

DIAGRAMMATICS

JAN DEREZIŃSKI

Dep. of Math. Meth. in Phys.

Faculty of Physics

University of Warsaw

Based on joint work with
Christian Gérard

The diagrammatic method is one of the most powerful and beautiful tools of theoretical physics. It allows us to organize efficiently perturbative computations in **statistical physics**, **quantum many-body theory** and **quantum field theory**. It often plays a rather fundamental role, especially when the non-perturbative theory is unknown. Its formal structure is quite interesting and complex, and involves some nontrivial combinatorial facts.

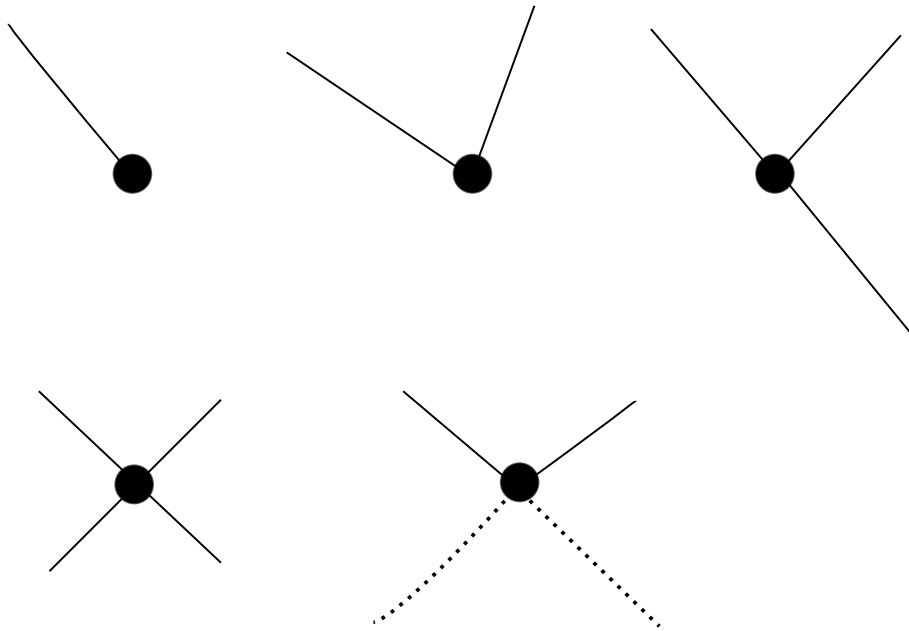
Diagrams seem to be often despised by rigorous people. In reality, diagrams are rigorous themselves. Of course, they are inherently perturbative. But many important physical theories seem to exist only in the perturbative sense. Diagrams can then be used to **define** such theories. The diagrammatic method is also useful in suggesting the right physical quantities to look for and compute.

There exist several kinds of diagrams. We will try to present them in a systematic way. This corresponds to a division of our presentation in 3 sections:

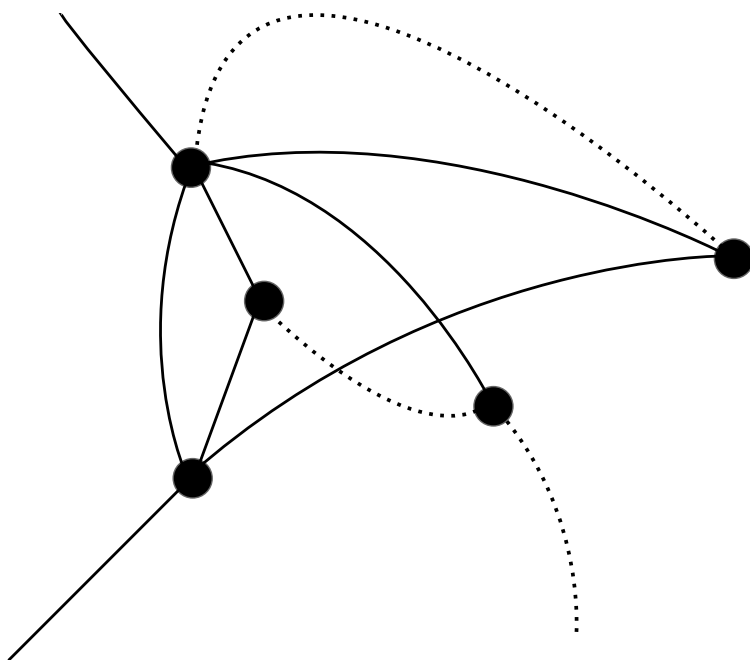
1. **Diagrams and gaussian integration;**
2. **Friedrichs diagrams and the scattering operator;**
3. **Feynman diagrams and the scattering operator.**

In Sect. 1 we present a diagrammatic formalism whose goal is to organize integration with respect to a **gaussian measure**. This formalism is used extensively in **classical statistical physics**. It also plays an important role in quantum physics, especially in the **euclidean approach**, since many quantum quantities can be expressed in terms of gaussian integrals over classical variables.

We use the term “gaussian integration” in a rather broad sense. Beside commuting, “**bosonic**” variables, we also consider anticommuting “**fermionic**” variables, where we use the **Berezin integral** with respect to a gaussian weight.



Various vertices

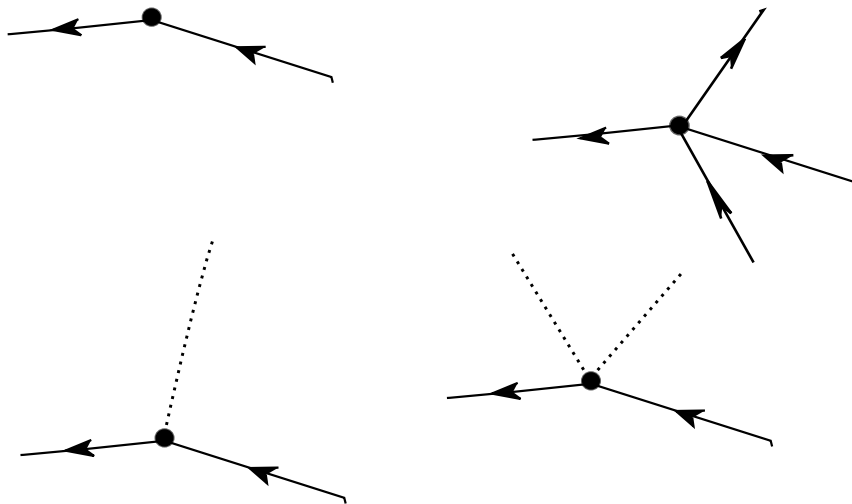


Connected diagram without self-lines

Even in the case of commuting variables, the “gaussian integral” is not necessarily meant in the sense of measure theory. It denotes an algebraic operation performed on polynomials (or formal power series), which in the case of a positive definite covariance coincides with the usual integral with a gaussian weight. One can allow, however, the covariance to be complex, or even negative definite. There is no need to insist that the operation has a measure theoretic meaning.

One can distinguish two kinds of vector spaces, on which one perform gaussian integrals: **real and complex**. Of course, the difference between the real (ie. **neutral**), and the complex (ie. **charged**) formalism is mainly that of a different notation. In particular, charged lines are equipped with an **arrow**, whereas neutral lines need not.

The terminology used in diagrams is inspired by quantum field theory. Therefore, the variables that enter the integral are associated to “particles”, they are divided into “bosons” and “fermions”, each subdivided into “neutral” and “charged” particles.



Various vertices involving charged particles

In Sections 2 and 3 we describe diagrams in the framework of **2nd quantization**. They are used in **many-body quantum theory** and **quantum field theory**. The main aim is the computation of the **scattering operator** and the **energy shift of the ground state**.

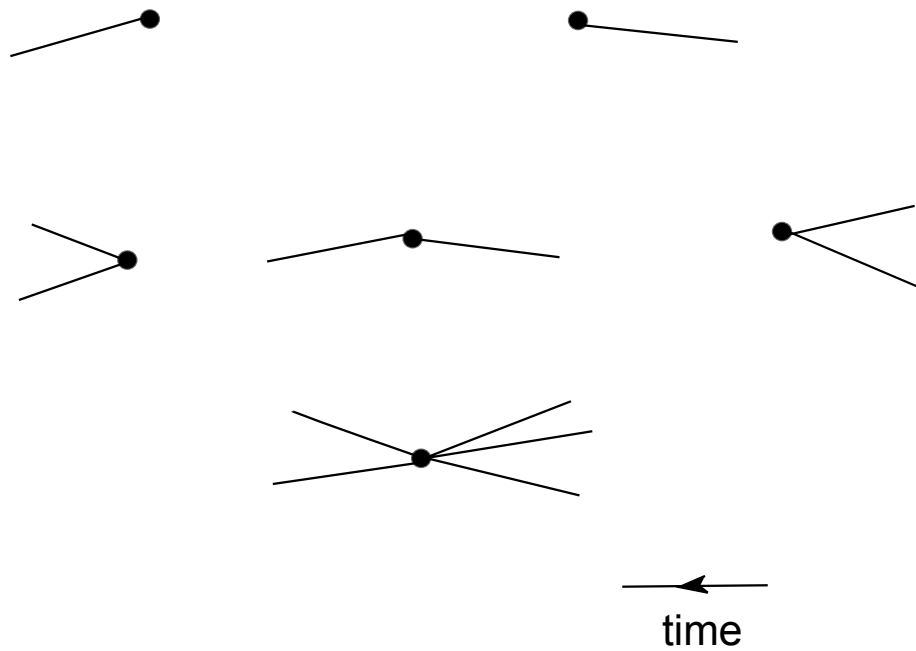
There are two natural diagrammatic formalisms in this context. Historically the first, and the most widely used, is that of **Feynman diagrams**. We start however with the less widely used, but easier to explain, the formalism of **Friedrichs diagrams**.

Friedrichs diagrams appear naturally when we want to compute the **Wick symbol** of a product of **Wick ordered** operators. An algorithm for its computation is usually called the **Wick theorem**.

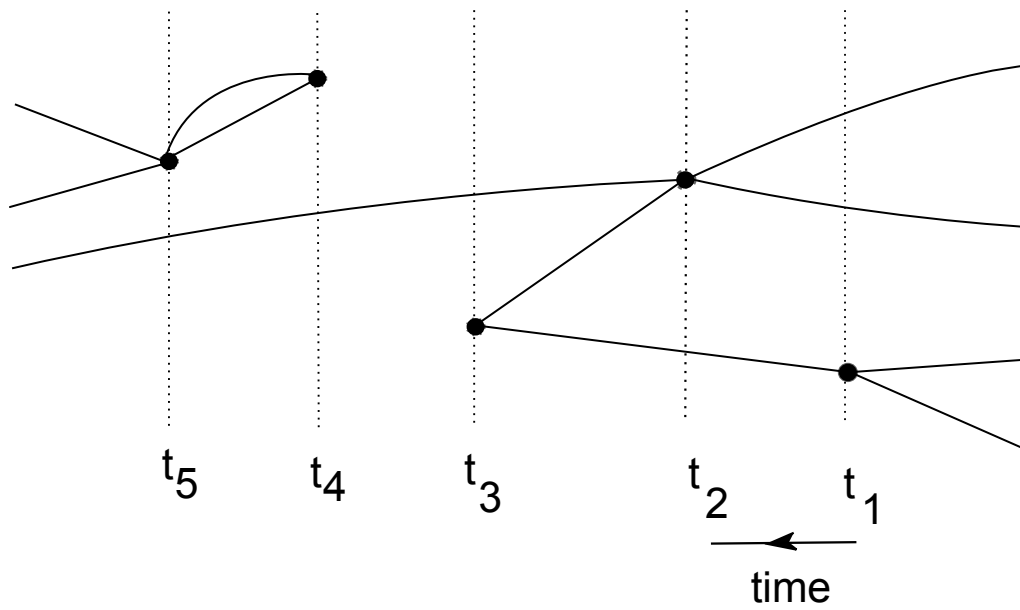
In the formalism of Friedrichs diagrams, a vertex represents a Wick monomial. It has two kinds of legs: **incoming legs** representing annihilation and **outgoing legs** representing creation operators. We draw the former on the right of a vertex and the latter on the left.

