Quantum measurements in continuous time and non-Markovian evolutions

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1) The problem. Connections among master equations, unravelling and observation in continuous time in the Markov and non-Markov cases.

2) From a class of master equations with memory (the Lindblad rate equation — Budini, Breuer, Petruccione...) to a jump/diffusion unravelling with measurement interpretation. An example: the spectrum of a 2-level atom in a structured bath. (work with Pellegrini)

3) From non-Markov stochastic Schrödinger equations to the theory of measurements in continuous time. Possible introduction of coloured noises, measurement based feedback,... (works with Di Tella, Pellegrini, Petruccione, Holevo).

The Markov case. a) We have a stochastic Schrödinger equation (SSE) for a vector state $\psi(t)$ in a Hilbert space $\mathcal{H}$; part of the noises represent the observed output. This measurement interpretation is shown to be consistent with the axiomatic of quantum mechanics: positive operator valued measures, instruments,...

b) By taking the conditional expectation of $|\psi(t)\rangle\langle\psi(t)|$ on the σ-algebra generated by the output we get the stochastic master equation (SME) for the conditional statistical operator, a stochastic equation in the trace-class $\mathcal{T}(\mathcal{H})$.

c) By expectation we get a master equation (ME) with a generator in Lindblad form: a completely positive (CP) dynamics.

$$\begin{tikzcd} \boxed{\text{SSE}} \arrow[rr, "\text{from } \mathcal{H} \text{ to } \mathcal{T}(\mathcal{H})", "+\ \text{conditional expectation}"'] && \boxed{\text{SME}} \arrow[rr, "\text{expectation}"] && \boxed{\text{ME}} \arrow[dll, bend left] \\ && \text{unravelling} \arrow[ull, bend left] && \end{tikzcd}$$

d) To construct a SSE compatible with a given master equation is called unravelling. Important also for numerical simulations.

The problem in the non-Markov case. Starting point: the ME

$$\begin{tikzcd}[column sep=huge] \boxed{\text{SSE}} \arrow[r, "\text{from } \mathcal{H} \text{ to } \mathcal{T}(\mathcal{H})", "+\text{ conditional expectation}"'] & \boxed{\text{SME}} \arrow[r, "\text{expectation}"] & \boxed{\text{ME}} \arrow[ll, bend left, "\text{unravelling}"] \end{tikzcd}$$

a) There is not a general theory of non-Markov master equations. The problem is to guarantee the complete positivity of the dynamics. Some classes of bona fide master equations have been constructed in the literature.

b) One can invent good unravelling for simulations, but a generic unravelling could be incompatible with the measurement interpretation.

c) Our work. Starting point: the Lindblad rate equation (non-Markov & CP)

c1) We construct unravellings of diffusion and jump type for which we can show compatibility with the structure of quantum mechanics.

c2) The interpretation induces restrictions on the possible unravellings and on the possible outputs.

c3) An example: modifications in the heterodyne spectrum of a 2-level system in a structured bath.

The problem in the non-Markov case. Starting point: the SSE

$$\begin{tikzcd}[column sep=huge] \boxed{\text{SSE}} \arrow[r, "\text{from } \mathcal{H} \text{ to } \mathcal{T}(\mathcal{H})", "+\text{ conditional expectation}"'] & \boxed{\text{SME}} \arrow[r, "\text{expectation}"] & \boxed{\text{ME}} \arrow[ll, bend left, "\text{unravelling}"] \end{tikzcd}$$

a) Easy to respect the CP property of the reduced dynamics.

b) The measurement interpretation gives rise to restrictions.

c) Our proposal: a SSE of diffusive type with memory, with measurement interpretation

d) Possibility of describing coloured baths and memory effects due to delayed feedback (closed loop control)

Lindblad rate equation. System space: $\mathcal{H}$, a separable complex Hilbert space

$\eta_i(t) \in \mathcal{T}(\mathcal{H})$ (trace class)

$$\frac{\mathrm{d}}{\mathrm{d}t}\eta_i(t) = \mathcal{L}_i[\eta_i(t)] + \mathcal{D}_i[\eta_1(t), \ldots, \eta_n(t)],$$

$\forall(\tau_1, \ldots, \tau_n)$, $\tau_i \in \mathcal{T}(\mathcal{H})$:

$$\mathcal{L}_i[\tau_i] := -\mathrm{i}[H^i, \tau_i] + \sum_{\alpha=1}^{d}\left(L^i_\alpha \tau_i L^{i\,*}_\alpha - \frac{1}{2}\left\{L^{i\,*}_\alpha L^i_\alpha, \tau_i\right\}\right),$$

