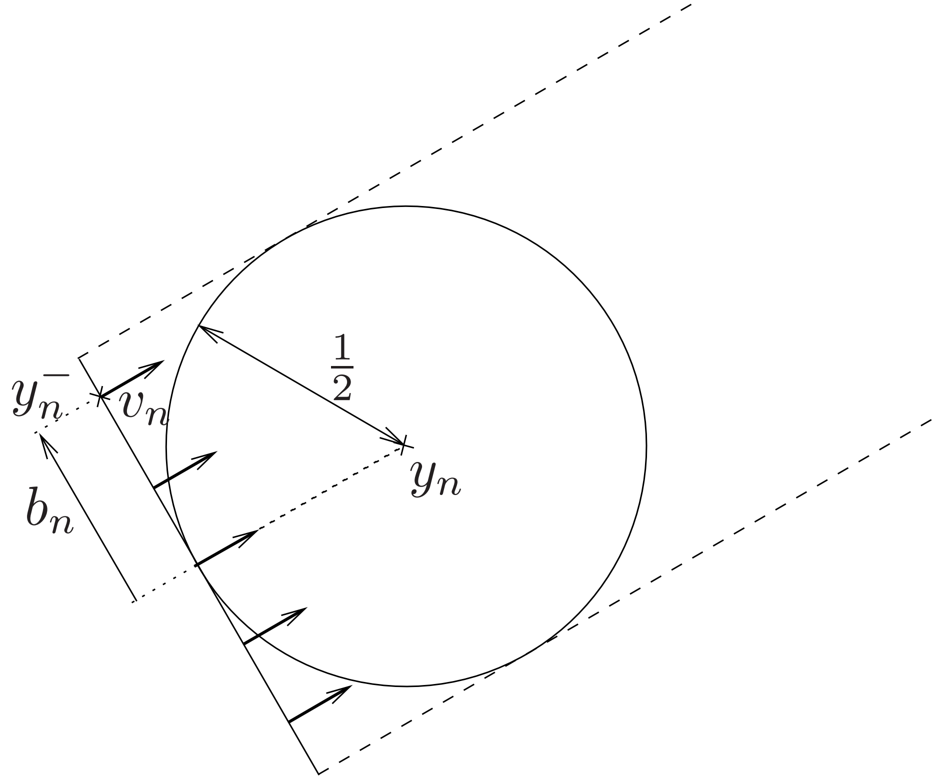


The motion A trajectory $=$ periods of free motion $+$ scattering events at instants t_n, centers $y_n := x_{N_n}$, where the particle is deviated by the local potential; $v_n = \|v_n\| e_n$ is the velocity just before the nth scattering event, at $y_n^- = x_{N_n} - \frac{1}{2} e_n + b_n$, where $b_n \cdot e_n = 0$, so that b_n is the (vector) impact parameter.

Analysis of energy growth

The velocity just after the nth scattering event is

$$v_{n+1} = v_n + \lambda R(v_n, b_n, \phi_n). \tag{1}$$

For all v, b with $v \cdot b = 0$, the impulse function is

$$R(v, b, \phi) = -\int_0^{+\infty} \mathrm{d}t \nabla W(y(t), \omega t + \phi) \tag{2}$$

in which $y(t)$ is the unique solution of

$$\ddot{y}(t) = -\nabla W(y(t), \omega t + \phi), \quad y(0) = b - \frac{1}{2}\frac{v}{\|v\|}, \quad \dot{y}(0) = v.$$

Hypothesis: The phases ϕ_n and impact parameters b_n encountered by the particle during its journey are i.d. and have short range correlations.

Conclusion: Equation (1) defines a random walk that allows to determine the large n behaviour of v_n, provided one understands the high velocity behaviour of R and of the energy transfer in a single collision, viewed as random variables in b and ϕ.

