OPEN QUANTUM SYSTEMS

(29 november - 2 december, 2010)

Université Joseph FOURIER

ANR HAM-MARK (N° ANR-09-BLAN-0098-01)

Non-analyticity of the binding energy and the ground state energy for Hydrogen in nonrelativistic QED

Jean-Marie Barbaroux

Joint works with Thomas Chen, Vitali Vugalter, Semjon Wugalter.

- "Non-analyticity of the ground state energy of the Hamiltonian for Hydrogen atom in non-relativistic QED", J. Phys. A : Math. Theor. 43 474004 (2010).
J.-M. B., Semjon Wugalter
- "Quantitative estimates on the binding energy for Hydrogen in non-relativistic QED", to appear in Ann. Henri Poincaré.
J.-M. B., Thomas Chen, Vitali Vougalter, Semjon Wugalter
- On the ground state energy of the translation invariant Pauli-Fierz model, Proc. Amer. Math. Soc., 136 (3), 1057-1064 (2008).
J.-M. B., Thomas Chen, Vitali Vougalter, Semjon Wugalter
- Quantitative estimates on the enhanced binding for the Pauli-Fierz operator, J. Math. Phys., vol. 46, no12 (2005).
J.-M. B., Helmut Linde, Semjon Wugalter
- Binding conditions for atomic N-electron systems in non-relativistic QED, Ann. Henri Poincaré. 4 (6), 1101 - 1136 (2003).
J.-M. B., Thomas Chen, Semjon Wugalter

Hamiltonian

One electron interacting with a pointwise nucleus of charge e, $(Z = 1)$.

$$H = -\Delta + V$$

Coulomb Potential : $V(x) = -\frac{e^2}{|x|}$
Electron mass $m = 1/2$; Planck constant $\hbar = 1$; velocity of light $c = 1$.
Fine structure constant : $\alpha = e^2 \approx 1/137$

Binding energy

Energy necessary to remove the electron to spatial infinity :

$$\Sigma(0) - \Sigma(V) = \inf \sigma(-\Delta) - \inf \sigma(-\Delta + V).$$

An elementary computation !

▷ Coulomb uncertainty principle : $\int_{\mathbb{R}^3} \frac{1}{|x|}|\psi(x)|^2 \mathrm{d}x \leq \|\nabla\psi\| \, \|\psi\|$

▷

$$\begin{aligned}\langle \psi, (-\Delta + V)\psi \rangle &\geq \|\nabla\psi\|^2 - \alpha\|\nabla\psi\| \, \|\psi\| \\ &= \underbrace{\left(\|\nabla\psi\| - \frac{\alpha}{2}\|\psi\|\right)^2}_{\geq 0} - \frac{\alpha^2}{4}\|\psi\|^2\end{aligned}$$

▷ $\Sigma(V) = \inf \sigma(-\Delta + V) = -\frac{\alpha^2}{4}$.

Binding energy

$$\Sigma(0) - \Sigma(V) = \inf \sigma(-\Delta) - \inf \sigma(-\Delta + V) = \frac{\alpha^2}{4} \, .$$

Coupling to the radiation field

$$H_\alpha = T + V \otimes \mathbb{I}_f \quad \text{sur} \quad \mathfrak{H} = L^2(\mathbb{R}^3) \otimes \mathfrak{F}_s$$

Self-energy : $T = (-i\nabla_x \otimes \mathbb{I}_f + \sqrt{\alpha}\mathbb{A}(x))^2 + \mathbb{I}_{el} \otimes H_f - c_{n.o.}\alpha$

- $L^2(\mathbb{R}^3)$: Hilbert space for one spinless electron.
- $\mathfrak{F}_s$: Symmetric photonic Fock space.

$$\mathfrak{F}_s = \bigoplus_{n=0}^{\infty} \underbrace{\bigotimes_{s}^{n} \left(\underbrace{L^2(\mathbb{R}^3)}_{\text{photon momentum space}} \otimes \underbrace{\mathbb{C}^2}_{\text{2 transv. polarizations}} \right)}_{\mathfrak{F}_s^{(n)}}$$

$$\mathfrak{F}_s^{(0)} = \Omega_f \mathbb{C}.$$

- Vacuum : Ω_f.