A typical Hamiltonian in many-body quantum physics and in quantum field theory can be written as the sum of a **quadratic term** of the form $d\Gamma(h)$ for some 1-particle Hamiltonian h and an **interaction** given by a Wick polynomial. One can use Friedrichs diagrams to compute the **scattering operator** for such Hamiltonians, as we describe in Sect. 2. A characteristic feature of this formalism is the presence of time labels on all vertices and the fact that diagrams with different time orderings are considered distinct.

Another application of Friedrichs diagrams is the formula for the **energy shift of the ground state**, which is attributed to **Goldstone**. In its derivation, it is convenient to use the **Sucher formula** involving **adiabatic scattering operators** and the so-called **Gell-Mann and Low Theorem**.



Various Friedrichs vertices



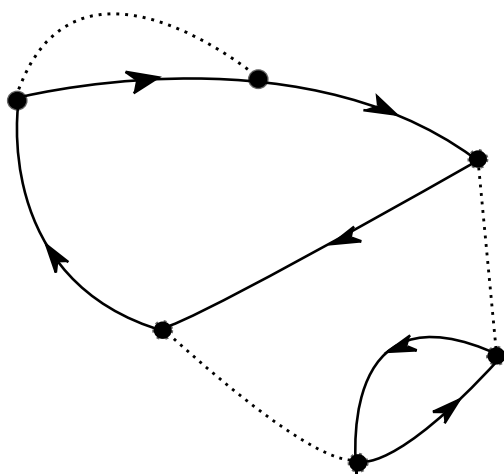
Friedrichs diagram

Naively, the formalism of Friedrichs diagrams seems very natural and physically intuitive. In fact, Friedrichs diagrams would provide natural illustration for typical computations of the early years of quantum field theory (even though apparently diagrams were not used in that time). Weinberg calls a formalism essentially equivalent to that of Friedrichs diagrams the **old-fashioned perturbation theory**.

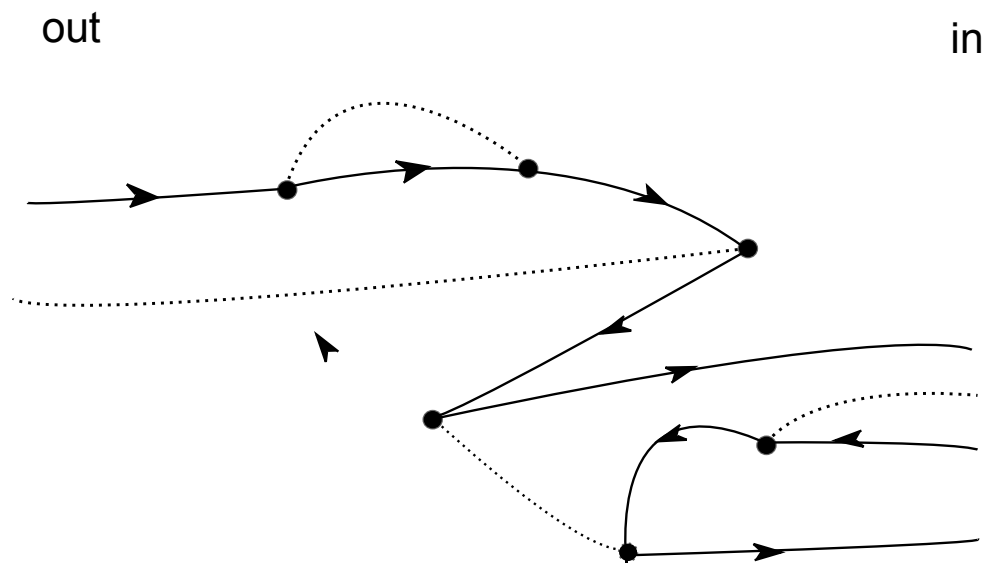
Since the late 40's, a different diagram formalism has been developed and since then it dominates the calculations of quantum field theory. It originated in the work of Stueckelberg, and, especially, of Feynman, and therefore is called the formalism of **Feynman diagrams**. Again, the main goal is to compute the scattering operator for a Hamiltonian of the form $d\Gamma(h)$ perturbed by a quantization of a Wick polynomial.

Feynman diagrams used to compute the vacuum expectation value of the scattering operator can be essentially interpreted as a special case of the diagrams described in Sect. 1 used to compute gaussian integrals. In this formalism, the order of times associated to individual vertices does not play any role. This allows us to cut down on the number of diagrams, as compared with Friedrichs diagrams. In relativistic theories, each Feynman diagram is manifestly covariant, which is not the case for Friedrichs diagrams. Therefore, Feynman diagrams are usually preferred for practical computations over Friedrichs diagrams.

Feynman diagrams used to compute the Wick symbol of a scattering operator have in addition external lines. These lines are either incoming or outgoing. The former are then paired with creation and the latter with annihilation operators. Again, the temporal order of vertices is not relevant.



Feynman diagram without external lines



Scattering type Feynman diagram

Even though Feynman diagrams dominate, especially in relativistic computations, Friedrichs diagrams are also useful in some situations. In particular, they can be used to compute **leading singularities** of certain terms of the perturbation expansion.

The main idea of passing from Friedrichs diagrams to Feynman diagrams consists in **combining the evolution going forwards and backwards in time in a single line**. It is done in a different way for neutral and charged particles. In the case of neutral particles, the lines with both time directions are combined into one unoriented line. In the case of charged lines, one combines particles going forwards and antiparticles going backwards in a single line decorated with an arrow pointing forwards. Similarly, one combines particles going backwards and antiparticles going forwards in a single line oriented backwards.

0.1 Gaussian integrals and diagrams

Let $\mathcal{V}$ be a **real vector space**. We will often represent an element ζ of $\mathcal{V}$ as a function on Ξ , that is $\Xi \ni \xi \mapsto \zeta(\xi) \in \mathbb{R}$. It is convenient to introduce functionals on $\mathcal{V}$ by

$$\langle \phi(\xi) | \zeta \rangle := \zeta(\xi).$$

Clearly, a **polynomial** on $\mathcal{V}$ can be written as

$$\sum_n \int \cdots \int v_n(\xi_n, \dots, \xi_1) \phi(\xi_n) \cdots \phi(\xi_1) d\xi_1 \cdots d\xi_n, \quad (0.1)$$

where v_n are symmetric functions on Ξ^n . Clearly, $v(0) = v_0$.

Sometimes we will consider **infinite polynomials**, that is formal infinite sums of n th degree homogeneous polynomials.

Polynomials can be **differentiated**:

$$\begin{aligned} & \nabla_{\phi(\xi)} \int \cdots \int v_n(\xi_n, \dots, \xi_1) \phi(\xi_n) \cdots \phi(\xi_1) d\xi_1 \cdots d\xi_n \\ &= n \int \cdots \int v_n(\xi, \xi_{n-1}, \dots, \xi_1) \phi(\xi_{n-1}) \cdots \phi(\xi_1) d\xi_1 \cdots d\xi_{n-1}. \end{aligned}$$

Note that

$$\nabla_{\phi(\xi)} \nabla_{\phi(\xi')} = \nabla_{\phi(\xi')} \nabla_{\phi(\xi)}.$$

We often smear the derivative with a $\zeta \in \mathcal{V}$. The **derivative in the direction of ζ** equals

$$\zeta \nabla_{\phi} = \int \zeta(\xi) \nabla_{\phi(\xi)} d\xi.$$

We fix σ , a **symmetric form** on the dual of $\mathcal{V}$, whose integral kernel is the symmetric function $\sigma(\cdot, \cdot)$. The form σ can be complex valued. We will need the following differential operator

$$\nabla_{\phi} \sigma \nabla_{\phi} = \int \int \nabla_{\phi(\xi)} \sigma(\xi, \xi') \nabla_{\phi(\xi')} d\xi d\xi',$$

If $v(\phi)$ is a polynomial, its **integral with covariance σ** is defined as

$$\int v(\phi) := \exp \left(\frac{1}{2} \nabla_{\phi} \sigma \nabla_{\phi} \right) v(0).$$

Note that if $\mathcal{V} = \mathbb{R}^d$ and $\operatorname{Re} \sigma$ is strictly positive, then $\int v(\phi)$ can be interpreted as the usual **integral with the probabilistic gaussian weight**. In fact,

$$\int v(\phi) = (2\pi)^{-d/2} (\det \sigma)^{-1/2} \int e^{-\frac{1}{2} \phi \sigma^{-1} \phi} v(\phi) d\phi, \quad (0.2)$$

where $d\phi$ is the usual Lebesgue measure on $\mathbb{R}^d$ and the integral has the usual meaning.

If $\mathcal{V}$ is infinite dimensional and $\operatorname{Re} \sigma$ is strictly positive, then (0.2) can still be interpreted in the sense of measure theory, although one needs a more sophisticated formalism.

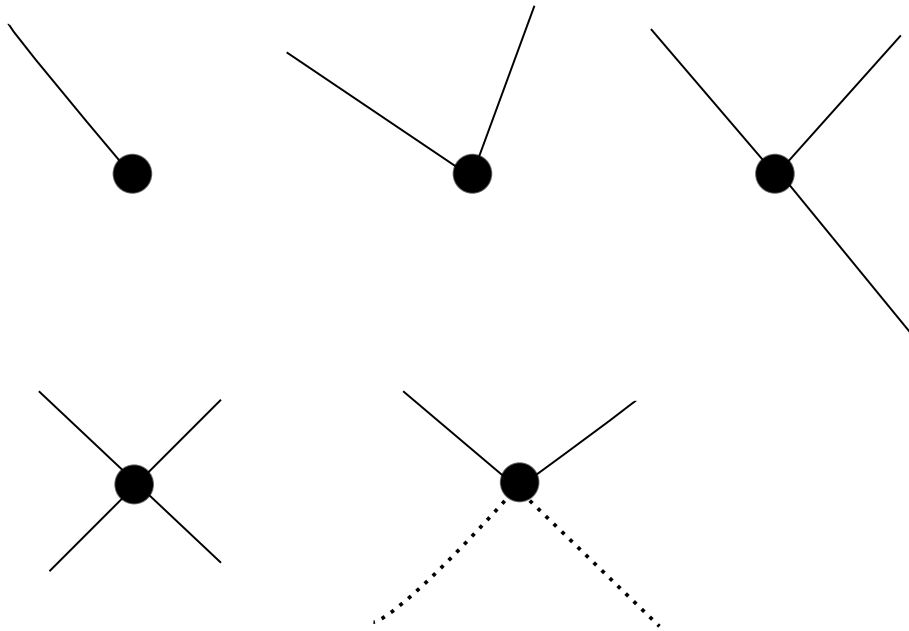
With every **homogeneous polynomial** of degree m we associate a **vertex**, that is a graphical symbol consisting of the **proper vertex**, usually a dot, and m **legs** (**prongs**).