$i = 1, \ldots, n$

$$\mathcal{D}_i[\tau_1, \ldots, \tau_n] := \sum_{\beta=1}^{m}\sum_{k=1}^{n}\left(R^{ik}_\beta \tau_k R^{ik\,*}_\beta - \frac{1}{2}\left\{R^{ki\,*}_\beta R^{ki}_\beta, \tau_i\right\}\right),$$

$H^i = H^{i\,*}$, L^i_α, $R^{ki}_\beta \in \mathcal{L}(\mathcal{H})$ (bounded linear operators)

Initial condition: $\eta_i(0) \in \mathcal{T}(\mathcal{H})$, $\eta_i(0) \geq 0$, $\sum_{i=1}^n \mathrm{Tr}\,\{\eta_i(0)\} = 1$.

System state: $\eta_S(t) = \sum_{i=1}^n \eta_i(t) \in \mathcal{S}(\mathcal{H}) \subset \mathcal{T}(\mathcal{H})$ (statistical operators).

The index i represents some degree of freedom of a structured environment, for instance the energy bands of a thermal bath.

The Lindblad rate equation is known to give rise to a completely positive (CP) dynamics and can be approximately deduced from the unitary system/environment dynamics by the correlated projection technique.

A dilation. $\mathcal{K} := \mathcal{H} \otimes \mathbb{C}^n$. e_i: canonical basis in $\mathbb{C}^n$. $\forall \tau \in \mathcal{T}(\mathcal{K})$:

$$\mathcal{L}[\tau] := -\mathrm{i}[H,\tau] + \sum_{\alpha=1}^{d} \left(V_\alpha \tau V_\alpha{}^* - \frac{1}{2}\{V_\alpha{}^* V_\alpha, \tau\} \right) + \sum_{\beta=1}^{m} \sum_{j=1}^{n} \left(S_\alpha^j \tau S_\alpha^j{}^* - \frac{1}{2}\left\{ S_\alpha^j{}^* S_\alpha^j, \tau \right\} \right).$$

$H := \sum_{i=1}^n H^i \otimes |e_i\rangle\langle e_i|$, $V_\alpha := \sum_{i=1}^n L_\alpha^i \otimes |e_i\rangle\langle e_i|$, $S_\beta^k := \sum_{i=1}^n R_\beta^{ik} \otimes |e_i\rangle\langle e_k|$.

1) $\mathrm{e}^{\mathcal{L}t}$ is a CP quantum dynamical semigroup on $\mathcal{T}(\mathcal{K})$

The degree of freedom i determines a superselection rule: the physical states are block-diagonal. Let $\mathcal{C} \subset \mathcal{T}(\mathcal{K})$ be defined by $\tau \in \mathcal{C} \Leftrightarrow \tau = (\tau_1, \ldots, \tau_n)$, with $\tau_j \in \mathcal{T}(\mathcal{H})$. Projection $\mathcal{P} : \mathcal{T}(\mathcal{K}) \to \mathcal{C}$ with $(\mathcal{P}[\tau])_j = \mathrm{Tr}_{\mathbb{C}^n}\{\tau(\mathbb{1} \otimes |e_j\rangle\langle e_j|\}$.

2) $\mathcal{T}(t) := \mathcal{P} \circ \mathrm{e}^{\mathcal{L}t}\big|_{\mathcal{C}}$ is a CP semigroup, in spite of the fact that $\mathrm{e}^{\mathcal{L}t}$ does not preserve the block diagonal form of the states.

3) The solution of the Lindblad rate equation is $\eta_j(t) = \big(\mathcal{T}(t)[\eta_1(0), \ldots, \eta_n(0)]\big)_j$

The degree of freedom i is unobservable.

Define the map $\mathcal{P}_S : \mathcal{C} \to \mathcal{T}(\mathcal{H})$ by $\mathcal{P}_S[\tau] = \sum_j \tau_j$.

4) The CP dynamics giving the system state is $\eta_S(t) = \mathcal{P}_S \circ \mathcal{T}(t)[\eta_1(0), \ldots, \eta_n(0)]$ and it is this dynamics which is non Markovian.

