Critical quench dynamics in confined quantum systems

Mario Collura and Dragi Karevski

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Qualitative picture

Time-dependent hamiltonian

$$H(t) = H_{critical} + g(t)V$$

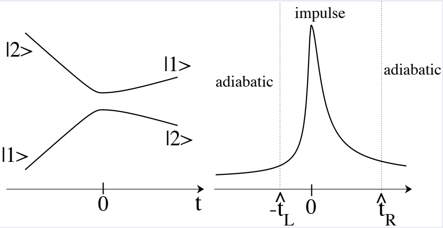
Power-law tuning parameter $g(t) \sim sgn(t)|t/\tau|^{\alpha} = sgn(t)v|t|^{\alpha}$ driving the system through the critical point.

- The system remains in the instantaneous ground state $|GS(t)\rangle$ as long as it is protected by a finite gap $\Delta(t)$ from the excited states.
- Breaking of the adiabaticity close to the critical point since the gap vanishes right at the QCP.

Kibble-Zurek mechanism

- Adiabatic: Sufficiently away from the critical point no transitions between instantaneous eigenstates
- Impulse: Sufficiently close to the critical point critical slowing down ⇒ no change in the wave function except for an overall phase factor

Adiabatic-Impulse approximation



$$
\begin{aligned}
t \in [-\infty, -\hat{t}_L] &\quad : \quad |\varphi(t)\rangle \approx \mathrm{e}^{-i\alpha(t)}|0(t)\rangle \\
t \in [-\hat{t}_L, \hat{t}_R] &\quad : \quad |\varphi(t)\rangle \approx \mathrm{e}^{-i\beta(t)}|0(-\hat{t}_L)\rangle \\
t \in [\hat{t}_R, +\infty] &\quad : \quad |\langle\varphi(t)|0(t)\rangle|^2 = \text{const.}
\end{aligned}
$$

Kibble-Zurek time-scale τ_{KZ}

Kibble-Zurek timescale τ_{KZ}

$$\tau_0/\Delta(\tau_{KZ}) = \Delta(\tau_{KZ})/|\dot{\Delta}(\tau_{KZ})|$$

with

$$\Delta(t) \sim |g(t)|^{\nu z} \sim v^{\nu z}|t|^{\nu z \alpha}$$

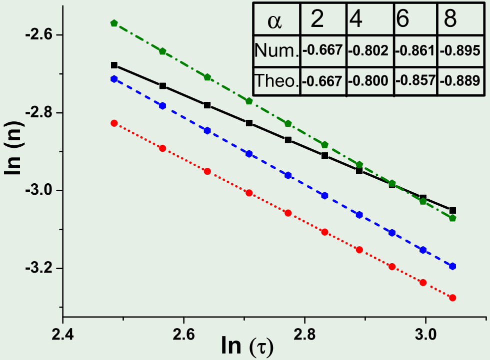
one has

$$\tau_{KZ} \sim v^{-\nu z/(1+\alpha\nu z)}; \quad \ell \sim \tau_{KZ}^{1/z}$$

Scaling for defect density

$$n \sim \ell^{-d} \sim v^{d\nu/(1+\nu z\alpha)}$$

A. Polkovnikov, PRB **72**, 161201(R) (2005)
W. H. Zurek, U. Dorner and P. Zoller, Phys. Rev. Lett. **95**, 105701 (2005).
B. Damski, Phys. Rev. Lett. **95**, 035701 (2005);
B. Damski and W. H. Zurek, Phys. Rev. A **73**, 063405 (2006);
ibid, Phys. Rev. Lett. **99**, 130402 (2007).



α	2	4	6	8
Num.	-0.667	-0.802	-0.861	-0.895
Theo.	-0.667	-0.800	-0.857	-0.889

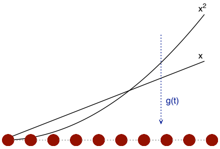
$$n \sim \tau^{-\alpha\nu d/(\alpha z\nu+1)} \sim v^{d\nu/(1+\nu z\alpha)}$$

A. Polkovnikov, PRB **72**, 161201(R) (2005)
D. Sen, K. Sengupta, S. Mondal, PRL **101**, 016806 (2008)
R. Barankov, A. Polkovnikov, PRL **101**, 076801 (2008)

Power-law spatial inhomogeneity

A power-law deviation in one direction of the quantum control parameter h form its critical value h_c:

$$\begin{aligned} \delta(x,t) &\equiv h(x,t) - h_c \simeq g(t)x^{\omega}, \quad x > 0 \\ g(t) &= v|t|^{\alpha}\mathrm{sgn}(t) \end{aligned}$$



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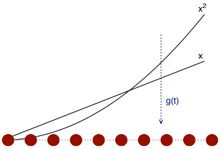

The perturbation introduces a crossover region in space-time (x,t) around the critical locus (0,0).