- Creation/annihilation operators : $a^*_\lambda(k)$, $a_\lambda(k)$. Fulfil the C.C.R : $[a_\lambda(k), a^*_{\lambda'}(k)] = \delta_{\lambda,\lambda'}\delta(k-k')$, $[a^\sharp_\lambda(k), a^\sharp_{\lambda'}(k)] = 0$.
- Photon field energy : $H_f = \sum_{\lambda=1,2}\int |k| a^*_\lambda(k) a_\lambda(k) \mathrm{d}k$
- Magnetic vector potential :

$$\begin{aligned}\mathbb{A}(x) &= \mathbb{A}^-(x) + \mathbb{A}^+(x) \\ &= \sum_{\lambda=1,2}\int \frac{\chi_\Lambda(|k|)}{2\pi|k|^{\frac{1}{2}}}\epsilon_\lambda(k)[\mathrm{e}^{ik.x}\otimes a_\lambda(k) + \mathrm{h.c.}]\mathrm{d}k\end{aligned}$$

- Polarization vectors : $\epsilon_\lambda(k)$, $k\cdot\epsilon_\lambda(k) = 0$, $\epsilon_1(k)\cdot\epsilon_2(k) = 0$.
- U.V. cutoff : $\chi_\Lambda(|k|)$
- Normal ordering : $-c_{n.o.}\alpha$: commutations of all creation operators to the left in the term $\mathbb{A}(x)^2$ (amount to subtract a constant due to C.C.R.)

Hamiltonian

$$H_\alpha = T + V \otimes \mathbb{I}_f \quad \text{sur} \quad \mathfrak{H} = L^2(\mathbb{R}^3) \otimes \mathfrak{F}$$

$$T = (-i\nabla_x \otimes \mathbb{I}_f + \sqrt{\alpha}\mathbb{A}(x))^2 + \mathbb{I}_{el} \otimes H_f - c_{n.o.}\alpha$$

Binding energy :

$$\Sigma_\alpha(0) - \Sigma_\alpha(V) = \inf \sigma(T) - \inf \sigma(T + V)$$

Remark :

$$H_\alpha = T + V$$

$$= -\Delta_x + V(x) + H_f + \underbrace{(-2\mathrm{Re}\sqrt{\alpha}\mathbb{A}(x) \cdot i\nabla_x + \alpha\mathbb{A}(x)^2 - c_{n.o.}\alpha)}_{:=W(\alpha)}$$

$$H_\alpha = -\Delta_x + V(x) + H_f + W(\alpha)$$

1. When the interaction $W(\alpha)$ is turned on, as the particle binds photons, it acquires an effective mass

$$m_{\text{eff}} = m_{\text{eff}}(\alpha) \nearrow \quad \text{as} \quad \alpha \nearrow \ .$$

2. The free particle binds a larger quantity of (low-energetic) photons than the confined particle, and thus has a larger effective mass than the confined particle.

1. Interaction with quantized radiation field increases binding abilities of a potential. (e.g., for some V, $-\Delta + V + H_f$ has no bound state, but $-\Delta + V + H_f + W(\alpha)$ has).
2. The binding energy should increase :

$$\Sigma_\alpha(0) - \Sigma_\alpha(V) > \Sigma(0) - \Sigma(V)$$

Several difficulties !

1. Both the Coulomb potential V and the interaction $W(\alpha)$ with the radiation field depend on the coupling constant α.
2. Usual Kato's perturbation theory failed : the G.S. energy of the noninteracting problem is not an isolated e.v.
3. The magnetic potential $\mathbb{A}(x)$ contains a frequency space singularity $\frac{1}{|k|^{\frac{1}{2}}}$ at the origin : Infrared divergence problem.