The linear stochastic Schrödinger equation (lSSE). $\quad \zeta(0) \in \mathcal{K}, \quad \|\zeta(0)\| = 1$

$$\mathrm{d}\zeta(t) = K\zeta(t_-)\mathrm{d}t + \sum_{\alpha=1}^{d'} \overline{h_\alpha(t)} V_\alpha \zeta(t_-)\mathrm{d}W_\alpha(t) + \sum_{\alpha=d'+1}^{d} \left(\frac{V_\alpha}{\sqrt{\lambda_\alpha}} - \mathbb{1} \right) \zeta(t_-)\mathrm{d}N_\alpha(t)$$

$$+ \sum_{\beta=1}^{m'} \sum_{k=1}^{n} \overline{h_\beta^k(t)} S_\beta^k \zeta(t_-)\mathrm{d}W_\beta^k(t) + \sum_{\beta=m'+1}^{m} \sum_{k=1}^{n} \left(\frac{S_\beta^k}{\sqrt{\lambda_\beta^k}} - \mathbb{1} \right) \zeta(t_-)\mathrm{d}N_\beta^k(t),$$

$$K = -\mathrm{i}H - \frac{1}{2}\sum_{\alpha=1}^{d} {V_\alpha}^* V_\alpha - \frac{1}{2}\sum_{\beta=1}^{m}\sum_{k=1}^{n} {S_\beta^k}^* S_\beta^k + \frac{\lambda}{2},$$

$$\lambda = \sum_{\alpha=d'+1}^{d} \lambda_\alpha + \sum_{\beta=m'+1}^{m}\sum_{k=1}^{n} \lambda_\beta^k, \qquad |h_\alpha(t)| = \left|h_\beta^k(t)\right| = 1.$$

$h_\alpha(t)$, $h_\beta^k(t)$ complex functions, continuous from the left with limits from the right.

W_α, W_β^k: standard Wiener processes; N_α, N_β^k: Poisson processes of intensities λ_α, λ_β^k. All the noises are independent and defined in a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{Q})$. The key properties: $\eta_j(t) = \mathbb{E}_\mathbb{Q}[|\zeta_j(t)\rangle\langle\zeta_j(t)|]$, $j = 1, \ldots, n$, satisfy the Lindblad rate equation and $\|\zeta(t)\|_\mathcal{K}^2$ is a mean-one martingale.

Change of probability and Girsanov transformation

$\mathbb{P}_T(\mathrm{d}\omega) = \|\zeta(T,\omega)\|_{\mathcal{K}}^2\, \mathbb{Q}(\mathrm{d}\omega)$, T is a fixed time horizon. Under $\mathbb{P}_T$:

1) $\psi(t) := \|\zeta(t)\|_{\mathcal{K}}^{-1}\, \zeta(t)$ satisfies the non-linear SSE, starting point for simulations (unravelling).

2) $\hat{W}_\alpha(t) := W_\alpha(t) - \int_0^t v_\alpha(s)\mathrm{d}s$, $\hat{W}^k_\beta(t) := W^k_\beta(t) - \int_0^t v^k_\beta(s)\mathrm{d}s$ are independent standard Wiener processes.

$$v_\alpha(t) := 2\,\mathrm{Re}\,\overline{h_\alpha(t)}\left\langle \psi(t_-)\middle| V_\alpha \psi(t_-)\right\rangle \equiv 2\sum_{j=1}^n \mathrm{Re}\,\overline{h_\alpha(t)}\left\langle \psi_j(t_-)\middle| L^j_\alpha \psi_j(t_-)\right\rangle$$

$$v^k_\beta(t) := 2\,\mathrm{Re}\,\overline{h^k_\beta(t)}\left\langle \psi(t_-)\middle| S^k_\beta \psi(t_-)\right\rangle \equiv 2\sum_{j=1}^n \mathrm{Re}\,\overline{h^k_\beta(t)}\left\langle \psi_j(t_-)\middle| R^{jk}_\beta \psi_k(t_-)\right\rangle$$

3) N_α, N^k_β become counting processes of stochastic intensities, respectively,

$$I_\alpha(t) := \left\|V_\alpha \psi(t_-)\right\|^2 \equiv \sum_{j=1}^n \left\|L^j_\alpha \psi_j(t_-)\right\|^2,$$

$$I^k_\beta(t) := \left\|S^k_\beta \psi(t_-)\right\|^2 \equiv \sum_{j=1}^n \left\|R^{jk}_\beta \psi_k(t_-)\right\|^2.$$

Possible outputs of the measurement: W_α, W^i_β, N_α, N^i_β — From the general theory (Markov case) it is possible to construct "positive operator valued measures" and "instruments" with these outputs — Formally, the structure of quantum mechanics is respected. Physically, no problem with $W_\alpha(t)$ and N_α:

$$W_\alpha(t) = \hat{W}_\alpha(t) + \int_0^t v_\alpha(s)\mathrm{d}s, \quad \mathbb{E}_{\mathbb{P}_T}[W_\alpha(t)] = 2\sum_{j=1}^n \mathrm{Re}\int_0^t \overline{h_\alpha(s)}\,\mathrm{Tr}\left\{L^j_\alpha \eta_j(s_-)\right\}\mathrm{d}s,$$

N_α has mean intensity $\mathbb{E}_{\mathbb{P}_T}[I_\alpha(t)] = \sum_{j=1}^n \mathrm{Tr}\{{L^j_\alpha}^* L^j_\alpha \eta_j(t_-)\}$.