High $\|v\|$ expansions:

$$R(v,\kappa) = \sum_{k=1}^{K} \frac{\alpha^{(k)}(e,\kappa)}{\|v\|^k} + \mathrm{O}(\|v\|^{-K-1}),\ e = \frac{v}{\|v\|},$$

with $\kappa = (b,\phi)$ and $\alpha^{(1)}(e,b,\phi) = -\int_{-\infty}^{+\infty} \mathrm{d}\lambda\, \nabla W((b+\lambda e),\phi)$. Then,

$$\Delta E(v,\kappa) = \frac{1}{2}\left((v+R(v,\kappa))^2 - v^2\right) = \sum_{\ell=0}^{L} \frac{\beta^{(\ell)}(e,\kappa)}{\|v\|^\ell} + \mathrm{O}(\|v\|^{-L-1}), \tag{3}$$

and with the suggestive notation $\partial_t := \omega \cdot \nabla_\phi$

$$\begin{aligned} \beta^{(0)}(e,\kappa) &= e\cdot\alpha^{(1)}(e,\kappa) = \mathbf{0} \\ \beta^{(1)}(e,\kappa) &= e\cdot\alpha^{(2)}(e,\kappa) = \int_{-\infty}^{+\infty} \mathrm{d}\lambda\, \partial_t W((b+\lambda e),\phi) \end{aligned}$$

So, $\Delta E(v,\kappa)$ is of order $\|v\|^{-1}$ in this situation. We need the following information:

Define: $\overline{f(v)} = \int \frac{\mathrm{d}b}{C_d} \int \mathrm{d}\phi f(v,b,\phi)$.

THEOREM $\overline{\Delta E(v)} = \frac{B}{\|v\|^4} + \mathrm{O}(\|v\|^{-5})$, $\overline{(\Delta E(v))^2} = \frac{D^2}{\|v\|^2} + \mathrm{O}(\|v\|^{-3})$,
where $B = (d-3)D^2/2$. In particular, for all unit vector $e \in \mathbb{R}^m$ and for $\ell = 0, 1, 2, 3$,

$$\overline{\beta^{(\ell)}(e)} = 0, \qquad B = \overline{\beta^{(4)}(e)} \quad \text{and} \quad D^2 = \overline{(\beta^{(1)}(e))^2} > 0.$$

Conclusion: the energy transfer in a single scattering event is a random variable with fluctuations of order $\|v\|^{-1}$ and a mean of order $\|v\|^{-4}$. Then, with

$$\xi_n = \frac{\|v_n\|^3}{3D}, \quad \epsilon_n = \frac{\beta_n^{(1)}}{D} \quad \text{and} \quad \gamma = \frac{1}{3}(\frac{B}{D^2} + \frac{1}{2}) = \frac{1}{6}(d-2) \geq -\frac{1}{6},$$

$$\Delta\xi_n = \epsilon_n + \frac{\gamma}{\xi_n} + \mathrm{O}_0(\xi_n^{-1/3}) + \mathrm{O}(\xi_n^{-4/3}).$$

Here the notation $\mathrm{O}_0(\|v_n\|^{-1})$ means the term is $\mathrm{O}(\|v_n\|^{-1})$ and of zero average. After scaling ξ_n/n^2 converges to the square of a Bessel process of dimension $\delta = 2\gamma + 1 > 2/3$ (See Thesis B. Aguer). This yields

$$\langle \xi_n^k \rangle \sim n^{k/2}, \quad \text{for all} \quad k > -1 \Rightarrow \langle \|v_n\|^\ell \rangle \sim n^{\ell/6}, \quad \ell > -3$$

The scattering times t_n satisfy

$$t_{n+1} = t_n + \frac{\eta_*}{\|v_{n+1}\|}, \quad \text{so that} \quad t_n = \sum_{k=1}^{n} \frac{\eta_*}{\|v_k\|},$$

$$\text{whence} \qquad \langle t_n \rangle \sim n^{5/6} \sim \langle \|v_n\| \rangle^5, \quad \text{and} \quad \langle \|v(t)\|^2 \rangle \sim t^{2/5}.$$

This is precisely what we found numerically.

Asymptotics of $\langle y^2(t)\rangle$: $d>1$

The successive scatterering centers visited by the particle are given by

$$y_{n+1} = y_n + v_{n+1}\Delta t_n = y_n + \frac{\eta_*}{\|v_{n+1}\|} v_{n+1} = y_n + \eta_* e_{n+1}.$$

To understand $\|y_n\|$ for large n, we need to know how the particle turns. From a further analysis of