- Systematical and rigorous study of Pauli-Fierz Hamiltonian for atoms and molecules initiated by Bach-Fröhlich-Sigal : [BFS'97, BFS'98, BFS'99, ...] (see also [Sp'95, DG'99]).

$\alpha \ll 1$:

- Self-adjointness of H_α and T.
- Existence of a ground state : $H_\alpha \Psi^{\mathrm{GS}} = \Sigma_\alpha(V)\Psi^{\mathrm{GS}}$
- Photon number bound : $\langle \Psi^{\mathrm{GS}},\ N_f \Psi^{\mathrm{GS}} \rangle = \mathcal{O}(\alpha)$.

- Binding condition : [GLL'01, LL'03, BCV'03]

For all α

- For hydrogen atom : $\Sigma_\alpha(0) - \Sigma_\alpha(V) \geq \alpha^2/4 = \Sigma(0) - \Sigma(V)$
- Similar a priori photon number bound for the ground state.

- Self-energy at fixed total momentum : [F'74], [C'01], [CF'07], [C'08].
  The operator $T = -\Delta + H_f + W(\alpha)$ commutes with the total momentum P_{tot},

$$P_{\text{tot}} = (-i\nabla_x \otimes \mathbb{I}_f) + (\mathbb{I}_{\text{el}} \otimes P_f), \quad P_f = \sum_\lambda \int k\, a_\lambda^*(k) a_\lambda(k) \mathrm{d}k$$

$$T \simeq \int_{\mathbb{R}^3} T(P) \mathrm{d}P$$

$\inf \sigma(T(0)) = \inf \sigma(T) = \Sigma_\alpha(0)$
$\Sigma_\alpha(0)$ is an eigenvalue of $T(0)$: $T(0)\Psi_0^{\text{GS}} = \Sigma_\alpha(0)\Psi_0^{\text{GS}}$.

- Study of $\Sigma_\alpha(0)$: [HVV'03], [H'03], [CH'04], [BLV'05], [BCVV'08], [BV'10]

[BCVV'08] : $\Sigma_\alpha(0)$ up to the order α^3, with error $\mathcal{O}(\alpha^4)$.

- [HiSp'01] (large α), [HS'03] (spin, αZ fixed)
  [CVV'03] : $\Sigma_\alpha(0) - \Sigma_\alpha(V) > \alpha^2/4$ (α small).
- [HHSp'05] : $\Sigma_\alpha(0) - \Sigma_\alpha(V)$ up to the order α^3, with error $\mathcal{O}(\alpha^{\frac{7}{2}} \log \alpha)$ (scalar boson)
- [BFP'06] : Expansion in α for $\inf \sigma(H_\alpha)$ and its associated ground state, for arbitrary N (using a scheme going back to early work of A. Pizzo)

$$\Sigma_\alpha(V) = \inf \sigma(H_\alpha) = \epsilon_0 + \sum_{k=1}^{2N} \epsilon_k(\alpha)\alpha^{k/2} + o(\alpha^N)$$

$\forall \delta > 0$, $\lim_{\alpha \to 0}(\alpha^\delta \epsilon_k(\alpha)) = 0$ ("typically" $\epsilon_k(\alpha) = \log \alpha$)

Remark : Scaling $X_{[\mathrm{BFP}]} = \alpha x$, $K_{[\mathrm{BFP}]} = \frac{1}{\alpha^2} k$

$$H_{[\mathrm{BFP}]} \simeq \alpha^{-2} H_\alpha$$

i.e., $\Lambda < \infty$ fixed in [BFP] corresponds to $\alpha^2 \Lambda$ fixed in [BCVV].

Conjecture

Occurrence of $\ln \alpha$ terms in the expansion for the binding energy $\Sigma_\alpha(0) - \Sigma_\alpha(V)$.