To specify a **diagram** we need

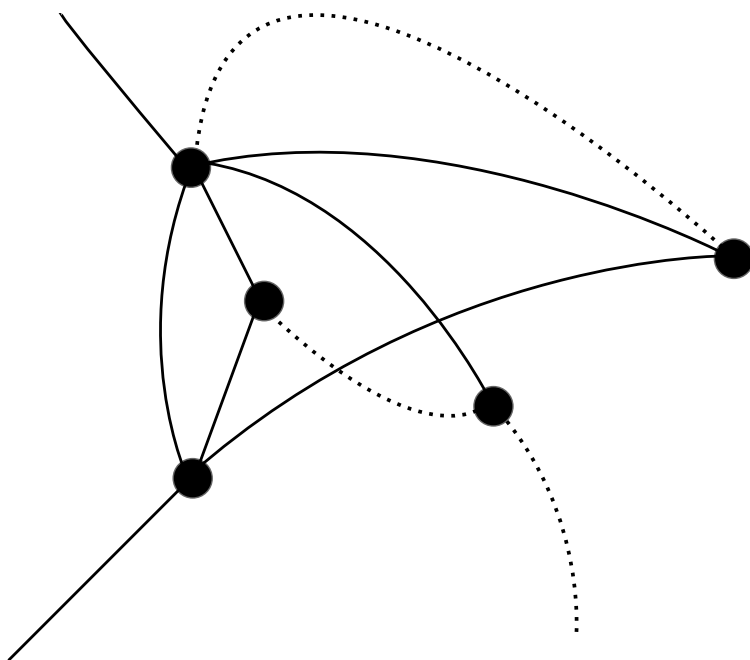
1. a sequence of vertices (the order in the sequence does not play a role);
2. a family of internal lines.

An **internal line** is a pair of legs, each belonging to one of the vertices, joined with a line. Each leg belongs to at most one internal line.

All legs that are not a part of an internal line are called **external lines**. Lines joining two legs of the same vertex are called **self-lines**.



Various vertices



Connected diagram without self-lines

Let δ be a diagram with vertices corresponding to $v_n(\phi), \dots, v_1(\phi)$. First assume it has no external legs. Its **evaluation** is the number

$$z_\delta := \int \dots \int \prod_{j=1}^n v_j(\xi_{j,m_j}, \dots, \xi_{j,1}) \prod_{l \in \text{int. lines}} \sigma(\xi_{1,l}, \xi_{2,l}) d\xi_{1,l} d\xi_{2,l}.$$

If δ is an arbitrary diagram, its evaluation is the monomial

$$z_\delta(\phi) := \int \dots \int \prod_{j=1}^n v_j(\xi_{j,m_j}, \dots, \xi_{j,1}) \prod_{l \in \text{int. lines}} \sigma(\xi_{1,l}, \xi_{2,l}) d\xi_{1,l} d\xi_{2,l} \prod_{k \in \text{ext. lines}} \phi(\xi_k) d\xi_k.$$

Diagrams can be **multiplied** – they are just juxtaposed. We can also consider formal **sums** of diagrams. Obviously, the evaluation of the product of two diagrams is the product of their evaluations. Likewise, the evaluation of the sum of diagrams is the sum of evaluations.

Later on we will need the **exponential** of a diagram, defined as the obvious power series.

If $v(\phi)$ is a polynomial, define

$$:v(\phi): := e^{\frac{1}{2}\nabla_{\phi}\sigma\nabla_{\phi}}v(\phi).$$

We will call $:v(\phi):$ the **Wick transformation of $v(\phi)$** . Note that $\int :v(\phi): = v(0)$.

The Wick transformation is a bijection. If $g(\phi)$ is a polynomial, we can write it as

$$g(\phi) = :v(\phi):.$$

We will say that $g(\phi)$ is **represented in the Wick form**.

If σ is strictly positive, then the Wick transformation can be interpreted in terms of the Gram-Schmidt orthogonalization. In fact, let P_n be the orthogonal projection onto the space of at most n th degree polynomials in $L^2(\mathcal{V}, e^{-\phi\sigma^{-1}\phi})$. If $v(\phi)$ is homogeneous of degree n , we have

$$:v(\phi): = v(\phi) - P_{n-1}v(\phi).$$

In particular, in the case of a single variable, $:\phi^n:$ is the **n th Hermite polynomial**.

Let $v_n(\phi), \dots, v_1(\phi)$ be homogeneous polynomials. To each of them we associate a vertex denoted by the same symbol. Diagrams with vertices $v_n, \dots, v_1$ can be used to compute various quantities.

First of all, we have the identity

$$\int v_n(\phi) \cdots v_1(\phi) = \sum_{\delta} z_{\delta}$$

where we sum over **all diagrams with no external lines**. More generally,

$$v_n(\phi) \cdots v_1(\phi) = : \sum_{\delta} z_{\delta}(\phi) :. \quad (0.3)$$

where we sum over **all diagrams**.

In practice, one often prefers polynomials in the Wick form. We have then the identity

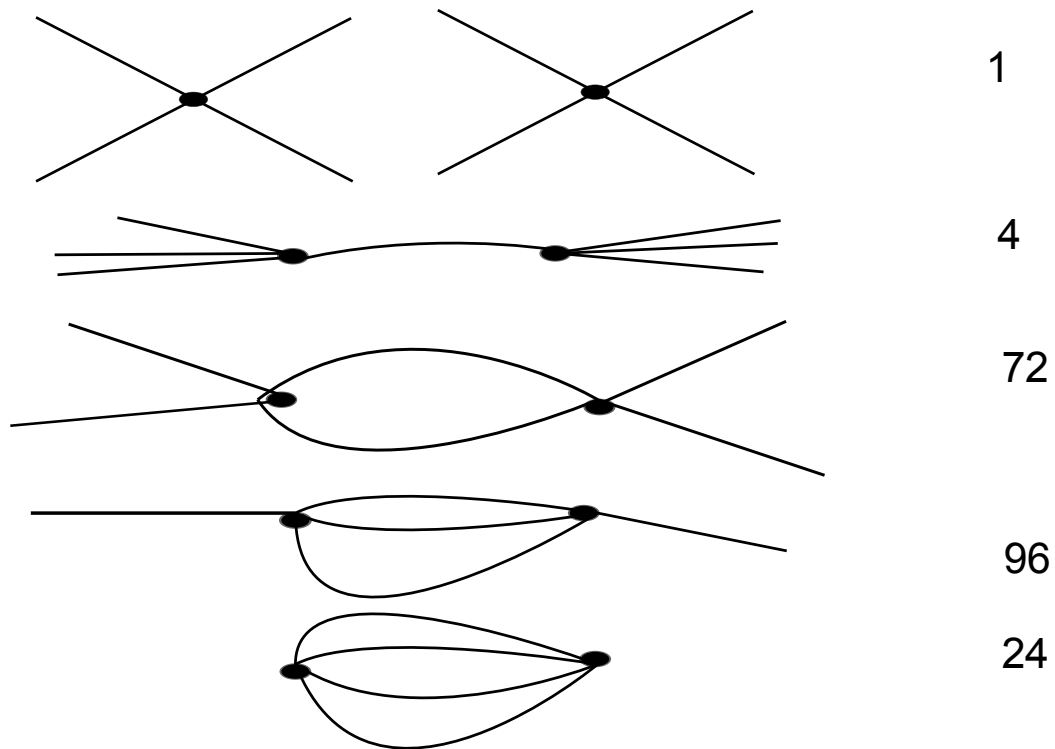
$$\int :v_n(\phi):\cdots:v_1(\phi): = \sum_{\delta} z_{\delta}$$

where we sum over **all diagrams without external legs and without self-lines**. More generally,

$$:v_n(\phi):\cdots:v_1(\phi): = : \sum_{\delta} z_{\delta}(\phi) :. \quad (0.4)$$

where we sum over **all diagrams without self-lines**.

From now on, we will consider only diagrams without self-lines.



$$: \phi^4 : : \phi^4 : = : \phi^8 : + 4 : \phi^6 : + 72 : \phi^4 : + 96 : \phi^2 : + 24, \quad (2\pi)^{-1/2} \int : \phi^4 : : \phi^4 : d\phi = 24.$$

Suppose that we have a (classical) Hamiltonian given by

$$H = \frac{1}{2} \phi \sigma^{-1} \phi - :v_n(\phi): - \cdots - :v_1(\phi):, \quad (0.5)$$

where $v_i(\phi)$, $i = 1, \dots, n$, are homogeneous polynomials. We would like to compute various quantities motivated by statistical physics related to this Hamiltonian. To this end we will need diagrams with vertices of type $v_n(\phi), \dots, v_1(\phi)$, with possible repetitions. $\int$ will denote the integral with covariance σ .

The **partition function** is given by the formula

$$\int e^{-H} d\phi = \int \exp(:v_n(\phi): + :v_1(\phi):) = \sum_{\delta} z_{\delta},$$

where we sum over **all diagrams with with no external lines**.

A combinatorial miracle happens if we consider the logarithm of the partition function.

$$\log \int \exp(:V_n: + :V_1:) = \sum_{\delta} z_{\delta}, \quad (0.6)$$

where we sum over **all connected diagrams without external lines**. This is a part of the famous **Linked Cluster Theorem**.

Suppose we are interested in the Gibbs state $e^{-H} / \int e^{-H} d\phi$. We would like to represent it in the Wick form. This can be also expressed in terms of diagrams.

In fact, first note that

$$\exp(:v_n(\phi): + :v_1(\phi):) = \sum_{\delta} :z_{\delta}(\phi):, \quad (0.7)$$

where we sum over **all diagrams, including those with external legs**.

This can be simplified.

$$\exp(:v_n(\phi): + :v_1(\phi):) = : \left(\exp \sum_{\delta} z_{\delta}(\phi) \right) :, \quad (0.8)$$

where we sum over **all connected diagrams**. This is one of versions of the **Linked Cluster Theorem**.

A **linked diagram** is a diagram whose all connected parts have at least one external line. Another version of the Linked Cluster Theorem says that

$$\frac{\exp(:v_n(\phi): + :v_1(\phi):)}{\int \exp(:v_n(\phi): + :v_1(\phi):)} = : \sum_{\delta} z_{\delta}(\phi) :, \quad (0.9)$$

where we sum over **all linked diagrams**.

An element of the dual of $\mathcal{V}$ can be written as

$$\phi(f) = \int f(\xi) \phi(\xi) d\xi.$$

The **generating function** is defined as

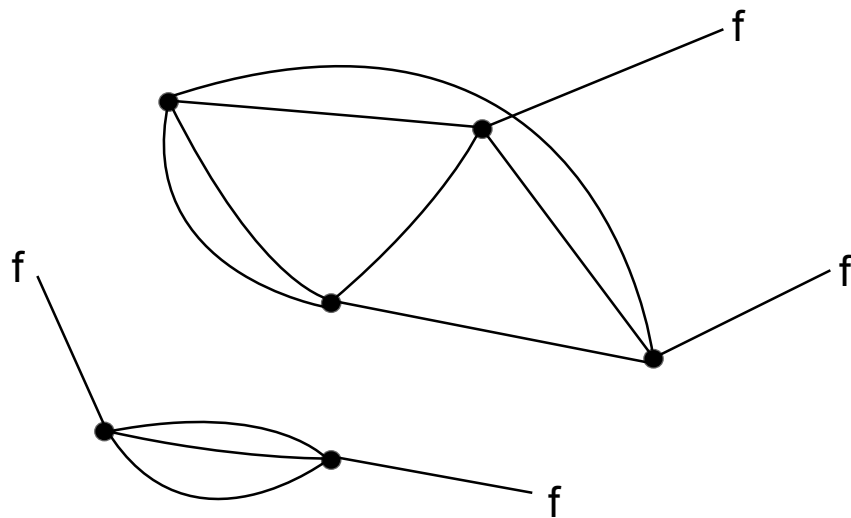
$$Z(f) := \frac{\int \exp(:v_n(\phi): + :v_1(\phi): + \phi(f))}{\int \exp(:v_n(\phi): + :v_1(\phi):)}$$

To obtain diagrams for the generating function one uses diagrams with vertices $v_n(\phi), \dots, v_1(\phi)$ and an **additional 1-legged vertex for the insertion f** . (Vertices can be repeated).