Lenght-scale

$$\ell(t) \sim \delta(\ell,t)^{-\nu} \rightarrow \ell(t) \sim |g(t)|^{-1/y_g}$$
$$y_g = (1+\nu\omega)/\nu$$

Time-scale

$$\tau \sim \ell(\tau)^z \rightarrow \tau \sim v^{-z/y_v}$$
$$y_v = y_g + z\alpha$$

The exponent y_v is the RG dimension of the perturbation field, such that under rescaling by a factor b the amplitude transforms as $v' = b^{y_v} v$.

Scaling arguments

Under rescaling, the profile $\varphi(x,t,v)$ associated to an operator φ with scaling dimension x_φ transform as

$$\varphi(x,t,v) = b^{-x_\varphi}\varphi(xb^{-1}, tb^{-z}, vb^{y_v})$$

Taking $b = v^{-1/y_v} \propto \ell \propto \ \tau^{1/z}$ one obtains

$$\varphi(x,t,v) = v^{x_\varphi/y_v}\Phi(xv^{1/y_v}, tv^{z/y_v})$$

- Trap-size scaling $\varphi \sim \ell^{-x_\varphi}$ associated to a finite size system with $\ell \sim v^{-1/y_v}$.

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B. Damski and W. H. Zurek, New J. Phys. **11** 063014 (2009)
M. Campostrini and E. Vicari, Phys. Rev. Lett. **102**, 240601 (2009)
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Adiabatic approximation

Time evolution of a quantum system described by a time-dependent Hamiltonian $\mathcal{H}(t)$

- The system is initially in the instantaneous ground state of the Hamiltonian $\mathcal{H}(t_0)$:

$$|\varphi(t_0)\rangle = |0(t_0)\rangle$$

- At time t

$$|\varphi(t)\rangle = \mathcal{U}(t, t_0)|0(t_0)\rangle$$

where the time evolution operator is

$$\mathcal{U}(t, t_0) = \hat{\mathrm{T}} \exp -i \int_{t_0}^{t} ds \mathcal{H}(s)$$

Adiabatic expansion in the instantaneous eigenbasis

Instantaneous eigenstates

$$\mathcal{H}(t)|k(t)\rangle = E_k(t)|k(t)\rangle$$

Adiabatic expansion up to first order

Rate of change of the Hamiltonian: $\partial_t \mathcal{H}(t) \sim \partial_t g(t) \sim v \to 0$

$$|\varphi(t)\rangle = e^{-i\int_{t_0}^{t} dsE_0(s)}|0(t)\rangle + \sum_{k\neq 0} e^{-i\int_{t_0}^{t} dsE_0(s)} a_k(t_0, t)|k(t)\rangle$$

Adiabatic expansion up to first order

$$|\varphi(t)\rangle = e^{-i\int_{t_0}^{t} dsE_0(s)}|0(t)\rangle + \sum_{k\neq 0} e^{-i\int_{t_0}^{t} dsE_0(s)} a_k(t_0,t)|k(t)\rangle$$

$$a_k(t_0,t) = \int_{g(t_0)}^{g(t)} dg \frac{\langle k(g)|\partial_g \mathcal{H}(g)|0(g)\rangle}{\delta\omega_{k0}(g)} \mathrm{e}^{-i\vartheta_k(g,g(t))}$$

where

$$\vartheta_k(x,y) = \frac{v^{-1/\alpha}}{\alpha}\int_x^y dg|g|^{1/\alpha-1}\delta\omega_{k0}(g)$$

$$\delta\omega_{k0}(g) = E_k(g) - E_0(g)$$

Adiabatic expansion up to first order

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$$a_k(t_0,t) = \int_{g(t_0)}^{g(t)} dg \frac{\langle k(g)|\partial_g \mathcal{H}(g)|0(g)\rangle}{\delta\omega_{k0}(g)} \mathrm{e}^{-i\vartheta_k(g,g(t))}$$

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$$\delta\omega_{k0}(g) = E_k(g) - E_0(g)$$