Theorem

$$\Sigma_\alpha(0) - \Sigma_\alpha(V)$$

$$= \underbrace{\frac{\alpha^2}{4}}_{\Sigma(0)-\Sigma(V)} + \underbrace{e^{(1)}\alpha^3}_{\text{increase of bind. energy}} + e^{(2)}\alpha^4 + e^{(3)}\underbrace{\alpha^5 \log\alpha}_{\text{infrared divergence}} + o(\alpha^5 \log\alpha)$$

$$e^{(1)} = \frac{1}{2\pi}\int_0^\infty \frac{\chi_\Lambda(t)}{1+t}\mathrm{d}t > 0$$

$$e^{(2)} = \frac{1}{6}\langle \mathbb{A}^-(0)(H_f + P_f^2)^{(-1)}\mathbb{A}^+(0)\cdot\mathbb{A}^-(0)\Omega_f, (H_f+P_f^2)^{-1}\Omega_f\rangle$$

$$+\frac{1}{12}\sum_{i=1}^{3}\|(P_f^2+H_f)^{-\frac{1}{2}}P_f^i(P_f^2+H_f)^{-1}\mathbb{A}^+(0)\cdot\mathbb{A}^+(0)\Omega_f\|^2 - \frac{1}{2}\|\mathbb{A}^-(0).(H_f+P_f^2)^{-1}\mathbb{A}^+(0)\Omega_f\|^2$$

$$+4a_0^2\|(-\Delta - \frac{1}{|x|} + \frac{1}{4})^{-\frac{1}{2}}Q^\perp\Delta f_1\|^2, \quad a_0 = \int \frac{k_1^2+k_2^2}{4\pi^2|k|^2}\frac{1}{|k|^2+|k|}\chi_\Lambda(|k|)\mathrm{d}k_1\mathrm{d}k_2\mathrm{d}k_3$$

$$e^{(3)} = \frac{4}{\pi}\|(-\Delta - \frac{1}{|x|} + \frac{1}{4})^{\frac{1}{2}}\nabla f_1\|^2 \neq 0$$

$$\mathcal{U} = \mathrm{e}^{iP_f \cdot x}$$

The e^- momentum variable is shifted by P_f.
The photon "position" is shifted by x.

- $\mathcal{U}(i\nabla_x)\mathcal{U}^* = i\nabla_x + P_f$
  ($i\nabla_x$ acquires now the meaning of the total momentum, i.e., momentum of particle + field).
- $\mathcal{U}\mathbb{A}(x)\mathcal{U}^* = \mathbb{A}(0)$.
- $\mathcal{U}T\mathcal{U}^* = (P - P_f - \sqrt{\alpha}\mathbb{A}(0))^2 + H_f - c_{n.o.}\alpha$, $P := i\nabla_x$.
- $\mathcal{U}(T+V)\mathcal{U}^* = \underbrace{(P - P_f - \sqrt{\alpha}\mathbb{A}(0))^2 + H_f}_{T} - \frac{\alpha}{|x|} - c_{n.o.}\alpha$

$$H := \mathcal{U}(T+V)\mathcal{U}^* = (P^2 - \frac{\alpha}{|x|}) + T(0) - 2\mathrm{Re}\, P.(P_f + \sqrt{\alpha}\mathbb{A}(0))$$

- Iterative procedure to estimate the binding energy $\Sigma_\alpha(0) - \Sigma_\alpha(V)$.
- No need to compute $\Sigma_\alpha(0)$ and $\Sigma_\alpha(V)$ separately up to the required order : E.g., to compute the difference

$$\Sigma_\alpha(0) - \Sigma_\alpha(V)$$

with the error $o(\alpha^5 \log \alpha)$ it suffices to know $\Sigma_\alpha(0)$ and $\Sigma_\alpha(V)$ with the error $\mathcal{O}(\alpha^4)$:

- Estimate up to the order α^3 with error $\mathcal{O}(\alpha^4 \log \alpha)$ improved to $\mathcal{O}(\alpha^4)$.
- Estimate up to the order $\alpha^5 \log \alpha$ with error $o(\alpha^5 \log \alpha)$.