We have

$$Z(f) = \sum z_\delta,$$

where we sum over **all diagrams without external lines with at least one insertion vertex**.



A diagram for the generating function of the $:\phi^4:$ theory

From the generating function we can compute N -point correlation functions

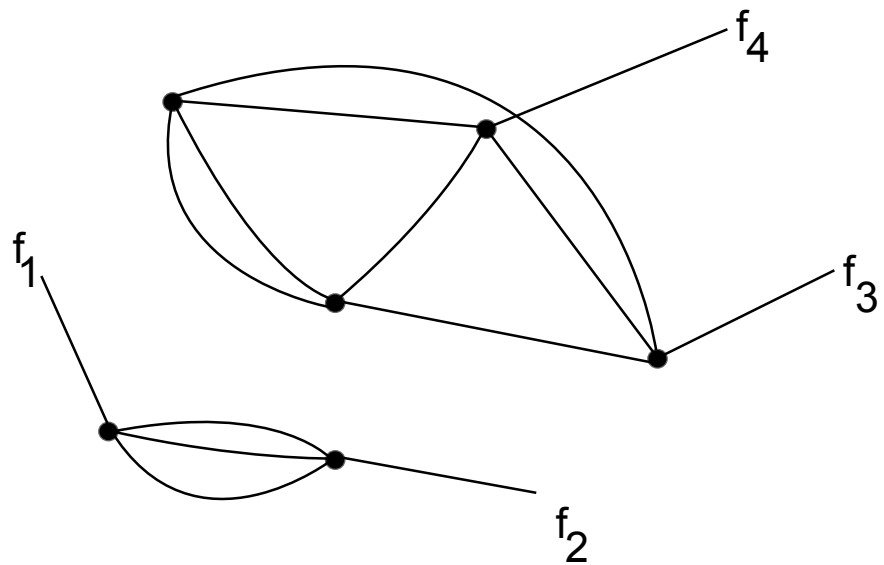
$$\begin{aligned} G(f_N, \dots, f_1) &= f_N \nabla_f \cdots f_1 \nabla_f Z(f) \Big|_{f=0} \\ &= \frac{\int \exp(:v_n(\phi): + :v_1(\phi):) \phi(f_N) \cdots \phi(f_1)}{\int \exp(:v_n(\phi): + :v_1(\phi):)} \end{aligned}$$

We have

$$G(f_N, \dots, f_1) = \sum z_\delta,$$

where we sum over all diagrams without external lines with precisely one insertion vertex for each f_j .

(Usually, the lines ending with f_j are called external lines).



A diagram for the 4-point function of the $:\phi^4:$ theory

The formalism that we described above is used in the case of **neutral particles**. In the case of **charged particles**, real variables are replaced with complex variables. Let us describe the complex (charged) formalism.

Consider a complex vector space $\mathcal{W}$. As usual, we will assume that elements of $\mathcal{W}$ can be represented as a functions on a certain set Ξ , that is $\Xi \ni \xi \mapsto \zeta(\xi) \in \mathbb{C}$.

We will use the complex linear functional on $\mathcal{W}$ denoted by $\psi(\xi)$ given by

$$\langle \psi(\xi) | \zeta \rangle := \zeta(\xi).$$

We also have its complex conjugation

$$\langle \overline{\psi}(\xi) | \zeta \rangle := \overline{\zeta(\xi)}.$$

Polynomials on $\mathcal{W}$ can be expressed in ψ and $\bar{\psi}$:

$$w(\bar{\psi}, \psi) = \sum_{p,q} \int \cdots \int w_{p,q}(\xi'_1, \dots, \xi'_p, \xi_q, \dots, \xi_1) \\ \bar{\psi}(\xi'_1) \cdots \bar{\psi}(\xi'_p) \psi(\xi_q) \cdots \psi(\xi_1) d\xi'_1 \cdots d\xi'_p d\xi_q \cdots d\xi_1,$$

where w_n are functions separately symmetric on Ξ^p and Ξ^q .

We fix η , a **sesquilinear form** on the dual of $\mathcal{V}$, whose integral kernel is the function $\eta(\cdot, \cdot)$. We will need the differential operator

$$\nabla_{\bar{\psi}} \eta \nabla_{\psi} = \int \int \nabla_{\bar{\psi}(\xi)} \eta(\xi, \xi') \nabla_{\psi(\xi')} d\xi d\xi',$$

If $w(\bar{\psi}, \psi)$ is a polynomial, its **integral with covariance η** is defined as

$$\int w(\bar{\psi}, \psi) := \exp \left(\nabla_{\bar{\psi}} \eta \nabla_{\psi} \right) w(0, 0).$$

If w is a polynomial, define its **Wick transformation** as

$$:w(\bar{\psi}, \psi): := \exp(\nabla_{\bar{\psi}} \eta \nabla_{\psi}) w(\bar{\psi}, \psi).$$

Note that the above definitions are consistent with the treatment of $\mathcal{W}$ as a real space equipped with the symmetric form

$$f_1 \sigma f_2 = \bar{f}_1 \eta f_2 + \bar{f}_2 \eta f_1.$$

In the charged formalism, every vertex has two kinds of legs: **charge creating** with arrows pointing outward and **charge annihilating** with arrows pointing inward. Each internal line joins a charge creating and a charge annihilating leg. We can consistently decorate the lines with arrows.

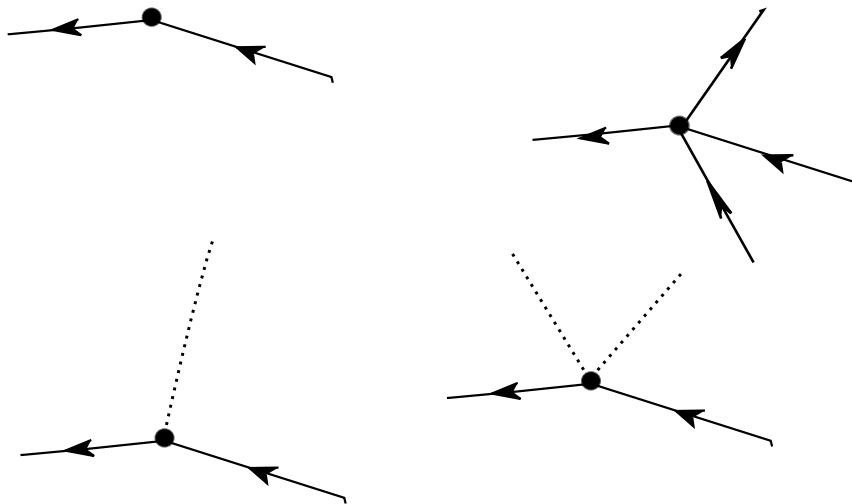
In most applications the vertices **preserve the charge** – the same number of legs create and annihilate the charge.

The generating function is defined using a pair of test functions

$$G(\bar{g}, g') := \frac{\int \exp(:W_n: + :W_1: + \psi(g) + \bar{\psi}(g'))}{\int \exp(:w_n(\bar{\psi}, \psi): + :w_1(\bar{\psi}, \psi):)}$$

Here

$$\psi(g) = \int \psi(\xi) \bar{g}(\xi) d\xi, \quad \bar{\psi}(g') = \int \bar{\psi}(\xi) g'(\xi) d\xi.$$



Various vertices involving charged particles

The diagrammatic method can be used for the evaluation of **antisymmetric gaussian integrals**. We first describe its **neutral** version.

Again we consider a real vector space $\mathcal{V}$. Elements of the antisymmetric tensor algebra over the dual of $\mathcal{V}$ will be called **antisymmetric polynomials**. Every antisymmetric polynomial can be written as

$$v(\phi) = \sum_{j=0}^n \int \cdots \int v_j(\xi_j, \dots, \xi_1) \phi(\xi_j) \cdots \phi(\xi_1) d\xi_1 \cdots d\xi_j,$$

where $v_j(\cdots)$ is an antisymmetric function on Ξ^j .

If $v(\phi) = \sum_{n=0}^{\infty} v_n(\phi)$ is an antisymmetric polynomial decomposed into its homogeneous components, then by definition $v(0) := v_0$.

We can apply to antisymmetric polynomials the **left derivative**:

$$\begin{aligned} & \nabla_{\phi(\xi)} \int \cdots \int v_n(\xi_n, \dots, \xi_1) \phi(\xi_n) \cdots \phi(\xi_1) d\xi_1 \cdots d\xi_n \\ &= n \int \cdots \int v_n(\xi, \xi_{n-1}, \dots, \xi_1) \phi(\xi_{n-1}) \cdots \phi_1(\xi_1) d\xi_1 \cdots d\xi_{n-1}. \end{aligned}$$

Note that

$$\nabla_{\phi(\xi)} \nabla_{\phi(\xi')} = -\nabla_{\phi(\xi')} \nabla_{\phi(\xi)}.$$

We often smear the derivative with a $\zeta \in \mathcal{V}$. The **antisymmetric left derivative in the direction of ζ** equals

$$\zeta \nabla_{\phi} = \int \zeta(\xi) \nabla_{\phi(\xi)} d\xi.$$

If $\mathcal{V}$ is finite dimensional and its dimension equals n , then we define the **Berezin integral over $\mathcal{V}$** as follows. We fix a basis $e_1, \dots, e_n$ of $\mathcal{V}$. Then the Berezin integral of $v(\phi)$ equals

$$\int v(\phi) d\phi := e_1 \nabla_\phi \cdots e_n \nabla_\phi v(\phi).$$

Equivalently, we decompose $v(\phi)$ into the sum of homogeneous elements. The highest order element has the form $v_n \phi_n \cdots \phi_1$. We set

$$\int v(\phi) d\phi = v_n.$$

We fix σ , an **antisymmetric form** on the dual of $\mathcal{V}$, whose integral kernel is the antisymmetric function $\sigma(\cdot, \cdot)$. We will need the following differential operator

$$\nabla_\phi \sigma \nabla_\phi = \int \int \nabla_{\phi(\xi)} \sigma(\xi, \xi') \nabla_{\phi(\xi')} d\xi d\xi',$$

If $v(\phi)$ is a polynomial, its **integral with covariance σ** is defined as

$$\int v(\phi) := \exp \left(\frac{1}{2} \nabla_\phi \sigma \nabla_\phi \right) v(0).$$

Equivalently, in terms of the Berezin integral,

$$\int v(\phi) = \text{Pf}(\sigma) \int \exp \left(\frac{1}{2} \phi \sigma^{-1} \phi \right) v(\phi) d\phi.$$

If $v(\phi)$ is a polynomial, define

$$:v(\phi): := e^{\frac{1}{2}\nabla_\phi\sigma\nabla_\phi}v(\phi).$$

We will call $:v(\phi):$ the **Wick transformation of $v(\phi)$** .

All the diagrammatic methods that we described in the symmetric case remain valid, with obvious modifications, if we restrict ourselves to even antisymmetric polynomials.

Actually, in the antisymmetric case the neutral formalism is rarely used. One prefers the **charged formalism**. Its starting point is a complex vector space $\mathcal{W}$ equipped with a sesquilinear form η . To go back to the neutral formalism one considers $\mathcal{W}$ as a real space equipped with the antisymmetric form

$$f_1 \sigma f_2 = \overline{f_1} \eta f_2 - \overline{f_2} \eta f_1.$$

0.2 Scattering operator and Friedrichs diagrams

Let $\mathcal{Z}$ be a Hilbert space. We will assume that it can be identified with $L^2(\Xi)$. We are interested in operators on $\Gamma_{s/a}(\mathcal{Z})$, treating in parallel the bosonic and fermionic case.