- For $v \ll 1$, $a_k \simeq 0$: instantaneous ground state
- For $v \gg 1$, $\exp(-i\vartheta_k) \sim 1$: sudden quench

Density of defects

Density of defects

$$n = \sum_{k \neq 0} |a_k|^2$$

General scaling arguments ($\ell \sim g^{-1/y_g}$):

$$\delta\omega_{k0} \sim \ell^{-z}\Omega(\ell^{-z}/k^z); \quad \langle k(g)|\partial_g \mathcal{H}(g)|0(g)\rangle \sim \ell^{-z+y_g} G(\ell^{-z}/k^z)$$

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- For a quench crossing the QCP, in order that the integral converges at $g = 0$ the scaling function $G(u)/\Omega(u) = uf(u)$ at small u.

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- In the inhomogeneous case the convergence close to the critical point is not garanted.

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Inhomogeneous QCP

$$\tau_{KZ} \sim \left(\frac{\tau_0}{\Omega_0}\frac{z\alpha}{y_g}\right)^{y_g/y_v} v^{-z/y_v}$$

$$n \sim [\Delta(\tau_{KZ})]^{d/z} \sim \left(\frac{\tau_0}{\Omega_0}\frac{z\alpha}{y_g}\right)^{d\alpha/y_v} v^{d/y_v}$$

Ising quantum chain in time-dependent inhomogeneous transverse field

$$\mathcal{H}(t) = -\frac{1}{2}\sum_{n=1}^{L-1}\sigma_n^x\sigma_{n+1}^x - \frac{1}{2}\sum_{n=1}^{L} h_n(g)\sigma_n^z$$

$$h_n(g) = 1 + g(t)n^{\omega}, \quad g(t) = v|t|^{\alpha}\mathrm{sgn}(t)$$

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- Introducing the $2L$-component real Majorana field $\mathbf{\Gamma}^{\dagger} = \left(\mathbf{\Gamma}^{1\dagger}, \quad \mathbf{\Gamma}^{2\dagger} \right)$ with components

$$\Gamma_n^1 = \prod_{j=1}^{n-1}(-\sigma_j^z)\sigma_n^x, \; \Gamma_n^2 = -\prod_{j=1}^{n-1}(-\sigma_j^z)\sigma_n^y$$

$$\mathcal{H}(t) = \frac{1}{4}\mathbf{\Gamma}^{\dagger}\mathbf{T}(g)\mathbf{\Gamma}$$

where $\mathbf{T}(g)$ is a $2L \times 2L$ hermitian matrix.

Bogoliubov time-dependent transformation

The Hamiltonian is diagonalized in terms of Dirac fermionic algebra $\{\eta_p^\dagger(g), \eta_q(g)\} = \delta_{pq}$ through the mapping

$$\eta_p(g) = \frac{1}{2}\sum_n \{\phi_p(n,g)\Gamma_n^1 + i\psi_p(n,g)\Gamma_n^2\}$$
$$\eta_p^\dagger(g) = \frac{1}{2}\sum_n \{\phi_p(n,g)\Gamma_n^1 - i\psi_p(n,g)\Gamma_n^2\}$$

$$\Gamma_n^1 = \sum_p \phi_p(n,g)\left[\eta_p(g) + \eta_p^\dagger(g)\right]$$
$$\Gamma_n^2 = -i\sum_p \psi_p(n,g)\left[\eta_p(g) - \eta_p^\dagger(g)\right]$$

with real Bogoliubov coefficients ϕ and ψ. One has

$$\mathcal{H}(t) = \sum_p \epsilon_p(g)\left[\eta_p^\dagger(g)\eta_p(g) - 1/2\right]$$

where $\epsilon_p(g)$ are the L-positive eigenvalues of $\mathbf{T}(g)$.

In the scaling limit $g \to 0$, $L \to \infty$ while keeping gL^{ω} costant, the Bogoliubov coefficients $\phi_p(x)$ and $\psi_p(x)$ are solutions of the differential set

$$\left[\frac{d^2}{du^2} + \Omega_p^2 - \mathrm{sgn}(g)\omega u^{\omega-1} - u^{2\omega}\right]\tilde{\phi}_p(u) = 0,\ \partial_u\tilde{\phi}_p|_0 = 0,\ \tilde{\phi}_p(\infty) = 0$$

$$\left[\frac{d^2}{du^2} + \Omega_p^2 + \mathrm{sgn}(g)\omega u^{\omega-1} - u^{2\omega}\right]\tilde{\psi}_p(u) = 0,\ \tilde{\psi}_p(0) = 0,\ \partial_u\tilde{\psi}_p|_\infty = 0$$

with rescaled variables

$$x = |g|^{-1/y_g}u, \qquad \epsilon_p = |g|^{1/y_g}\Omega_p,$$

$$\phi_p(x) = |g|^{1/2y_g}\tilde{\phi}_p(u), \qquad \psi_p(x) = |g|^{1/2y_g}\tilde{\psi}_p(u),$$