Proof - Increase of the binding energy

- Let f_α be the GS of $(-\Delta - \alpha/|x|)$, with e.v. $-e_0 = -\alpha^2/4$.
- Let Ψ_0^{GS} be the G.S. of the operator $T(0) = (-P_f - \sqrt{\alpha}\mathbb{A}(0))^2 + H_f$
- Normalized trial state : $\Theta^{\mathrm{trial}} = f_\alpha(x)\Psi_0^{\mathrm{GS}} \in L^2(\mathbb{R}^3) \otimes \mathfrak{F}$

$$\Sigma_\alpha(V) \leq \langle \Theta^{\mathrm{trial}}, \underbrace{H}_{\mathcal{U}(T+V)\mathcal{U}^*} \Theta^{\mathrm{trial}} \rangle$$

$$= \langle \Theta^{\mathrm{trial}}, \left(: (P - P_f - \sqrt{\alpha}\mathbb{A}(0))^2 : + H_f - \frac{\alpha}{|x|} \right) \Theta^{\mathrm{trial}} \rangle$$

$$= \underbrace{\langle (P^2 - \frac{\alpha}{|x|}) f_\alpha \Psi_0^{\mathrm{GS}}, f_\alpha \Psi_0^{\mathrm{GS}} \rangle}_{-\alpha^2/4} + \underbrace{\langle f_\alpha \Psi_0^{\mathrm{GS}}, T(0) f_\alpha \Psi_0^{\mathrm{GS}} \rangle}_{\Sigma_\alpha(0)}$$

$$- \underbrace{2\mathrm{Re}\, \langle P \cdot (P_f + \sqrt{\alpha}\mathbb{A}(0)) f_\alpha \Psi_0^{\mathrm{GS}}, f_\alpha \Psi_0^{\mathrm{GS}} \rangle}_{0 \text{ by sym.}}$$

Proof - increase of the binding energy

Thus

$$\Sigma_\alpha(0) - \Sigma_\alpha(V) \geq \frac{\alpha^2}{4} = \Sigma(0) - \Sigma(V)$$

Improved trial function : add a state orthogonal to u_α. :

$$\Theta^{\text{trial}} = f_\alpha(x)\Psi_0^{GS} + g\,.$$

A priori estimates for states orthogonal to f_α

Assume that $g \in \mathfrak{H}$ is such that for all $k \geq 0$ we have $\langle \Pi_k g, f_\alpha \rangle_{L^2(\mathbb{R}^3, \mathrm{d}x)} = 0$. Then there exists $\nu > 0$ and $\alpha_0 > 0$ such that for all $0 < \alpha < \alpha_0$ we have

$$\langle g,\, H\, g \rangle \geq (\Sigma_\alpha(0) - e_0)\|g\|^2 + 3/32\,\alpha^2\|g\|^2 + \nu\|H_f^{\frac{1}{2}} g\|^2\,.$$

Photon number bound

Let Ψ^{GS} be the GS of $H := \mathcal{U} H_\alpha \mathcal{U}^*$. Then there exists $c > 0$ such that for all α small enough

$$\langle \Psi^{\text{GS}},\, N_f \Psi^{\text{GS}} \rangle \leq c\alpha^2 \log(\alpha^{-1})$$

Translationnaly invariant case/free case [BCVV'08]

$$\Phi_2 := -(H_f + P_f^2)^{-1}\mathbb{A}^+(0)\cdot\mathbb{A}^+(0)\Omega_f \ ,$$
$$\Phi_3 := -(H_f + P_f^2)^{-1}P_f\cdot\mathbb{A}^+(0)\Phi_2 \ ,$$
$$\Phi_1 := -(H_f + P_f^2)^{-1}P_f\cdot\mathbb{A}^-(0)\Phi_2 \ .$$

$$\Theta_0^{\text{trial}} := \Omega_f + \alpha\Phi_2 + 2\alpha^{\frac{3}{2}}\Phi_1 + 2\alpha^{\frac{3}{2}}\Phi_3$$