$W^i_\beta(t) = \hat{W}^i_\beta(t) + \int_0^t v^i_\beta(s)\mathrm{d}s$ does not respect the superselection rule and it is not a physical observable: for instance, its mean value is not expressible with $\eta_1, \ldots, \eta_n$

$$\mathbb{E}_{\mathbb{P}_T}[W^i_\beta(t)] = 2\sum_{j=1}^n \mathrm{Re}\int_0^t \overline{h^i_\beta(s)}\,\mathbb{E}_{\mathbb{P}_T}\left[\left\langle \psi_j(s_-)\middle| R^{ji}_\beta \psi_i(s_-)\right\rangle\right]\mathrm{d}s.$$

N^i_β has mean intensity $\mathbb{E}_{\mathbb{P}_T}[I^i_\beta(t)] = \sum_{j=1}^n \mathrm{Tr}\{{R^{ji}_\beta}^* R^{ji}_\beta \eta_i(t_-)\}$. The counting process N^i_β gives information on the index i which is assumed to be unobservable.

Physically possible output: $M_\beta(t) := \sum_{i=1}^n N_\beta^i(t)$, counting process of stochastic intensity $J_\beta(t) = \sum_{i=1}^n I_\beta^i(t)$, whose mean is $\mathbb{E}_{\mathbb{P}_T}[J_\beta(t)] = \sum_{i,j=1}^n \operatorname{Tr}\{R_\beta^{ji\,*} R_\beta^{ji} \eta_i(t_-)\}$.

Let $\{\mathcal{G}_t,\ t \geq 0\}$ be the augmented natural filtration generated by the set of the observed processes $W_\alpha,\ \alpha = 1, \ldots, d'$, $N_\alpha,\ \alpha = d'+1, \ldots, d$, $M_\beta,\ \beta = m'+1, \ldots, m$

A posteriori states: the conditional state having observed the outputs up to time t

$$\rho_i(t) = \mathbb{E}_{\mathbb{P}_T}\left[|\psi_i(t)\rangle\langle\psi_i(t)|\middle|\mathcal{G}_t\right], \qquad \rho_S(t) = \sum_{i=1}^n \rho_i(t)$$

Conditional intensities:

$$m_\alpha(t) := \mathbb{E}_{\mathbb{P}_T}\left[v_\alpha(t)\middle|\mathcal{G}_t\right] = 2\operatorname{Re}\textstyle\sum_{j=1}^n \overline{h_\alpha(t)}\operatorname{Tr}_{\mathcal{H}_S}\left\{L_\alpha^j \rho_j(t_-)\right\},$$
$$J_\alpha^1(t) := \mathbb{E}_{\mathbb{P}_T}\left[I_\alpha(t)\middle|\mathcal{G}_t\right] = \textstyle\sum_{j=1}^n \operatorname{Tr}_{\mathcal{H}_S}\left\{L_\alpha^{j\,*} L_\alpha^j \rho_j(t_-)\right\},$$
$$J_\beta^2(t) := \mathbb{E}_{\mathbb{P}_T}\left[J_\beta(t)\middle|\mathcal{G}_t\right] = \textstyle\sum_{i,j=1}^n \operatorname{Tr}_{\mathcal{H}_S}\left\{R_\beta^{ji\,*} R_\beta^{ji} \rho_i(t_-)\right\}.$$

The non linear stochastic master equation (SME). Under the physical probability $\mathbb{P}_T$:

$$\begin{aligned}
\mathrm{d}\rho_i(t) = {} & \left(\mathcal{L}_i[\rho_i(t)] + \mathcal{D}_i[\rho_1(t), \ldots, \rho_n(t)]\right) \mathrm{d}t \\
& + \sum_{\alpha=1}^{d'} \left(\overline{h_\alpha(t)}\, L_\alpha^i \rho_i(t_-) + h_\alpha(t) \rho_i(t_-) {L_\alpha^i}^* - m_\alpha(t)\rho_i(t_-) \right) \mathrm{d}\hat{W}_\alpha(t) \\
& + \sum_{\alpha=d'+1}^{d} \left(\frac{L_\alpha^i \rho_i(t_-) {L_\alpha^i}^*}{J_\alpha^1(t)} - \rho_i(t_-) \right) \left(\mathrm{d}N_\alpha(t) - J_\alpha^1(t)\mathrm{d}t\right) \\
& + \sum_{\beta=m'+1}^{m} \left(\frac{\sum_{j=1}^{n} R_\beta^{ij} \rho_j(t_-) {R_\beta^{ij}}^*}{J_\beta^2(t)} - \rho_i(t_-) \right) \left(\mathrm{d}M_\beta(t) - J_\beta^2(t)\mathrm{d}t\right).
\end{aligned}$$