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$$\phi_p(x) = |g|^{1/2y_g}\tilde{\phi}_p(u), \qquad \psi_p(x) = |g|^{1/2y_g}\tilde{\psi}_p(u),$$

All the dependence on g is inside the rescaled variables.

Hamiltonian derivative

In terms of fermions $\partial_g \mathcal{H}(g)$ takes the form

$$\partial_g \mathcal{H}(g) = \frac{1}{2} \sum_{p,q} X^{\omega}_{pq}(g)[\eta^{\dagger}_p(g) + \eta_p(g)][\eta^{\dagger}_q(g) - \eta_q(g)]$$

with $X^{\omega}_{pq}(g) = \sum_n \phi_p(n,g) n^{\omega} \psi_q(n,g)$.

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with $X^{\omega}_{pq}(g) = \sum_n \phi_p(n,g) n^{\omega} \psi_q(n,g)$.

The system deviates from the adiabatic ground state $|0(g)\rangle$ by tansitions to the two-particles states $|pq(g)\rangle = \eta^{\dagger}_q(g)\eta^{\dagger}_p(g)|0(g)\rangle$ only with

$$\langle pq(g)|\partial_g \mathcal{H}(g)|0(g)\rangle = [X^{\omega}_{qp}(g) - X^{\omega}_{pq}(g)]/2$$

Continuum limit

$$X^{\omega}_{pq}(g) = |g|^{-\omega/(1+\omega)}(\tilde{\phi}_p, u^{\omega}\tilde{\psi}_q)$$

with the scalar product $(f,g) = \int_0^{\infty} f^*(u)g(u)du$.

For $\omega = 1$ the previous differential equations reduce to a harmonic oscillator problem.

$$
\begin{aligned}
\phi_p(x) &= |g|^{1/4}\sqrt{2}\chi_{2p}(u) \\
\psi_p(x) &= \mathrm{sgn}(g)|g|^{1/4}\sqrt{2}\chi_{2p+\mathrm{sgn}(g)}(u) \\
\epsilon_p &= |g|^{1/2}\sqrt{4p+1+\mathrm{sgn}(g)}
\end{aligned}
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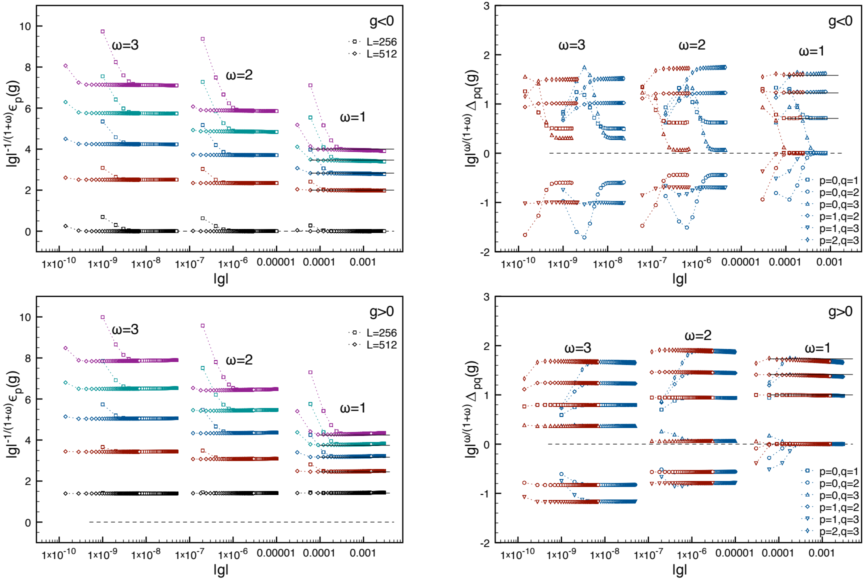
$$