Then

$$\Sigma_\alpha(0) = \inf\sigma(T) = \inf\sigma(T(0))$$
$$= \langle\Theta_0^{\text{trial}}, T(0)\Theta_0^{\text{trial}}\rangle + \mathcal{O}(\alpha^4)$$
$$= -\alpha^2\|\Phi_2\|_*^2 + \alpha^3(2\|\mathbb{A}^-(0)\Phi_2\|^2 - 4\|\Phi_3\|_*^2 - 4\|\Phi_1\|_*^2) + \mathcal{O}(\alpha^4) \ ,$$

where $\langle\phi, \psi\rangle_* = \langle\phi, (H_f + P_f^2)\psi\rangle$.
For the true GS Ψ_0^{GS} of $T(0)$, $\Psi_0^{\text{GS}} = \Theta_0^{\text{trial}} + R_0$, with $\|R_0\| = \mathcal{O}(\alpha)$, $\|R_0\|_* = \mathcal{O}(\alpha^2)$.

Proof - where does $log\alpha$ term come from ?

$$\left\langle (f_\alpha \Psi_0^{GS} + g),\, H\,(f_\alpha \Psi_0^{GS} + g) \right\rangle$$

$$= (\Sigma_0(\alpha) - \frac{\alpha^2}{4}) \underbrace{+\langle g,\, H\, g\rangle + 2\mathrm{Re}\,\langle f_\alpha \Psi_0^{GS},\, H\, g\rangle}_{\text{responsible for the increase of the binding energy}}$$

$$\underbrace{=}_{g \perp f_\alpha} (\Sigma_0(\alpha) - \frac{\alpha^2}{4}) + \langle g\,, H\, g\rangle - 4\mathrm{Re}\,\langle (P_f + \sqrt{\alpha} A(0)).P\, f_\alpha \Psi_0^{GS},\, g\rangle$$

The largest contribution to the cross-term comes from the projection of Ψ_0^{GS} onto the vacuum-vector Ω_f

$$-4\mathrm{Re}\,\sqrt{\alpha}\langle A^+(0).P\, f_\alpha \Omega_f\,,\, \Pi_1 g\rangle$$

Proof - where does $log\alpha$ term come from ?

- The term $-4\mathrm{Re}\,\sqrt{\alpha}\langle A^+(0).P\, f_\alpha\Omega_f\,,\,\Pi_1 g\rangle$ is estimated together with

$$\langle (H+e_0)\Pi_1 g,\Pi_1 g\rangle = \underbrace{\langle (-\Delta-\frac{\alpha}{|x|}+e_0)\Pi_1 g,\Pi_1 g\rangle}_{>0 \text{ since } g\perp f_\alpha} + \langle (H_f+P_f^2)\Pi_1 g,\Pi_1 g\rangle$$

$$= \langle \underbrace{(-\Delta-\frac{\alpha}{|x|}+e_0+|k|+|k|^2)}_{=:B^2}\,\Pi_1 g,\Pi_1 g\rangle$$

- We can minimize w.r.t. $\Pi_1 g$ $(-2ba+b^2\geq -a^2)$

$$-2\mathrm{Re}\,\sqrt{\alpha}\langle 2B^{-1}A^+(0).P\, f_\alpha\Omega_f,\ B\Pi_1 g\rangle + \langle B\Pi_1 g,\ B\Pi_1 g\rangle$$

$$\geq \underbrace{-\alpha\langle 2B^{-1}A^+(0).P\, f_\alpha\Omega_f,\ 2B^{-1}A^+(0).P\, f_\alpha\Omega_f\rangle}_{\text{Leading correction term to the binding energy.}}$$

- Forgetting coef. and polarization vectors, and rescaling in x yields

$$-\alpha^3\left\|\frac{\chi(|k|)}{|k|^{\frac{1}{2}}\left(|k|+|k|^2+\alpha^2(-\Delta-\frac{1}{|x|}+\frac{1}{4})\right)^{\frac{1}{2}}}\ \frac{\partial f_1}{\partial x_1}\right\|^2,$$

where f_1 is the ground state of $(-\Delta-1/|x|)$.