We introduce notation for the following linear functionals (= homogeneous 1st order polynomials) on $\mathcal{Z}$:

$$\begin{aligned}\langle a(\xi)|\zeta\rangle &:= \zeta(\xi), \\ \langle \bar{a}(\xi)|\zeta\rangle &:= \overline{\zeta(\xi)}.\end{aligned}\tag{0.10}$$

A multilinear function of the form

$$v(\bar{a}, a) = \int \cdots \int v(\xi_1, \cdots, \xi_{m_+}, \xi'_{m_-}, \cdots, \xi'_1) \bar{a}(\xi_1) \cdots \bar{a}(\xi_{m_+}) a(\xi'_{m_-}) \cdots a(\xi'_1) d\xi_1 \cdots d\xi_{m_+} d\xi'_{m_-} \cdots d\xi'_1$$

symmetric/antisymmetric wrt the variables a and $\bar{a}$ separately, will be called a **Wick monomial of degree (m_+, m_-)** . Recall that $v(\xi_1, \cdots, \xi_{m_+}, \xi'_{m_-}, \cdots, \xi'_1)$ can be interpreted as the integral kernel of an operator from $\Gamma_{s/a}^{m_-}(\mathcal{Z})$ to $\Gamma_{s/a}^{m_+}(\mathcal{Z})$. The **Wick quantization of v** is denoted by

$$v(\hat{a}^*, \hat{a}) = \int v(\xi_1, \cdots, \xi_{m_+}, \xi'_{m_-}, \cdots, \xi'_1) \hat{a}^*(\xi_1) \cdots \hat{a}^*(\xi_{m_+}) \hat{a}(\xi'_{m_-}) \cdots \hat{a}(\xi'_1) d\xi_1 \cdots d\xi_{m_+} d\xi'_{m_-} \cdots d\xi'_1.$$

It is an operator on $\Gamma_{s/a}(\mathcal{Z})$.

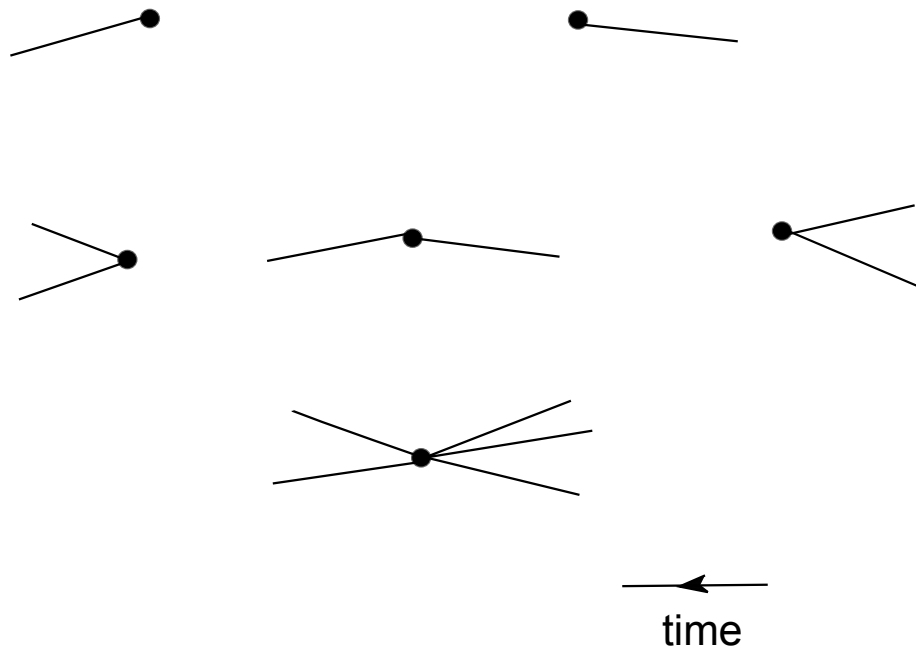
Graphically, to each Wick monomial of order (m_-, m_+) we associate a **Friedrichs vertex** with m_- **incoming legs** (on the right) and m_+ **outgoing legs** (on the left). To a leg with label l we associate a variable $\xi_l \in \Xi$.

An n th order **Friedrichs diagram** is determined if we specify

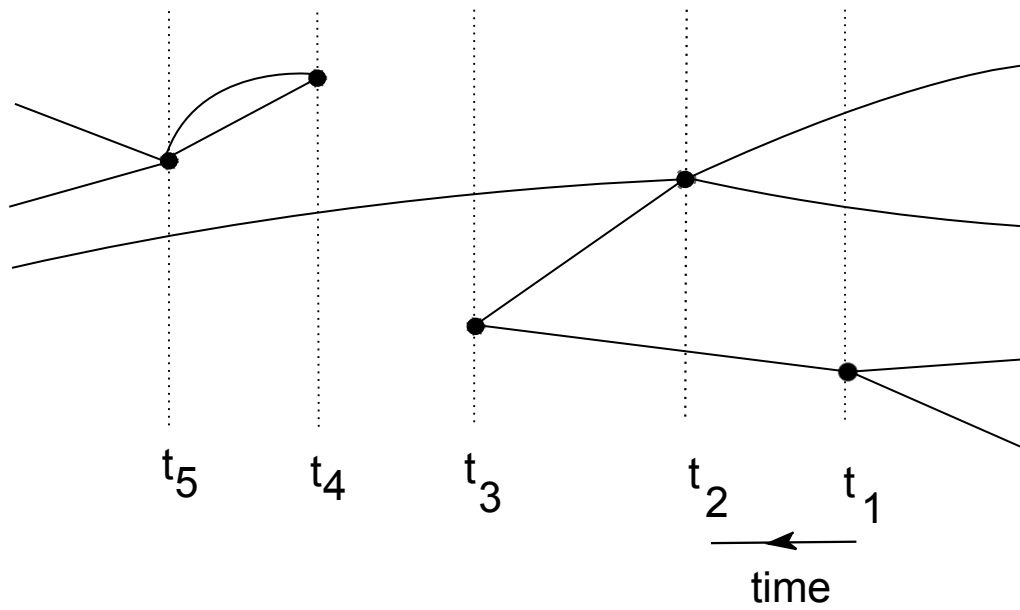
1. $v_n, \dots, v_1$, a sequence of vertices (the order in the sequence plays a role);
2. a family of internal lines.

An **internal line** is a pair of legs belonging to $v_n, \dots, v_1$ joined by a line. They should belong to two distinct vertices v_{j_+} and v_{j_-} , where $j_+ > j_-$. The leg belonging to v_{j_-} should be outgoing and the leg belonging to v_{j_+} should be incoming. Each leg belongs to at most one internal line. We do not put arrows on the lines – they have a natural direction from the right to the left, obvious from the picture.

All outgoing/incoming legs that are not a part of an internal line are called **external outgoing/incoming lines**.



Various Friedrichs vertices



Friedrichs diagram

Friedrichs diagrams will be used to compute perturbatively a number of various quantities. To a Friedrichs diagram δ with $m^+(\delta)$ outgoing and $m^-(\delta)$ incoming legs we will associate a Wick monomial of order (m_-, m_+) according to various rules.

The first quantity that we consider will be Wick ordered products of quantized Wick monomials $v_n(\hat{a}^*, \hat{a}), \dots, v_1(\hat{a}^*, \hat{a})$.

Rules for evaluating a Friedrichs diagram to compute the product

1. Write the product

$$\prod_{j=1}^n v_j(\xi_{j,1}^+, \dots, \xi_{j,m_+(j)}^+; \xi_{j,m_-(j)}^-, \dots, \xi_{j,1}^-).$$

2. For each internal line joining the vertices j_- and j_+ with the variables ξ_- and ξ_+ integrate

$$\int \int \delta(\xi_+, \xi_-) d\xi_+ d\xi_-.$$

3. For each incoming external line with label ξ_- write $\int \hat{a}(\xi_-) d\xi_-$.

4. For each outgoing external line with label ξ_+ write $\int \hat{a}^*(\xi_+) d\xi_+$.

5. Multiply by $(-1)^p$ where p is the number of crossings of fermionic lines.

The resulting quantized Wick monomial is denoted $p_\delta(\hat{a}^*, \hat{a})$.

Now,

$$(\Omega|v_n(\hat{a}^*, \hat{a}) \cdots v_1(\hat{a}^*, \hat{a})\Omega) = \sum p_\delta,$$

where we sum over **all Friedrichs diagrams without external lines**. Moreover,

$$v_n(\hat{a}^*, \hat{a}) \cdots v_1(\hat{a}^*, \hat{a}) = \sum p_\delta(\hat{a}^*, \hat{a}),$$

where we sum over **all Friedrichs diagrams**.

The above statements often go under the name of the **Wick Theorem**.

We will describe now a more interesting application of Friedrichs diagrams. Consider a **free Hamiltonian**

$$H_0 := \int \omega(\xi) \hat{a}^*(\xi) \hat{a}(\xi) d\xi,$$

perturbed by a **sum of quantized Wick monomials**

$$V(t) := \sum_{\alpha} v_{\alpha}(t; \hat{a}^*, \hat{a}),$$

where $v_{\alpha}(t; \hat{a}^*, \hat{a})$ is a quantized Wick monomial of degree $(m_+(\alpha), m_-(\alpha))$. The corresponding vertex will be called α . If we want, we can assume that $V(t)$ is (formally) hermitian. We do not insist that $H_0 + V(t)$ is selfadjoint, nor that the dynamics is well-defined.

We would like to compute the **scattering operator**

$$\begin{aligned} S &= \lim_{t_+, -t_- \rightarrow \infty} e^{it_+ H_0} \text{Texp} \left(-i \int_{t_-}^{t_+} V(t) dt \right) e^{-it_- H_0} \\ &= \text{Texp} \left(-i \int_{-\infty}^{\infty} e^{itH_0} V(t) e^{-itH_0} dt \right), \end{aligned}$$

as a formal infinite expansion, possibly divergent.

Let δ be a Friedrichs diagram with vertices $\alpha_n, \dots, \alpha_1$.

Rules for evaluating a Friedrichs diagram to compute the scattering operator

1. Write the product

$$(-i)^n \prod_{j=1}^n v_{\alpha_j}(t_j; \xi_{j,1}^+, \dots, \xi_{j,m_+(\alpha_j)}^+; \xi_{j,m_-(\alpha_j)}^-, \dots, \xi_{j,1}^-).$$

2. For each external incoming line attached to the vertex α_j and with the variable ξ_- put

$$\int e^{-i\omega(\xi_-)t_j} \hat{a}(\xi_-) d\xi_-.$$

3. For each external outgoing line attached to the vertex α_j and with the variable ξ put

$$\int e^{i\omega(\xi_+)t_j} \hat{a}^*(\xi_+) d\xi_+.$$

4. For each internal line joining the vertices α_{j_-} and α_{j_+} with the variables ξ_- and ξ_+ integrate

$$\int \int e^{i\omega(\xi_+)(t_{j_+}-t_{j_-})} \delta(\xi_+, \xi_-) d\xi_+ d\xi_-.$$

5. Integrate with respect to time

$$\int \cdots \int_{t_n \geq \cdots \geq t_1} dt_n \cdots dt_1.$$

6. Multiply by $(-1)^p$ where p is the number of crossings of fermionic lines.

The resulting quantized Wick monomial is denoted $s_\delta(\hat{a}^*, \hat{a})$.

The **scattering operator** equals

$$S = \sum_{\delta} s_{\delta}(\hat{a}^*, \hat{a}), \quad (0.11)$$

where we sum over **all Friedrichs diagrams**.