The outputs are $W_\alpha(t) = \hat{W}_\alpha(t) + \int_0^t m_\alpha(s)\mathrm{d}s$, $N_\alpha(t)$, $M_\beta(t)$.

The mean state $\eta_i(t) = \mathbb{E}_{\mathbb{P}_T}[\rho_i(t)]$, $i = 1, \ldots, n$, satisfies the Lindblad rate equation by construction.

A two-level system in a structured bath. A single qubit in a two-band environment or an optically active molecule, as the fluorophore system, in a local nanoenvironment. Output: only heterodyne detection of the fluorescence light.

$\mathcal{H} = \mathbb{C}^2$, $n = 2$, $d = d' = 1$, $m' = m = 2$.

Parameters: $\omega_i > 0$, $i = 1, 2$; $\gamma_i > 0$, $i = 0, 1, 2$; $\varkappa > 0$; $\nu \in \mathbb{R}$.

$L_1^1 = L_1^2 = \sqrt{\gamma_0}\,\sigma_-$, $h_1(t) = \mathrm{e}^{-\mathrm{i}\nu t}$; $\quad R_1^{11} = R_1^{22} = R_1^{12} = 0$, $R_1^{21} = \sqrt{\gamma_0\varkappa}\,\mathbb{1}$;
$R_2^{11} = R_2^{22} = 0$, $R_2^{12} = \sqrt{\gamma_2}\,\sigma_+$, $R_2^{21} = \sqrt{\gamma_1}\,\sigma_-$; $\quad H^i = \frac{\omega_i}{2}\,\sigma_z$.

$$\mathrm{d}\rho_i(t) = \left(\gamma_0\left(\sigma_-\rho_i(t)\sigma_+ - \frac{1}{2}\{\sigma_+\sigma_-, \rho_i(t)\}\right) + \mathcal{D}_i[\rho_1(t), \rho_2(t)]\right)\mathrm{d}t$$
$$- \frac{\mathrm{i}\omega_i}{2}\,[\sigma_z, \rho_i(t)]\,\mathrm{d}t + \sqrt{\gamma_0}\left(\mathrm{e}^{\mathrm{i}\nu t}\sigma_-\rho_i(t) + \mathrm{e}^{-\mathrm{i}\nu t}\rho_i(t)\sigma_+ - m(t)\rho_i(t)\right)\mathrm{d}\hat{W}(t),$$
$$m(t) = 2\,\mathrm{Re}\left(\mathrm{e}^{\mathrm{i}\nu t}\,\mathrm{Tr}_{\mathcal{H}_S}\left\{\sigma_-\left(\rho_1(t_-) + \rho_2(t_-)\right)\right\}\right).$$

$$\mathcal{D}_1[\rho_1(t), \rho_2(t)] = \gamma_2\sigma_+\rho_2(t)\sigma_- - \frac{\gamma_1}{2}\{\sigma_+\sigma_-, \rho_1(t)\} - \gamma_0\varkappa\rho_1(t)$$

$$\mathcal{D}_2[\rho_1(t), \rho_2(t)] = \gamma_1\sigma_-\rho_1(t)\sigma_+ - \frac{\gamma_2}{2}\{\sigma_-\sigma_+, \rho_2(t)\} + \gamma_0\varkappa\rho_1(t).$$

Output current: $J(t)=\sqrt{k}\int_0^t \mathrm{e}^{-k(t-s)/2}\,\mathrm{d}W(s), \quad k>0.$

The power of the output current produced by the detector is proportional to $J(t)^2$ and the mean power at large times is proportional to

$P(\nu)=\lim_{t\to+\infty}\mathbb{E}_{\mathbb{P}_t}[J(t)^2]=1+4\pi\Sigma(\nu)$: white noise (shot noise) + spectrum