The operator $(-\Delta - 1/|x| + 1/4)$ acts on $\partial f/\partial x_1$. For simplicity, we replace it by a constant C :

$$-\alpha^3 \int \frac{\chi(|k|)^2\, d^3k}{|k|(|k| + |k|^2 + \alpha^2 C)}$$
$$= -\alpha^3 \int \frac{\chi(|k|)^2\, d^3k}{|k|(|k| + |k|^2)} + \alpha^3 \int \frac{\alpha^2 C \chi(|k|)^2\, d^3k}{|k|(|k| + |k|^2 + \alpha^2 C)(|k| + |k|^2)}$$
$$= c_1\alpha^3 + \underbrace{c_2\alpha^5 \int_0^1 \frac{d|k|}{|k| + \alpha^2 C}}_{=c_3\alpha^5 \log\alpha} + \mathcal{O}(\alpha^5)$$

In the case of a *sharp infrared cutoff* or a sufficiently strong *infrared regularization*, or *photons with mass*, or *large Z limit* (i.e. $\alpha Z =: \beta$ fixed and independent of α in the Coulomb potential), this $\log\alpha$ term would not exist here.

Ground state energy of $T(0)$ - improved estimate

We have

$$\Sigma_\alpha(0) \;=\; d^{(0)}\alpha^2 \;+\; d^{(1)}\alpha^3 \;+ d^{(2)}\alpha^4 \;+\; \mathcal{O}(\alpha^5)\,,$$

with

$$d^{(0)} := -\|\Phi_2\|_*^2, \quad d^{(1)} := 2\|A^-\Phi_2\|^2 - 4\|\Phi_3\|_*^2 - 4\|\Phi_1\|_*^2$$

$$d^{(2)} := -\left(\frac{2\|A^-\Phi_2\|^2 - 4\|\Phi_1\|_*^2 - 4\|\Phi_3\|_*^2}{\|\Phi_2\|_*}\right)^2$$

$$+8\Re\langle\Phi_1, A^- \cdot A^-\Phi_3\rangle + 8\|A^-\Phi_1\|^2 + 8\|A^-\Phi_3\|^2 - 16\|\tilde{\Phi}_2\|_*^2 - 16\|\Phi_4\|_*^2 + \|\Phi_2\|^2\|\Phi_2\|_*^2\,,$$

and

$$\Phi_2 := -(H_f + P_f^2)^{-1}A^+ \cdot A^+\Omega_f, \quad \Phi_3 := -(H_f + P_f^2)^{-1}P_f \cdot A^+\Phi_2\,, \quad \Phi_1 := -(H_f + P_f^2)^{-1}P_f \cdot A^-\Phi_2\,,$$

$$\tilde{\Phi}_2 := -P_{\Phi_2}{}^{\perp}(H_f + P_f^2)^{-1}\left(P_f \cdot A^+\Phi_1 + P_f \cdot A^-\Phi_3 + \frac{1}{2}A^+ \cdot A^-\Phi_2\right)$$

$$\Phi_4 := -(H_f + P_f^2)^{-1}\left(P_f \cdot A^+\Phi_3 + \frac{1}{4}A^+ \cdot A^+\Phi_2\right)\,,$$

where $P_{\Phi_2}{}^{\perp}$ is the orthogonal projection onto $\{\varphi \in \mathfrak{F} \mid \langle\varphi,\, \Phi_2\rangle_* = 0\}$.