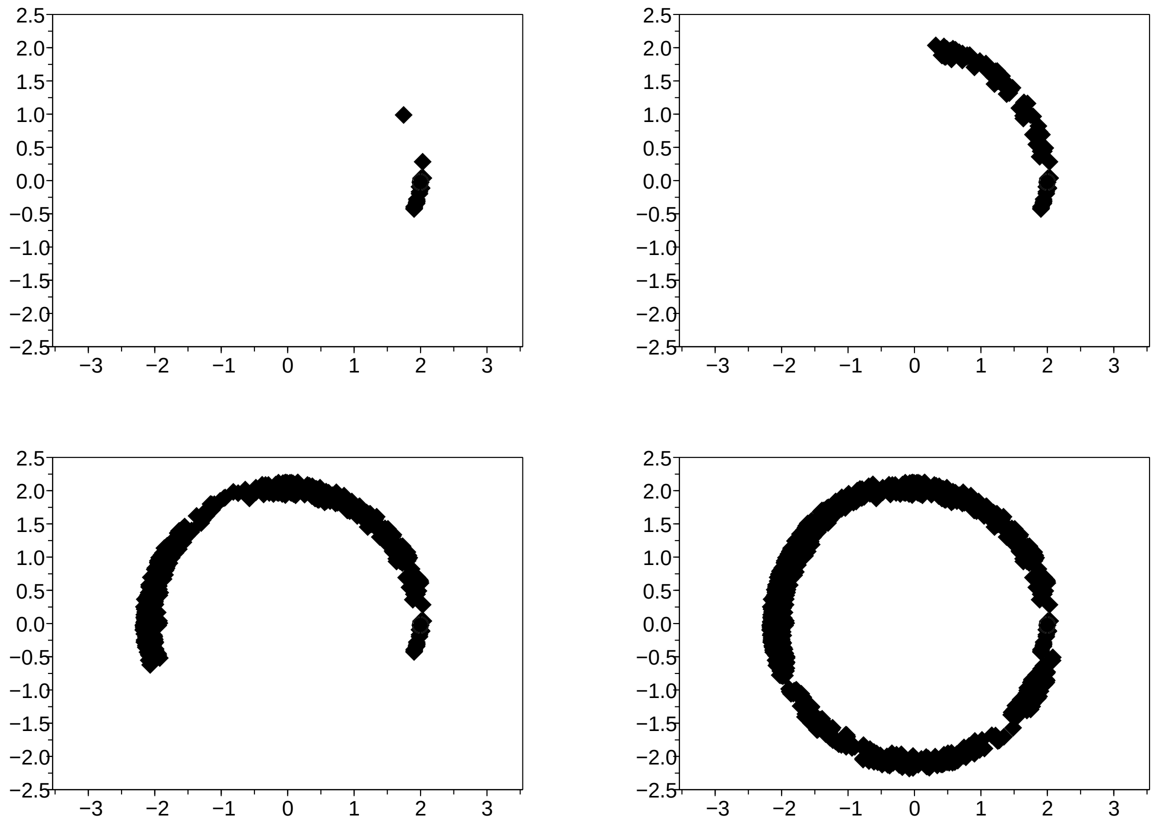
$$v_{n+1} = v_n + R(v_n, \kappa_n) \simeq v_n + \frac{\alpha_n^{(1)}}{\|v_n\|} + \dots$$

one shows the e_n execute a random walk on the unit sphere

$$e_{n+1} = e_n + \delta_n = e_n + \delta_n^{\perp} + \mu_n e_n, \|\delta_n\| = \frac{\|\alpha_n^{(1)}\|}{\|v_n\|^2} + \mathrm{O}(\|v_n\|^{-3}) = \|\delta_n^{\perp}\|,$$

with $|\mu_n| = \mathrm{O}(\|v_n\|^{-4})$ so that

$$\langle \|e_{n+m} - e_n\|^2 \rangle \simeq m \frac{\langle \|\alpha_0^{(1)}\|^2 \rangle}{\|v_n\|^4} \sim 1 \quad \text{provided} \quad m \sim M_*(\|v_n\|) := \|v_n\|^4 \sim n^{2/3}$$



Spreading of the velocity vectors for an initial distribution concentrated on a fixed direction.