To prove (0.11), we compute

$$S = \sum_{n=0}^{\infty} \int \cdots \int_{t_n \geq \cdots \geq t_1} dt_n \cdots dt_1 (-i)^n e^{-it_n H_0} V e^{i(t_n - t_{n-1})H_0} \cdots e^{i(t_2 - t_1)H_0} V e^{it_1 H_0}.$$

We use the Wick Theorem and

$$e^{itH_0} \hat{a}^*(\xi) e^{-itH_0} = e^{it\omega(\xi)} \hat{a}^*(\xi), \quad e^{itH_0} \hat{a}(\xi) e^{-itH_0} = e^{-it\omega(\xi)} \hat{a}(\xi).$$

The vacuum expectation value of the scattering operator equals

$$(\Omega|S\Omega) = \sum_{\delta} s_{\delta}. \quad (0.12)$$

where we sum over all Friedrichs diagrams without external lines.

The Linked Cluster Theorem allows to simplify (0.12):

$$\log(\Omega|S\Omega) = \sum_{\delta} s_{\delta}$$

where we sum over all connected Friedrichs diagrams without external lines.

We can multiply Friedrichs polynomials. It involves the juxtaposition of diagrams, preserving however the order of times. Similarly, we can define $v(\bar{a}, a)^n$ and

$$\exp v(\bar{a}, a) := \sum_{n=1}^{\infty} \frac{1}{n!} v(\bar{a}, a)^n.$$

One of versions of the **Linked Cluster Theorem** says that

$$S = \exp \left(\sum_{\delta} s_{\delta} \right) (\hat{a}^*, \hat{a}).$$

where we sum over **all connected Friedrichs diagrams**.

Therefore,

$$\frac{S}{(\Omega|S\Omega)} = \exp \left(\sum_{\delta} s_{\delta} \right) (\hat{a}^*, \hat{a}),$$

where we sum over **all connected Friedrichs diagrams with at least one external line**. Equivalently,

$$\frac{S}{(\Omega|S\Omega)} = \sum_{\delta} s_{\delta}(\hat{a}^*, \hat{a}),$$

where we sum over **all linked Friedrichs diagrams**, that is, Friedrichs diagrams **whose each connected part has at least one external line**.

Assume now that v_α are **time independent**. In this case, it is convenient to use the **adiabatic switching** of the dynamics, by setting for $\epsilon > 0$

$$V_\epsilon(t) := e^{-\epsilon|t|}V.$$

We introduce the **adiabatic scattering operator**

$$S_\epsilon := \text{Texp} \left(-i \int_{-\infty}^{\infty} e^{itH_0} V_\epsilon(t) e^{-itH_0} dt \right).$$

and the **Gell-Mann and Low scattering operator**

$$S_{\text{GL}} := \text{w-} \lim_{\epsilon \searrow 0} \frac{S_\epsilon}{(\Omega | S_\epsilon \Omega)}.$$

First note that S_{GL} commutes with H_0 . Therefore, it can be decomposed in the energy, yielding $S_{\text{GL}} = \int^{\oplus} S_{\text{GL}}(E) dE$.

Rules for evaluating a Friedrichs diagram to compute Gell-Mann and Low scattering operator

1. Write the product

$$(-i)^n \prod_{j=1}^n v_{\alpha_j}(t_j; \xi_{j,1}^+, \dots, \xi_{j,p_+(\alpha_j)}^+; \xi_{j,p_-(\alpha_j)}^-, \dots, \xi_{j,1}^-).$$

2. Suppose that between the vertex t_j and t_{j+1} there are the lines with the variables $\xi_1, \dots, \xi_j$.

Then put

$$(\omega(\xi_1) + \dots + \omega(\xi_j) - E + i0)^{-1}.$$

3. If $\xi_{m_-}^-, \dots, \xi_1^-$ are incoming momenta, put

$$\int \dots \int \delta(\omega(\xi_{m_-}^-) + \dots + \omega(\xi_1^-) - E) \hat{a}(\xi_{m_-}^-) \dots \hat{a}(\xi_1^-) d\xi_{m_-}^- \dots d\xi_1^-.$$

4. If $\xi_{m_+}^+, \dots, \xi_1^+$ are outgoing momenta, put

$$\int \cdots \int \delta(\omega(\xi_{m_+}^+) + \cdots + \omega(\xi_1^+) - E) \hat{a}^*(\xi_{m_+}^+) \cdots \bar{a}^*(\xi_1^+) d\xi_{m_+}^+ \cdots d\xi_1^+.$$

5. For each internal line joining the vertices α_{j_-} and α_{j_+} with the variables ξ_- and ξ_+ integrate

$$\int \delta(\xi_+, \xi_-) d\xi_+ d\xi_-.$$

6. Multiply by $(-1)^p$ where p is the number of crossings of fermionic lines.

The resulting quantized Wick monomial is denoted $s_\delta(E, \hat{a}^*, a^*)$.

Using the Linked Cluster Theorem we obtain a formula

$$S_{\text{GL}}(E) = \sum_{\delta} s_\delta(E, \hat{a}^*, \hat{a}),$$

where we sum over **all linked Friedrichs diagrams**.

We still assume that v_α are time independent. There exists a diagrammatic formula that allows us to compute the **energy of the ground state**. This formula goes under the name of the **Goldstone Theorem**. Friedrichs diagrams in this context are often called **Goldstone diagrams**.

A convenient starting point for calculation of the energy shift is the **Sucher formula**.

$$E = \lim_{\epsilon \searrow 0} \frac{i\epsilon\lambda}{2} \partial_\lambda \log(\Omega | S_\epsilon \Omega). \quad (0.13)$$

Under rather general conditions one can show that the Sucher formula gives the correct ground state energy (noting that the unperturbed ground state energy is 0).

Let δ be a Goldstone diagram without external lines.

Rules for evaluating a Goldstone diagram to compute the ground state energy.

1. Write the product

$$(-i)^n \prod_{j=1}^n v_{\alpha_j}(\xi_{j,1}^+, \dots, \xi_{j,m_+(\alpha_j)}^+; \xi_{j,m_-(\alpha_j)}^-, \dots, \xi_{j,1}^-).$$

2. Suppose that between the vertex α_j and α_{j+1} there are lines with the variables $\xi_1, \dots, \xi_j$.
Then put

$$(\omega(\xi_1) + \dots + \omega(\xi_j))^{-1}. \quad (0.14)$$

3. For each internal line joining the vertices α_{j-} and α_{j+} with the variables ξ_- and ξ_+ we integrate

$$\int \int \delta(\xi_+, \xi_-) d\xi_+ d\xi_-.$$

4. Multiply by $(-1)^p$ where p is the number of crossings of fermionic lines.

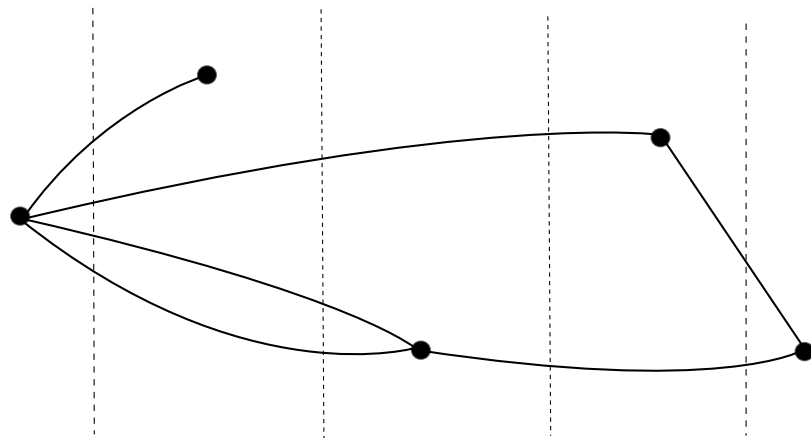
As a result we obtain a number denoted E_δ .

Now the Goldstone Theorem says that

$$E = \sum_{\delta} E_{\delta},$$

where we sum over all connected Goldstone diagrams with no external lines.

This formula is used also even if some 1-particle energies are not positive. In this case we additionally put $+i0$ in the denominators (0.14). We obtain in general a complex number with a negative imaginary part, which describes an excited metastable state.



Goldstone diagram

0.3 Scattering operator and Feynman diagrams

We begin with the **neutral bosonic formalism**. We consider the one-particle space $\mathcal{Z} = L^2(K, dk)$. The corresponding classical functionals are introduced as in (0.10) and are denoted $\bar{a}(k)/a(k)$. The variable k will have the interpretation of the 1-particle momentum, supplemented perhaps with inner degrees of freedom such as the species of a particle and the spin. Thus we will use consistently the so-called **momentum space formalism**.

We also consider the corresponding bosonic Fock space $\Gamma_s(\mathcal{Z})$. The creation/annihilation operators are denoted $\hat{a}^*(k)/\hat{a}(k)$.

We fix a **free Hamiltonian**

$$H_0 = \int \omega(k) \hat{a}^*(k) \hat{a}(k) dk$$

We consider the operator valued measure, called a **quantum field**

$$K \ni k \mapsto \hat{\phi}(k) := \frac{1}{\sqrt{2\omega(k)}}(\hat{a}(k) + \hat{a}^*(k)),$$

and the corresponding **classical field**

$$K \ni k \mapsto \phi(k) := \frac{1}{\sqrt{2\omega(k)}}(a(k) + \bar{a}(k)),$$

These fields are **neutral**: quantum fields are **self-adjoint** and classical are **real**.

A polynomial in the (classical) real variables $\phi(\mathbf{k})$

$$v(\phi) = \int \cdots \int v(\mathbf{k}_1, \dots, \mathbf{k}_m) \phi(\mathbf{k}_1) \cdots \phi(\mathbf{k}_m) d\mathbf{k}_1 \cdots d\mathbf{k}_m \quad (0.15)$$

will be called a **Feynman monomial**. It can be written as a **Wick monomial**:

$$\int \cdots \int v(\mathbf{k}_1, \dots, \mathbf{k}_m) \frac{1}{\sqrt{2\omega(\mathbf{k}_1)}} (a(\mathbf{k}_1) + \bar{a}(\mathbf{k}_1)) \cdots \frac{1}{\sqrt{2\omega(\mathbf{k}_m)}} (a(\mathbf{k}_m) + \bar{a}(\mathbf{k}_m)) d\mathbf{k}_1 \cdots d\mathbf{k}_m.$$

The Wick quantization of the above monomial will be denoted by

$$:v(\hat{\phi}): \quad := \int \cdots \int v(\mathbf{k}_1, \dots, \mathbf{k}_m) : \hat{\phi}(\mathbf{k}_1) \cdots \hat{\phi}(\mathbf{k}_m) : d\mathbf{k}_1 \cdots d\mathbf{k}_m \quad (0.16)$$

and is an operator on the Hilbert space $\Gamma_s(\mathcal{Z})$.

To an operator of this form we associate a **Feynman vertex**, which has m **legs**.

Suppose that $v_\alpha(t; \phi)$ are time-dependent Feynman monomials. Consider H_0 perturbed by a sum of **Wick quantized Feynman monomials**

$$V(t) := \sum_{\alpha} :v_{\alpha}(t; \hat{\phi}):.$$

We would like to compute the **scattering operator**

$$S = \text{Texp} \left(-i \int_{-\infty}^{\infty} e^{itH_0} V(t) e^{-itH_0} dt \right).$$

An n th order **Feynman diagram** δ is determined if we specify

1. $\alpha_n, \dots, \alpha_1$ – an n -element combination with repetitions of vertices (we represent it by a sequence, but its order is not important);
2. a family of internal lines.

An **internal line** is a pair of legs belonging to two vertices α_i, α_j for distinct i and j joined by a line. Each leg belongs to at most one internal line.