Rescaled matrix elements: $G_{pq}(g) = |g|^{1/2}\langle pq(g)|\partial_g \mathcal{H}(g)|0(g)\rangle$
Rescaled Bohr frequencies: $\Omega_{pq}(g) = |g|^{-1/2}(\epsilon_p(g) + \epsilon_q(g))$

$$
\begin{aligned}
G_{pq}(g) &= \sqrt{\frac{p+q+H(g)}{2}}\,[\delta_{p\,q-1} - \delta_{p\,q+1}] \\
\Omega_{pq}(g) &= \sqrt{4p+2H(g)} + \sqrt{4q+2H(g)}
\end{aligned}
$$


g<0
L=256
L=512
ω=3
ω=2
ω=1
$|g|^{-1/(1+\omega)}\epsilon_p(g)$
$|g|^{\omega/(1+\omega)}\Delta_{pq}(g)$
p=0,q=1
p=0,q=2
p=0,q=3
p=1,q=2
p=1,q=3
p=2,q=3
1x10-10
1x10-9
1x10-8
1x10-7
1x10-6
0.00001
0.0001
0.001
|g|
g>0

The transition amplitude $a_{pq}(t_0, t)$ for a quench starting at a value $g_0 = g(t_0)$ and ending at a new value $g_t = g(t)$ is obtained from the first order adiabatic approximation if the quench parameter stays sufficiently far away from the critical locus (which is set at $t = 0$).

Quenches that do not cross the critical point

$$a_{pq}(t_0, t) = F_{pq} A_{\phi_{pq}} \left(|g_0|, |g(t)|\right) \mathrm{e}^{i\Theta_{pq}(t)}$$

where

$$\begin{aligned}
\Theta_{pq}(t) &= \pi H(-g_0) + \phi_{pq} |g(t)|^{\frac{2+\alpha}{2\alpha}} \\
\phi_{pq} &= -2\Omega_{pq} \frac{v^{-1/\alpha}}{\alpha + 2} \mathrm{sgn}(g_0) \\
A_\phi(x, y) &= \frac{2\alpha}{2 + \alpha} \left[\mathrm{E}_1\left(i\phi x^{\frac{2+\alpha}{2\alpha}} \right) - \mathrm{E}_1\left(i\phi y^{\frac{2+\alpha}{2\alpha}} \right) \right]
\end{aligned}$$

The spatial inhomogeneity modifies the dependence on g of the scaling function $F_{pq}(g) = G_{pq}(g)/(2\Omega_{pq}(g))$ close to $g = 0$ such that it leads to a complete breakdown of the approximation

Up to the first order correction we can write the evolution of the Ising chain ground state $|0(g_0)\rangle$ as

$$|\varphi(t)\rangle \approx |0(g)\rangle + \sum_{pq} a_{pq}(t_0, t)\eta_q^\dagger(g)\eta_p^\dagger(g)|0(g)\rangle$$

Using the properties of the Fermion's operators, one obtains for the adiabatic occupation numbers $n_p = \langle\varphi(t)|\eta_p^\dagger(g)\eta_p(g)|\varphi(t)\rangle$

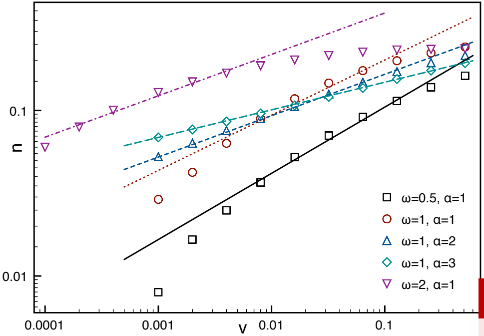
Occupation numbers

$$n_p \approx 4\sum_q |a_{pq}(t_0, t)|^2$$

The first two levels

$$\begin{aligned} n_0 &\approx 4|a_{01}(t_0, t)|^2 \\ n_1 &\approx 4[|a_{12}(t_0, t)|^2 - |a_{10}(t_0, t)|^2] \end{aligned}$$

Finnegans Wake... crossing the QCP



Defect density

$$n \sim v^{1/(1+\omega+\alpha)}$$

Conclusion

- We have presented a theory of the non-linear quench of a power-law perturbation, such as a confining potential, close to a critical point
- We have determined the scaling properties of such a theory
- Power law behavior of the density of defects with the ramping rate with an exponent which depends on the space-time properties of the potential.
- First order adiabatic calculation and exact results on an inhomogeneous transverse field Ising chain

What should be looked at...

A relevant extension of this work would be the study of the influence of a finite temperature on the scaling properties.