This suggests the particle goes straight for $N_1 \sim \|v_0\|^4$ steps, then turns in a random direction, goes straight for $N_1^{2/3}$ steps etc.:

$$N_{k+1} = N_k + N_k^{\frac{2}{3}} \;\Rightarrow\; N_k \sim k^3.$$

After N_k collisions, the particle has turned k times over a macroscopic angle.
Between N_k et N_{k+1} it goes straight. So

$$y_{N_{k+1}} = y_{N_k} + \eta_*(N_{k+1} - N_k)e_{N_k} \Rightarrow \langle \| y_{N_k} \|^2 \rangle \sim \sum_{\ell=1}^{k} \ell^4 \sim k^5 \sim N_k^{5/3}.$$

Interpolating between the N_k, and using $t_n \sim n^{5/6}$,

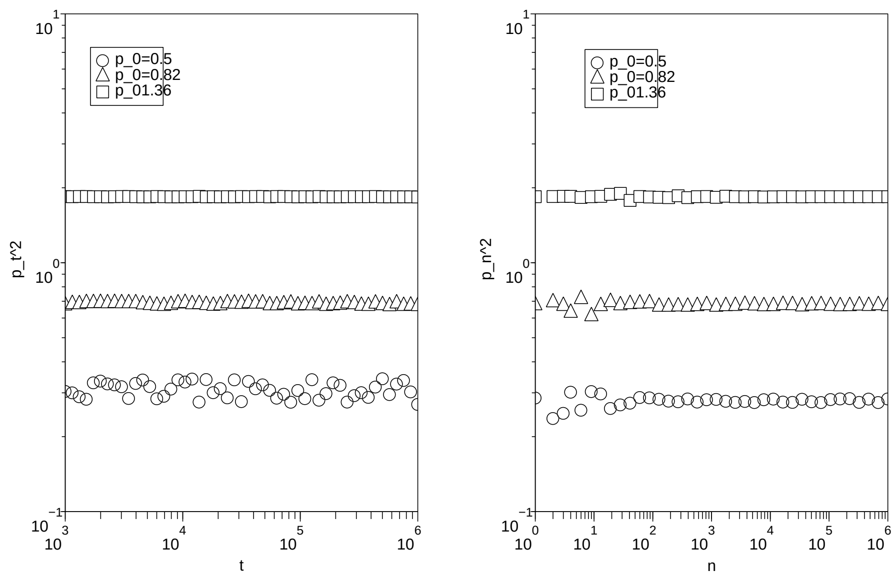
$$\langle \| y_n \|^2 \rangle \sim n^{5/3} \;\Rightarrow\; \langle \| y^2(t) \| \rangle \sim t^2.$$

The motion is ballistic in $d > 1$, although the particle accelerates.

In one dimension

The preceding analysis does not apply when $d = 1$: no random impact parameters!

The energy now remains bounded and $\langle q^2(t) \rangle \sim t^2$.



REMARK: With random coupling constants and phases, one has $\langle v^2(t) \rangle \sim t^{2/5}$.

$V(q,t)$ space and time periodic, $\int_0^1 \mathrm{d}q V(q,t) = 0$, for all t, $v_0 >> 1, q_0 = 0$.