All legs that are not a part of an internal line are called **external lines**.

The **propagator** is defined as

$$\begin{aligned}
 D(t; \mathbf{k}) &:= \left(\Omega | T(\hat{\phi}(t, \mathbf{k}), \hat{\phi}(0, \mathbf{k})) \Omega \right) \\
 &= \theta(t) \left(\Omega | \hat{\phi}(t, \mathbf{k}) \hat{\phi}(0, \mathbf{k}) \Omega \right) + \theta(-t) \left(\Omega | \hat{\phi}(0, \mathbf{k}) \hat{\phi}(t, \mathbf{k}) \Omega \right) \\
 &= \frac{1}{2\omega(\mathbf{k})} \left(\theta(t) e^{i\omega(\mathbf{k})t} + \theta(-t) e^{-i\omega(\mathbf{k})t} \right).
 \end{aligned}$$

Note that the propagator is a **Green's function** or a **fundamental solution** of the “**wave equation**”:

$$(\partial_t^2 + \omega(\mathbf{k})^2) D(t, \mathbf{k}) = i\delta(t).$$

Rules for evaluating Feynman diagrams without external lines.

1. Write the product

$$(-i)^n \prod_{j=1}^n v_{\alpha_j}(t_j; k_1, \dots, k_{m(\alpha_j)})$$

2. For each internal line joining the vertices α_j and $\alpha_{j'}$ with the variables k and k' , integrate

$$\int \int D(t_j - t_{j'}, k) \delta(k, k') dk dk'.$$

3. Integrate the expression with respect to time

$$\int dt_n \cdots \int dt_1.$$

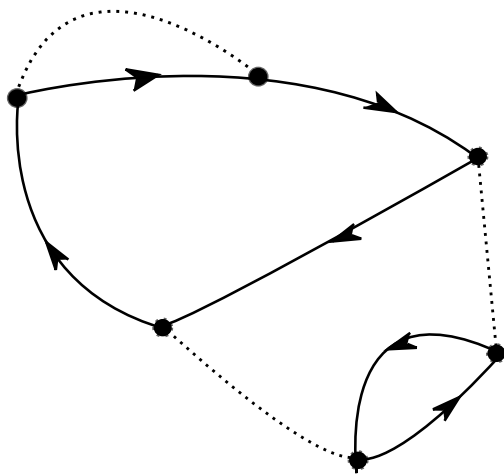
The vacuum expectation value of the scattering operator can be computed as

$$(\Omega|S\Omega) = \sum_{\beta} s_{\beta}.$$

where we sum over all Feynman diagrams without external lines. The Linked Cluster Theorem says that

$$\log(\Omega|S\Omega) = \sum_{\beta} s_{\beta}.$$

where we sum over all connected Feynman diagrams without external lines



Feynman diagram without external lines

Suppose that we are given an n th order Feynman diagram with m external lines. Before evaluating the diagram, we divide the set of its external lines in two subsets: m_- **incoming external lines**, pointing to the right, and m_+ **outgoing external lines**, pointing to the left. Clearly, $m = m_+ + m_-$. Diagrams with legs divided into two such subset will be called **scattering type Feynman diagrams**. We will associate to them Wick monomials of order $(m_+(\delta), m_-(\delta))$.

We have the expansion

$$S = \sum_{\beta} s_{\beta}(\hat{a}^*, \hat{a}),$$

where we sum over **all scattering type Feynman diagrams**.

Rules for evaluating a scattering type Feynman diagram

Let β be a scattering type Feynman diagram. We proceed as for Feynman diagrams without external lines. In addition,

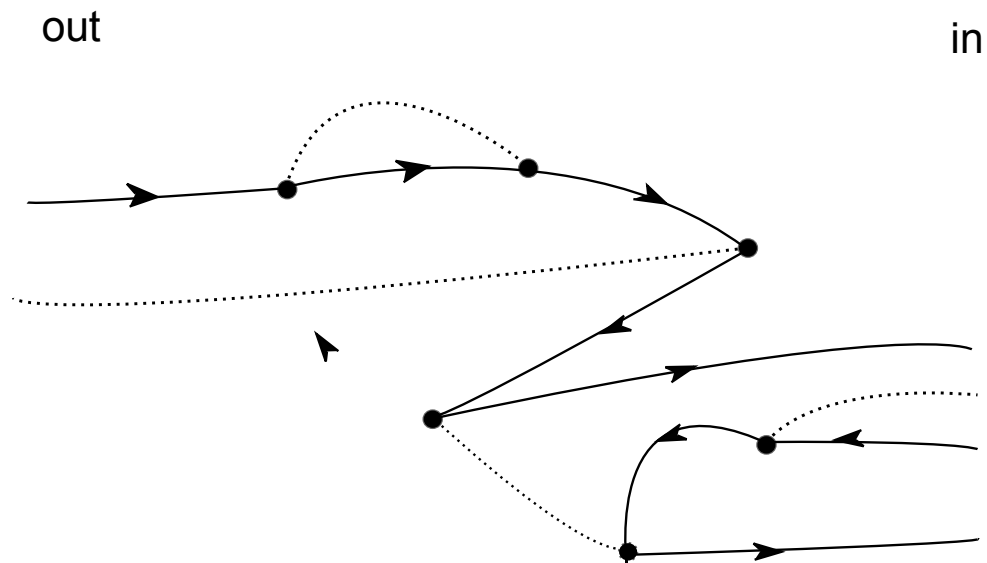
1. for each external incoming line attached to the vertex α_j , with the variable k_- , put

$$\int \frac{1}{\sqrt{2\omega(k)}} e^{i\omega(k_-)t_j} \hat{a}(k_-) dk_-.$$

2. for each external outgoing line attached to the vertex α_j , with the variable k_+ we put

$$\int \frac{1}{\sqrt{2\omega(k)}} e^{-i\omega(k_+)t_j} \hat{a}^*(k_+) dk_+.$$

The resulting quantized Wick monomial is denoted $s_\beta(\hat{a}^*, \hat{a})$.



Scattering type Feynman diagram

The formalism of Feynman diagrams can be considered as a special case of the formalism of diagrams for the **gaussian integration**. In fact, one needs to consider the space of functions on $\mathbb{R} \times K$, the first variable having the interpretation of the **time**. Elements of this space are called **paths**. Then $\phi(t, k)$ has the interpretation of a linear functional on this space. To each time-dependent polynomial $v(t; \phi)$ we can associate a function on the path space:

$$\int v(t; \phi) dt = \sum_{\alpha} \int dt \int dk_{\alpha, m(\alpha)} \cdots \int dk_{\alpha, 1} v_{\alpha}(t; k_{\alpha, m(\alpha)}, \dots, k_{\alpha, 1}) \phi(t, k_{\alpha, m(\alpha)}) \cdots \phi(t, k_{\alpha, 1}).$$

We consider the differential operator

$$\int \nabla_{\phi(t, k)} D(t - s, k) \delta(k - k') \nabla_{\phi(s, k')}.$$

This operator defines a Wick operation and a gaussian integral on the space of paths.

We then have the identity

$$\left(\Omega | T \exp \left(-i \int :v(t; \hat{\phi}): dt \right) \Omega \right) = \int \exp \left(-i \int v(t; \phi) dt \right). \quad (0.17)$$

On the left we use the **operator language**, whereas on the right we use the language of the **gaussian integrals on the space of paths**.

Identity (0.17) is the essence of the **path integral approach to QFT**. In this approach one tries to give a method to compute scattering amplitudes without speaking about operators.

The perturbations considered so far are restricted to a commutative algebra generated by $\hat{\phi}(\mathbf{k})$. Sometimes more general perturbations arise. To treat them one introduces the **conjugate fields**

$$K \ni \mathbf{k} \mapsto \hat{\pi}(\mathbf{k}) := \left. \partial_t \hat{\phi}(t, \mathbf{k}) \right|_{t=0} = \frac{\omega(\mathbf{k})}{\sqrt{2}} (\hat{a}(\mathbf{k}) - i\hat{a}^*(\mathbf{k})).$$

We have the canonical commutation relations

$$\begin{aligned} [\hat{\phi}(\mathbf{k}), \hat{\phi}(\mathbf{k}')] &= [\hat{\pi}(\mathbf{k}), \hat{\pi}(\mathbf{k}')] = 0, \\ [\hat{\phi}(\mathbf{k}), \hat{\phi}(\mathbf{k}')] &= \delta(\mathbf{k}, \mathbf{k}'). \end{aligned}$$

In the case of π -dependent perturbations, one needs to modify the path integration formalism. Instead of **paths in the configuration space**, given just by $t \mapsto \phi(t, \mathbf{k})$ one needs **paths in the phase space**, involving $t \mapsto \phi(t, \mathbf{k}), \pi(t, \mathbf{k})$. One also needs additional propagators:

$$\begin{aligned}
 \partial_t D(t, \mathbf{k}) &= \left(\Omega | T(\hat{\pi}(t, \mathbf{k}), \hat{\phi}(0, \mathbf{k})) \Omega \right) \\
 &= \frac{i}{2} \left(\theta(t) e^{i\omega(\mathbf{k})t} - \theta(-t) e^{-i\omega(\mathbf{k})t} \right), \\
 -\partial_t^2 D(t, \mathbf{k}) &= (\Omega | T(\hat{\pi}(t, \mathbf{k}), \hat{\pi}(0, \mathbf{k})) \Omega) \\
 &= \frac{\omega(\mathbf{k})}{2} \left(\theta(t) e^{i\omega(\mathbf{k})t} + \theta(-t) e^{-i\omega(\mathbf{k})t} \right) + i\delta(t).
 \end{aligned}$$

In the case of **charged particles** one uses a slightly different formalism. To describe it we will assume that the free system is charge reversal invariant.

Let the one-particle space be $\mathcal{Z}^{(+)} \oplus \mathcal{Z}^{(-)} = L^2(P, dp) \oplus L^2(P, dp)$. The first copy $L^2(P, dp)$ is called the **particle space** and the second the **antiparticle space**. The creation operators for particles are denoted $\hat{a}^*(p)$ and for antiparticles $\hat{b}^*(p)$.

Consider a **free Hamiltonian**

$$\hat{H}_0 = \int E(p) \hat{a}^*(p) \hat{a}(p) dp + \int E(p) \hat{b}^*(p) \hat{b}(p) dp$$

We consider operator valued measures, called **quantum fields**

$$\begin{aligned}P \ni p &\mapsto \hat{\psi}(p) = \frac{1}{\sqrt{2E(p)}}\hat{a}(p) + \frac{1}{\sqrt{2E(p)}}\hat{b}^*(p), \\P \ni p &\mapsto \hat{\psi}^*(p) = \frac{1}{\sqrt{2E(p)}}\hat{a}^*(p) + \frac{1}{\sqrt{2E(p)}}\hat{b}(p).\end{aligned}$$

and the corresponding **classical fields**

$$\begin{aligned}P \ni p &\mapsto \psi(p) = \frac{1}{\sqrt{2E(p)}}a(p) + \frac{1}{\sqrt{2E(p)}}\bar{b}(p), \\P \ni p &\mapsto \bar{\psi}(p) = \frac{1}{\sqrt{2E(p)}}\bar{a}(p) + \frac{1}{\sqrt{2E(p)}}b(p).\end{aligned}$$

These fields are **charged**: quantum fields are **non-self-adjoint** and classical are **complex**.