Corollary

The ground state energy $\Sigma_\alpha(V)$ of H fulfils

$$\Sigma_\alpha(V) = \tilde{d}^{(0)}\alpha^2 + \tilde{d}^{(1)}\alpha^3 + \tilde{d}^{(2)}\alpha^4 + \tilde{d}^{(3)}\alpha^5 \log\alpha + o(\alpha^5 \log\alpha)\,.$$

Photon number bound

Let Ψ^{GS} be the true GS of $\mathcal{U} H_\alpha \mathcal{U}^*$.
Then there exists $c > 0$ such that for all α small enough

$$\langle \Psi^{\mathrm{GS}}, N_f \Psi^{\mathrm{GS}} \rangle \leq c\alpha^2 \log(\alpha^{-1})$$

Using $[a_\lambda(k),\, H_f] = |k|$, and

$$[a_\lambda(k),\, \underbrace{(-i\nabla - \sqrt{\alpha}\mathbb{A}(x))}_{v}] = \frac{\epsilon_\lambda(k)}{2\pi|k|^{\frac{1}{2}}}\chi_\Lambda(|k|)\mathrm{e}^{ik\cdot x},$$

Pull-through formula

$$a_\lambda(k)E\Psi^{\mathrm{GS}} = a_\lambda(k)H_\alpha\Psi^{\mathrm{GS}}$$

$$= \left((H_f + |k|)a_\lambda(k) - \frac{\alpha}{|x|}a_\lambda(k) + [v,\, a_\lambda(k)]v + v[v,\, a_\lambda(k)]\right)\Psi^{\mathrm{GS}}$$

Thus

$$a_\lambda(k)\Psi^{\mathrm{GS}} = \frac{-\sqrt{\alpha}\chi_\Lambda(|k|)}{2\pi|k|^{\frac{1}{2}}}\,\frac{1}{H_\alpha + |k| - E}\,2(-i\nabla - \sqrt{\alpha}\mathbb{A}(x))\cdot\epsilon_\lambda(k)\mathrm{e}^{ik\cdot x}\Psi^{\mathrm{GS}}\,,$$

which implies

$$\|a_\lambda(k)\Psi^{\mathrm{GS}}\|^2 \leq \frac{\alpha\chi_\Lambda(|k|)}{|k|^3}\Big(\underbrace{\langle\Psi^{\mathrm{GS}}, H_\alpha\Psi^{\mathrm{GS}}\rangle}_{\mathcal{O}(\alpha)} + \underbrace{\langle\Psi^{\mathrm{GS}}, \frac{\alpha}{|x|}\Psi^{\mathrm{GS}}\rangle}_{\mathcal{O}(\alpha)}\Big) \leq c\frac{\alpha^2\chi_\Lambda(|k|)}{|k|^3}$$

I.e.

$$\|a_\lambda(k)\Psi^{\mathrm{GS}}\|^2 \leq c\frac{\alpha^2\chi_\Lambda(|k|)}{|k|^3} \tag{1}$$

On the other hand ([BFS'99], [GLL'01])

$$\|a_\lambda(k)\Psi^{\mathrm{GS}}\| \leq c\frac{\sqrt{\alpha}\chi_\Lambda(|k|)}{|k|^{\frac{1}{2}}}\underbrace{\|\,|x|\Psi^{\mathrm{GS}}\,\|}_{\mathcal{O}(\alpha^{-\frac{5}{4}})}$$

Thus

$$\|a_\lambda(k)\Psi^{\mathrm{GS}}\|^2 \leq c\frac{\alpha^{-\frac{3}{2}}\chi_\Lambda(|k|)}{|k|}\,. \tag{2}$$

$$\langle\Psi^{\mathrm{GS}},\, N_f\Psi^{\mathrm{GS}}\rangle = \int \|a_\lambda(k)\Psi^{\mathrm{GS}}\|^2\mathrm{d}k = \int_{|k|<\alpha^{7/4}} + \int_{|k|\geq\alpha^{7/4}}$$

$$\leq_{(1)\text{ and }(2)} c\int_{|k|<\alpha^{7/4}}\frac{\alpha^{-\frac{3}{2}}\chi_\Lambda(|k|)}{|k|}\mathrm{d}k + c\int_{|k|\geq\alpha^{7/4}}\frac{\alpha^2\chi_\Lambda(|k|)}{|k|^3}\mathrm{d}k \leq c\alpha^2|\log\alpha|$$