$$q_n = n, \quad t_{n+1} \simeq t_n + \frac{1}{v_{n+1}}$$

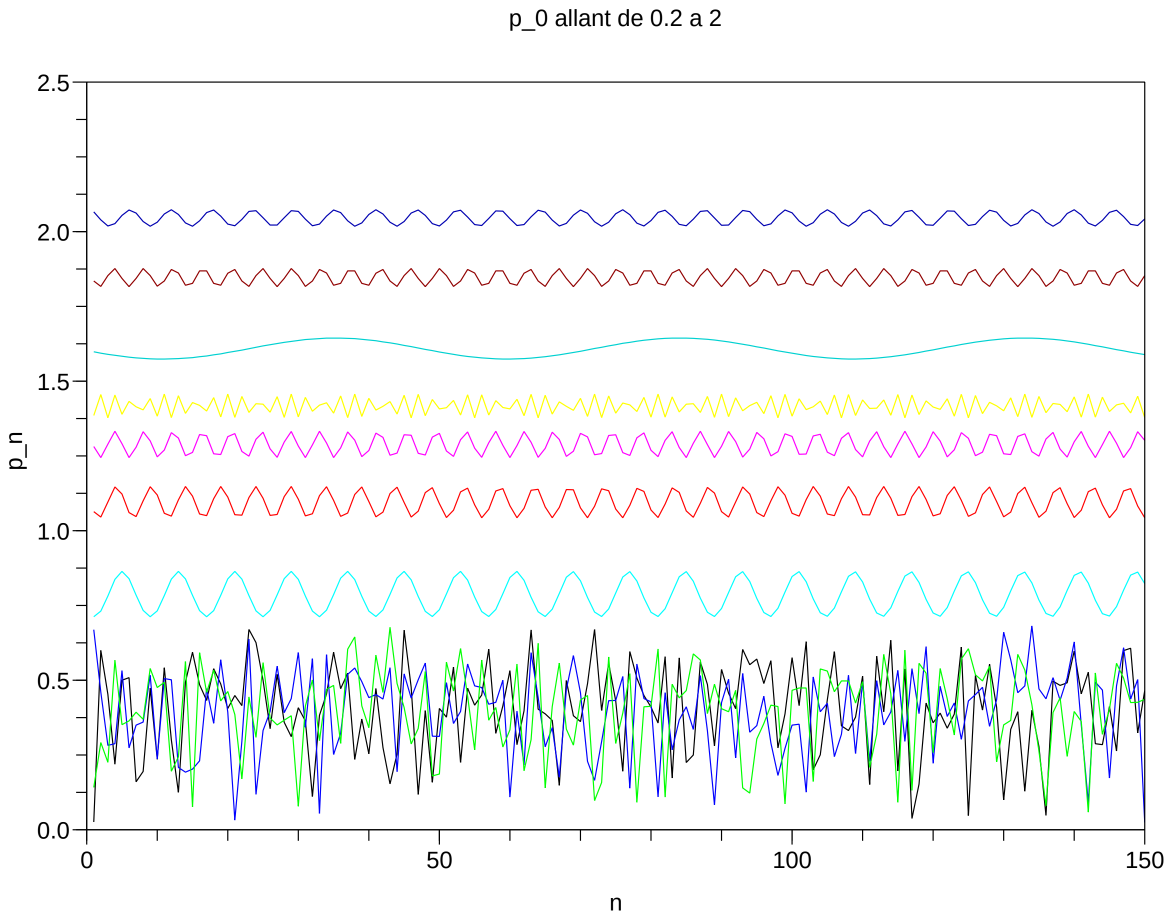
$$v_{n+1} = v_n + \frac{\alpha_1(q_n, v_n, t_n)}{v_n} + \frac{\alpha_2(q_n, v_n, t_n)}{v_n^2} + \mathcal{O}(v_n^{-3}),$$

where

$$\begin{aligned} \alpha_1(q_n, v_n, t_n) &= V(q_n, t_n) - V(q_n + v_n(t_{n+1} - t_n), t_n) = 0 \\ \alpha_2(q_n, v_n, t_n) &= -\partial_t V(q_n + v_n(t_{n+1} - t_n), t_n) + \int_0^{v_n(t_{n+1}-t_n)} \mathrm{d}\lambda \, \partial_t \, V(q_0 + \lambda, t_n \\ &= -\partial_t V(q_0, t_n), \end{aligned}$$

so that

$$\begin{aligned} v_n &\simeq v_0 + \frac{1}{v_0^2}\sum_{k=0}^{n-1}\partial_t V(0,t_k) + \mathcal{O}(v_0^{-3}) \simeq v_0 + \frac{1}{v_0}\int_0^{n/v_0} \mathrm{d}t\partial_t V(0,t) \\ &= v_0 + \frac{1}{v_0}\left(V(0,\frac{n}{v_0}) - V(0,0)\right). \end{aligned}$$



p_0 allant de 0.2 a 2
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2.0
1.5
1.0
0.5
0.0
p_n
0
50
100
150
n

Kicked rotors

It is instructive to notice the difference in behaviour of the pulsed rotors considered here with kicked rotors, in which the time dependence of the potentials is very singular, of the form

$$V(q,t) = \lambda \sum_n \delta(t-n) v(q). \tag{4}$$

For such systems the Floquet transformation which gives the evolution of the system over a period of the potential, is easily written down:

$$\Phi(q,p) = (q', p'), \quad \text{where} \quad p' = p - \lambda \nabla v(q),\ q' = q + p'. \tag{5}$$

In that case, one finds for λ sufficiently large, that

$$\langle p^2(t) \rangle \sim t, \qquad \langle q^2(t) \rangle \sim t^3 CHECK. \tag{6}$$

(Is there a difference between $d = 1$ and $d > 1$? CHECK) This is again easily understood in terms of the random walk picture we develop in Section **??**. The main difference with the case of pulsed systems resides in the observation that, whereas for kicked systems the momentum change undergone by a particle in one period of the potential is of order 1, independently of the size of the initial momentum of the particle, this momentum change is of order $\|p\|^{-1}$ for pulsed systems. This fully explains the slower energy growth observed in pulsed systems, as well as the slower growth of the mean squared displacement.