A polynomial in the (classical) complex variables $\bar{\psi}(p)$, $\psi(p)$ of the form

$$\begin{aligned}
 w(\bar{\psi}, \psi) = & \int \cdots \int w(p_1^+, \dots, p_{m(+)}^{(+)}; p_{m(-)}^{(-)}, \dots, p_1^{(-)}) \\
 & \times \bar{\psi}(p_1^{(+)}) \cdots \bar{\psi}(p_{m(+)}^{(+)}) \psi(p_{m(-)}^{(-)}) \cdots \psi(p_1^{(-)}) \\
 & \times dp_1^+ \cdots dp_{m(+)}^{(+)} dp_{m(-)}^{(-)} \cdots dp_1^{(-)}.
 \end{aligned} \tag{0.18}$$

will be called a **Feynman monomial**.

It corresponds to a **Wick monomial** given by

$$\begin{aligned}
& \int \cdots \int w(p_1^{(+)}, \dots, p_{m_{(+)}}^{(+)}; p_{m_{(-)}}^{(-)}, \dots, p_1^{(-)}) \left(\frac{1}{\sqrt{2E(p_1)}} a(p_1^{(+)}) + \frac{1}{\sqrt{2E(p_1)}} \bar{b}(p_1^{(+)}) \right) \cdots \\
& \left(\frac{1}{\sqrt{2E(p_{m_{(+)}})}} a(p_{m_{(+)}}^{(+)}) + \frac{1}{\sqrt{2E(p_{m_{(+)}})}} \bar{b}(p_{m_{(+)}}^{(+)}) \right) \\
& \times \left(\frac{1}{\sqrt{2E(p_{m_{(+)}})}} \bar{a}(p_{m_{(+)}}^{(+)}) + \frac{1}{\sqrt{2E(p_{m_{(+)}})}} b(p_{m_{(+)}}^{(+)}) \right) \cdots \\
& \left(\frac{1}{\sqrt{2E(p_1)}} \bar{a}(p_1^{(+)}) + \frac{1}{\sqrt{2E(p_1)}} b(p_1^{(+)}) \right) dp_1^{(+)} \cdots dp_{m_{(+)}}^{(+)} dp_{m_{(-)}}^{(-)} \cdots dp_1^{(-)}.
\end{aligned}$$

The Wick quantization of the above monomial will be denoted by

$$\begin{aligned}
:w(\hat{\psi}^*, \hat{\psi}): &= \int \cdots \int w(p_1^{(+)}, \dots, p_{m_{(+)}}^{(+)}; p_{m_{(-)}}^{(-)}, \dots, p_1^{(-)}) \\
&\times : \hat{\psi}^*(p_1^{(+)}) \cdots \hat{\psi}^*(p_{m_{(+)}}^{(+)}) \hat{\psi}(p_{m_{(-)}}^{(-)}) \cdots \hat{\psi}(p_1^{(-)}) : \\
&\times dp_1^{(+)} \cdots dp_{m_{(+)}}^{(+)} dp_{m_{(-)}}^{(-)} \cdots dp_1^{(-)},
\end{aligned} \tag{0.19}$$

and is an operator on the Hilbert space $\Gamma_s(\mathcal{Z}^{(+)} \oplus \mathcal{Z}^{(-)})$.

To an operator of this form we associate a **Feynman vertex**, to which we attach $m_{(-)}$ **charge annihilating legs** and $m_{(+)}$ **charge creating legs**. On charge annihilating/creating legs we put arrows pointing towards/away from the vertex.

We consider H_0 perturbed by a sum of Wick quantizations of Feynman monomials

$$V(t) = \sum_{\alpha} :w_{\alpha}(t; \hat{\psi}^*, \hat{\psi}):,$$

We would like to compute the scattering operator S .

An n th order Feynman diagram δ for the scattering operator is determined similarly as for neutral particles. We need to remember that a charge creating leg should be connected to a charge annihilating leg, forming a charged line. Charged lines are decorated with an arrow pointing in the direction of the flow of the charge.

The **propagator** in the charged case is

$$\begin{aligned}
 D(t, p) &:= \left(\Omega | T(\hat{\psi}(t, p), \hat{\psi}^*(0, p)) \Omega \right) \\
 &= \theta(t) \left(\Omega | \hat{\psi}(t, p) \hat{\psi}^*(0, p) \Omega \right) + \theta(-t) \left(\Omega | \hat{\psi}^*(0, p) \hat{\psi}(t, p) \Omega \right) \\
 &= \frac{1}{2E(p)} \theta(t) e^{iE(p)t} + \frac{1}{2E(p)} \theta(-t) e^{-iE(p)t}.
 \end{aligned}$$

The propagator is the **Green's function** of the “**wave equation**”:

$$(\partial_t^2 - E(p)^2) D(t, p) = i\delta(t).$$

Rules for evaluating Feynman diagrams without external lines are the same as in the neutral case, except for obvious modifications:

1. Write the product

$$(-i)^n \prod_{j=1}^n w_{\alpha_j}(t_j; p_1^{(+)}, \dots, p_{m_{(+)}(\alpha_j)}^{(+)}; p_{m_{(-)}(\alpha_j)}^{(-)}, \dots, p_1^{(-)})$$

2. For each internal line joining the vertices α_{j-} and α_{j+} with the variables p_- and p_+ integrate

$$\int \int D(t_{j+} - t_{j-}, p_+) \delta(p_+, p_-) dp_+ dp_-.$$

3. Integrate with respect to time

$$\int dt_n \cdots \int dt_1.$$

For the n th order Feynman diagram with m external lines, as in the neutral case, we divide the set of its external lines in two subsets: m_- incoming external lines, pointing to the right, and m_+ outgoing external lines, pointing to the left. Clearly, $m = m_+ + m_-$. Note that now external lines are divided into 4 categories:

1. incoming particle lines,
2. incoming antiparticle lines,
3. outgoing particle lines,
4. outgoing antiparticle lines.

A Feynman diagram together with the assignment of incoming and outgoing lines will be called a **scattering type Feynman diagram**.

Rules for evaluating scattering type Feynman diagrams

Let β be a scattering type Feynman diagram. We follow the rules we specified for diagrams without external lines, supplemented by the following rules:

1. (i) For each external incoming charge annihilating line attached to the vertex α_j , with the variable p_- , put

$$\int \frac{1}{\sqrt{2E(p_-)}} e^{iE(p_-)t_j} \hat{a}(p_-) dp_-.$$

-
- (ii) For each external incoming charge creating line attached to the vertex α_j , with the variable p_- , put

$$\int \frac{1}{\sqrt{2E(p_-)}} e^{iE(p_-)t_j} \hat{b}(p_-) dp_-.$$

2. (i) For each external outgoing charge annihilating line attached to the vertex α_j , with the variable p_+ , put

$$\int \frac{1}{\sqrt{2E(p_+)}} e^{-iE(p_+)t_j} \hat{a}^*(p_+) dp_+.$$

- (ii) For each external outgoing charge creating line attached to the vertex α_j , with the variable p_+ , put

$$\int \frac{1}{\sqrt{2E(p_+)}} e^{-iE(p_+)t_j} \hat{b}^*(p_+) dp_+.$$

The resulting quantized Wick polynomial is denoted $s_\beta(\hat{a}^*, \hat{b}^*, \hat{a}, \hat{b})$. Now

$$S = \sum_{\beta} s_\beta(\hat{a}^*, \hat{b}^*, \hat{a}, \hat{b}),$$

where we sum over **all scattering type Feynman diagrams**.

Let us now discuss the **fermionic** case.

We begin with the **neutral formalism**. We assume that $\mathcal{Z} = L^2(K, dk)$. The creation operators on $L^2(K, dk)$ are denoted $\hat{a}^*(k)$.

Consider a **free Hamiltonian**

$$H_0 = \int \omega(k) \hat{a}^*(k) \hat{a}(k) dk$$

We consider the pair of **neutral fields**

$$\begin{aligned} K \ni k &\mapsto \hat{\phi}_0(k) := \hat{a}(k) + \hat{a}^*(k), \\ K \ni k &\mapsto \hat{\phi}_1(k) := i\hat{a}(k) - i\hat{a}^*(k). \end{aligned}$$

Note that all operators can be expressed in terms of $\phi_i(k)$, $i = 1, 2$. We have the relations

$$[\hat{\phi}_i(k), \hat{\phi}_{i'}(k')]_+ = 2\delta(k, k')\delta_{i,i'}.$$

Set $\hat{\phi}(t, \mathbf{k}) := e^{itH} \hat{\phi}(\mathbf{k}) e^{-itH}$. The **propagator** is defined as

$$\begin{aligned} S(t, \mathbf{k}) &:= \left(\Omega | T(\hat{\phi}(t, \mathbf{k}), \hat{\phi}(0, \mathbf{k})) \Omega \right) \\ &= \theta(t) \left(\Omega | \hat{\phi}(t, \mathbf{k}) \hat{\phi}(0, \mathbf{k}) \Omega \right) - \theta(-t) \left(\Omega | \hat{\phi}(0, \mathbf{k}) \hat{\phi}(t, \mathbf{k}) \Omega \right) \\ &= \theta(t) \begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix} e^{it\omega(\mathbf{k})} - \theta(-t) \begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix} e^{-it\omega(\mathbf{k})}. \end{aligned}$$

The propagator is the **Green's function** of the “**Dirac equation**”:

$$(-i\partial_t + H) S(t, \mathbf{k}) = \delta(t).$$

where

$$H = \begin{bmatrix} 0 & -i\omega(\mathbf{k}) \\ i\omega(\mathbf{k}) & 0 \end{bmatrix}.$$

Fermions are usually described in the **charged formalism**. We will describe it assuming the charge reversal invariance of the free Hamiltonian. The one-particle space is $\mathcal{Z}^{(+)} \oplus \mathcal{Z}^{(-)} = L^2(P, dp) \oplus L^2(P, dp)$. The creation operators for **particles** are denoted $\hat{a}^*(k)$ and for **antiparticles** $\hat{b}^*(p)$. Consider a **free Hamiltonian**

$$\hat{H}_0 = \int E(p) \hat{a}^*(p) \hat{a}(p) dp + \int E(p) \hat{b}^*(p) \hat{b}(p) dp$$

Consider a pair of **charged fields**.

$$\begin{aligned} P \ni p &\mapsto \hat{\psi}(p) = \hat{a}(p) + \hat{b}^*(p), \\ P \ni p &\mapsto \hat{\psi}(p) = i\hat{a}(p) - i\hat{b}^*(p). \end{aligned}$$

They satisfy the anticommutation relations

$$\begin{aligned} [\hat{\psi}_i(p), \hat{\psi}_{i'}(p')]_+ &= 0, \\ [\hat{\psi}_i(p), \hat{\psi}_{i'}^*(p')]_+ &= \delta(p, p') \delta_{i, i'}. \end{aligned}$$

Set $\hat{\psi}(t, \mathbf{p}) := e^{itH} \hat{\psi}(\mathbf{p}) e^{-itH}$. The **propagator** in the charged case is

$$\begin{aligned} S(t, \mathbf{p}) &:= \left(\Omega | T(\hat{\psi}(t, \mathbf{p}), \hat{\psi}^*(0, \mathbf{p})) \Omega \right) \\ &= \theta(t) \left(\Omega | \hat{\psi}(t, \mathbf{p}) \hat{\psi}^*(0, \mathbf{p}) \Omega \right) - \theta(-t) \left(\Omega | \hat{\psi}^*(0, \mathbf{p}) \hat{\psi}(t, \mathbf{p}) \Omega \right) \\ &= \theta(t) \begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix} e^{iE(\mathbf{p})t} - \theta(-t) \begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix} e^{-iE(\mathbf{p})t}. \end{aligned}$$

The propagator is the **Green's function** of a “**Dirac equation**”:

$$(-i\partial_t + H) S(t, \mathbf{p}) = \delta(t).$$

where

$$H = \begin{bmatrix} 0 & iE(\mathbf{p}) \\ -iE(\mathbf{p}) & 0 \end{bmatrix}.$$