Heterodyne spectrum, for $k\downarrow 0$:

$$\Sigma(\nu)=D\gamma_0\varkappa\left\{\frac{\gamma_0(1+\varkappa)+\gamma_1}{4\left(\nu-\omega_1\right)^2+\Gamma_1^2}+\frac{\varkappa\gamma_2}{4\left(\nu-\omega_2\right)^2+\Gamma_2^2}\right.$$
$$\left.+\frac{\gamma_0\varkappa\left(\Gamma_1+\Gamma_2\right)^2}{\left[4\left(\nu-\omega_1\right)^2+\Gamma_1^2\right]\left[4\left(\nu-\omega_2\right)^2+\Gamma_2^2\right]}\right\},$$

$$\Gamma_1:=\gamma_0+\gamma_1+2\gamma_0\varkappa, \qquad \Gamma_2:=\gamma_0+\gamma_2,$$
$$D=\frac{2/\pi}{1+\varkappa\gamma_1/\gamma_2+\varkappa(1+\varkappa)(1+\gamma_0/\gamma_2)}.$$

A structured spectrum is a signature of the non-Markovian dynamics.

Coloured noises & feedback The linear stochastic Schrödinger equation (lSSE). For simplicity, only diffusive contributions

$$\mathrm{d}\psi(t) = K(t)\psi(t)\mathrm{d}t + \sum_{j=1}^{d} R_j(t)\psi(t)\mathrm{d}W_j(t), \qquad K(t) = -\mathrm{i}H(t) - \frac{1}{2}\sum_{j=1}^{d} R_j(t)^* R_j(t),$$

$\psi(0) = \psi_0 \in \mathcal{H}$ (Hilbert space of the system), $\qquad \|\psi_0\| = 1$

$H(t)$, $L_l(t)$, $R_j(t)$ are random bounded operators with $H(t) = H(t)^*$, say predictable càglàd processes in $\big(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{Q}\big)$; $\quad \forall T > 0$,

$$\sup_{\omega\in\Omega}\sup_{t\in[0,T]} \left\| \sum_{j=1}^{d} R_j^*(t,\omega) R_j(t,\omega) \right\| \leq L(T) < \infty, \qquad \sup_{\omega\in\Omega}\sup_{t\in[0,T]} \|H(t,\omega)\| \leq M(T) < \infty$$

The key property: $\|\psi(t)\|^2$ is a mean-one martingale. This allows for a change of probability and a Girsanov transformation; under the new probability $\psi(t)/\|\psi(t)\|$ satisfies the "non-linear SSE".

Call $A(t,s)$ the fundamental solution of the lSSE (= the propagator): $A(t,s)\psi(s) = \psi(t)$

The linear stochastic master equation (LSME).

$\sigma(t) = A(t,0)\varrho_0 A(t,0)^*$ Initial condition: $\sigma(0) = \varrho_0 \in \mathcal{S}(\mathcal{H})$

$\mathrm{d}\sigma(t) = \mathcal{L}(t)[\sigma(t)]\mathrm{d}t + \sum_{j=1}^{d} \mathcal{R}_j(t)[\sigma(t)]\mathrm{d}W_j(t),$ $\mathcal{R}_j(t)[\rho] := R_j(t)\rho + \rho R_j(t)^*$ random

The random Liouville operator:

$$\mathcal{L}(t)[\rho] = -\mathrm{i}[H(t), \rho] + \sum_{j=1}^{d} \left(R_j(t)\rho R_j(t)^* - \frac{1}{2}\left\{R_j(t)^* R_j(t), \rho\right\} \right)$$

Key properties.

$\mathrm{Tr}\,\sigma(t)$ is a mean-one martingale.

The propagator of the lSME $\mathcal{A}(t,s)[\rho] = A(t,s)\rho A(t,s)^*$ is CP

Instruments, POV measures, probabilities.

The output is the set of the first m $(1 \leq m \leq d)$ components of W, or, better, the set of its increments. The space of the observed events from s to t: the σ-algebra generated by the increments and the null sets

$\mathcal{G}_t^s = \sigma\{W_j(u) - W_j(s),\, u \in [s,t],\, j = 1, \ldots, m\} \vee \mathcal{N} \qquad \mathcal{N} = \{A \in \mathcal{F} : \mathbb{Q}(A) = 0\}$

$\mathcal{A}(t,s)$: propagator (the fundamental solution of the LSME) $\qquad \sigma(t) = \mathcal{A}(t,s)[\sigma(s)]$

$$\mathcal{I}_t(G)[\rho] := \mathbb{E}_{\mathbb{Q}}\left[1_G \mathcal{A}(t,0)[\rho]\right], \qquad G \in \mathcal{G}_t^0$$

This equation defines an instrument $\mathcal{I}_t$ on the value space $(\Omega, \mathcal{G}_t^0)$. $\mathcal{I}_t(\bullet)^*[\mathbb{1}]$ is a Positive Operator Valued measure on $(\Omega, \mathcal{G}_t^0)$.

For $t > s > r \geq 0$, we have time ordering, but not a simple composition law:
$\forall G_1 \in \mathcal{G}_s^r, \quad \forall G_2 \in \mathcal{G}_t^s$

$$\mathcal{I}_t(G_1 \cap G_2) = \mathbb{E}_{\mathbb{Q}}\Big[\mathbb{E}_{\mathbb{Q}}\big[\mathbb{E}_{\mathbb{Q}}\left[1_{G_2}\mathcal{A}(t,s)|\mathcal{F}_s\right]1_{G_1}\mathcal{A}(s,r)\big|\mathcal{F}_r\big]\mathcal{A}(r,0)\Big]$$

Probability of observing the result $G \in \mathcal{G}_t^0$ in the time interval $[0,t]$:
$\mathbb{P}_t[G] = \mathrm{Tr}\{\mathcal{I}_t(G)^*[\mathbb{1}]\varrho_0\} = \mathrm{Tr}\{\mathcal{I}_t(G)[\varrho_0]\} = \mathbb{E}_{\mathbb{Q}}[1_G \, \mathrm{Tr}\{\sigma(t)\}]$

$\tilde{\sigma}(t) = \mathbb{E}\left[\sigma(t)\middle|\mathcal{G}_t^0\right]$ does not satisfy a closed SDE

$p_t := \mathrm{Tr}\{\tilde{\sigma}(t)\}$, $t \geq 0$, is a mean one martingale (from the linear SME)
$\mathbb{P}_t(\mathrm{d}\omega) = p_t(\omega)\mathbb{Q}(\mathrm{d}\omega)\Big|_{\mathcal{G}_t^0}$ is the physical probability on $\left(\Omega, \mathcal{G}_t^0\right)$.
Consistency property: $\quad G \in \mathcal{G}_t^0\,,\ t \leq T \Rightarrow \mathbb{P}_T[G] = \mathbb{P}_t[G]$.

A posteriori states: $\rho(t) := \dfrac{1}{p_t}\,\tilde{\sigma}(t) \qquad \mathcal{I}_t(G)[\varrho_0] = \displaystyle\int_G \rho(t;\omega)\mathbb{P}_t(\mathrm{d}\omega), \qquad G \in \mathcal{G}_t^0$

A priori states: $\eta(t) := \mathbb{E}_{\mathbb{Q}}[\sigma(t)] = \mathbb{E}_{\mathbb{P}_t}[\rho(t)]$

No closed equation for a posteriori and a priori states. For instance,
$\dfrac{\mathrm{d}\eta(t)}{\mathrm{d}t} = \mathbb{E}_{\mathbb{Q}}\left[\mathcal{L}(t)[\sigma(t)]\right], \quad$ but both $\mathcal{L}(t)$ and $\sigma(t)$ are random

Better: at least heuristically some kind of generalised master equations can be obtained by using the Nakajima-Zwanzig projection technique

Output

$$v_j(t) := \frac{\mathrm{Tr}\,\{\mathcal{R}_j(t)[\sigma(t)]\}}{\mathrm{Tr}\,\{\sigma(t)\}} = 2\,\mathrm{Re}\,\frac{\mathrm{Tr}\,\{R_j(t)\sigma(t)\}}{\mathrm{Tr}\,\{\sigma(t)\}} \qquad B_j(t) := W_j(t) - \int_0^t v_j(s)\mathrm{d}s$$

By Girsanov theorem, it is possible to prove that under the physical probability the output $B(t)$ is a Wiener process, so that we have $\underbrace{\mathrm{d}W_j(t)}_{\text{output}} = \underbrace{\mathrm{d}B_j(t)}_{\text{noise}} + \underbrace{v_j(t)\mathrm{d}t}_{\text{signal}}$

In general noise and signal are correlated. Explicit expression for the moments of the output have been obtained by using characteristic functionals.

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\mathbb{E}_{\mathbb{P}_T}[W_j(t)] = \mathbb{E}_{\mathbb{P}_t}[v_j(t)] = \mathbb{E}_{\mathbb{Q}}\left[\,\mathrm{Tr}\,\{\mathcal{R}_j(t)[\sigma(t)]\}\,\right]$$

$$\frac{\mathrm{d}^2}{\mathrm{d}t\mathrm{d}s}\,\mathbb{E}_{\mathbb{P}_T}[W_j(t)W_i(s)] = \delta_{ij}\delta(t-s) + z_{ji}(t,s) + z_{ij}(s,t)$$

$$z_{ji}(t,s) = 1_{(0,+\infty)}(t-s)\,\mathbb{E}_{\mathbb{Q}}\left[\,\mathrm{Tr}\,\{\mathcal{R}_j(t)\circ\mathcal{A}(t,s)\circ\mathcal{R}_i(s)[\sigma(s)]\}\,\right]$$

where $\mathcal{A}(t,s)$ is the propagator associated with the lSME (the SDE for $\sigma(t)$)

The heterodyne spectrum of a single mode damped cavity driven by a coloured noisy coherent field. The lSSE

$$\mathrm{d}\psi_t = G\psi_t \mathrm{d}t - \mathrm{i}L\psi_t \mathrm{d}X(t) + R_0(t)\psi_t \mathrm{d}W_0(t) + R_1\psi_t \mathrm{d}W_1(t),$$

Emission of the light and heterodyne detection: $R_0(t) = \overline{\alpha_0}\, \mathrm{e}^{\mathrm{i}\nu t}\, a$, $\alpha_0 \in \mathbb{C}$, $\nu > 0$

Losses and incoherent dissipation: $R_1 = \overline{\alpha_1}\, a$, $\alpha_1 \in \mathbb{C}$

$H = \omega_0 a^\dagger a, \quad \omega_0 > 0,$ cavity mode frequency

$L = \overline{\beta} a + \beta a^\dagger$, $\beta \in \mathbb{C}$: $-\mathrm{i}L\psi_t \mathrm{d}X(t)$ noisy input field, X O-U process

$G = -\left(\mathrm{i}\omega_0 + \frac{\Gamma}{2}\right) a^\dagger a - \frac{1}{2}\, L^2, \qquad \Gamma = |\alpha_0|^2 + |\alpha_1|^2 .$

As before we compute the heterodyne power spectrum. Final result

$$P_{\mathrm{het}}(\nu)\Big|_{\kappa=0} = 1 + \frac{2\,|\alpha_0|^2\,|\beta|^2\,\nu^2}{\left[(\nu-\omega_0)^2 + \frac{\Gamma^2}{4}\right]\left[\nu^2 + \frac{\gamma^2}{4}\right]}$$

The spectrum is the product of a Lorentzian term due to the intrinsic dynamics and of the spectrum of the noise

Continuous measurements in the non-Markov case

- A. Barchielli, A. S. Holevo, *Constructing quantum measurement processes via classical stochastic calculus*, Stoch. Proc. Appl. **58** (1995) 293–317.
- A. Barchielli, C. Pellegrini, F. Petruccione, *Stochastic Schrödinger equations with coloured noise*, EPL **91** (2010) 24001; DOI: 10.1209/0295-5075/91/24001; arXiv:0911.2554v1 [quant-ph].
- A. Barchielli, P. Di Tella, C. Pellegrini, F. Petruccione, *Stochastic Schrödinger equations and memory*, to appear in QP-PQ: Quantum Probability and White Noise Analysis (ISSN: 1793-5121) Vol. 27; arXiv:1006.3647v1 [quant-ph].
- A. Barchielli, C. Pellegrini, *Jump-diffusion unravelling of a non Markovian generalized Lindblad master equation*, J. Math. Phys. 51 (2010) 112104; arXiv:1006.4527v1 [quant-ph].
- P. Di Tella, *Quantum trajectories: Memory and feedback*, Master thesis (2009), http://www1.mate.polimi.it/biblioteca/tesiview.php?id=302& L=i

Lindblad rate equations:

- A.A. Budini, *Lindblad rate equations*, Phys. Rev. A **74** (2006) 053815.
- H.-P. Breuer, *Non-Markovian generalization of the Lindblad theory of open quantum systems*, Phys. Rev. A **75** (2007) 022103.
- M. Moodley, F. Petruccione, *Stochastic wave-function unraveling of the generalized Lindblad master equation*, Phys. Rev. A **79** (2009) 042103.

Continuous measurements in the Markovian case:

- A. Barchielli, V. P. Belavkin, *Measurements continuous in time and a posteriori states in quantum mechanics*, J. Phys. A: Math. Gen. **24** (1991) 1495–1514; arXiv:quant-ph/0512189v1.
- A. Barchielli, *Continual Measurements in Quantum Mechanics and Quantum Stochastic Calculus*. In S. Attal, A. Joye, C.-A. Pillet (eds.), *Open Quantum Systems III*, Lect. Notes Math. **1882** (Springer, Berlin, 2006), pp. 207–291.
- A. Barchielli, M. Gregoratti, *Quantum Trajectories and Measurements in Continuous Time: The Diffusive Case*, Lect. Notes Phys. **782** (Springer, Berlin, 2009).