Continuous Limits of Classical Repeated Interactions Systems

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Introduction

- Context : System + Environment
  Ex : Object in contact with some thermal baths, charged particule...
- Quantum Mechanics (Attal, Pautrat)$\Rightarrow$ Repeated Interactions
- Questions : - Limit evolution of the system ?
  - Renormalization in Hamiltonian cases ?

Plan of the Talk

Discrete Time

- Dynamical System $\widehat{T}$ measurable application on $S \times E$

  measurable $\Rightarrow$ T on $\mathcal{L}^\infty(S \times E)$,

$$Tg(x,y) = g(\widehat{T}(x,y))$$

- Point of view of S

  - $f \in \mathcal{L}^\infty(S) \Rightarrow f \otimes \mathbb{1} \in \mathcal{L}^\infty(S \times E)$,

$$f \otimes \mathbb{1}(x,y) = f(x)$$

  - E is endowed with a probability measure μ

  Assumption : What the system S sees from the whole dynamics on $S \times E$ is an average on E along the probability measure μ.

Discrete Time

- $\forall f \in \mathcal{L}^\infty(S)$, $\forall x \in E$

$$Lf(x) = \int_E T(f \otimes \mathbb{1})(x, y)\, d\mu(y)$$

Question : What really is the operator L ?

Discrete Time

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Discrete Time

Theorem

- There exists a Markov transition kernel Π such that L is of the form
$$Lf(x) = \int_S f(z)\,\Pi(x, dz)\,,$$
for all $f \in \mathcal{L}^\infty(S)$.

- Conversely, if S is a Lusin space and Π is any Markov transition kernel on S, then there exist a probability space $(E, \mathcal{E}, \mu)$ and a dynamical system $\widehat{T}$ on $S \times E$ such that the operator
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Repeated Interactions

- Pb : Not a commuting diagram for all powers.

- $\widehat{T} : S \times E \longrightarrow S \times E$

$$(x, y) \longmapsto (U(x, y), V(x, y))$$

- Scheme of repeated interactions

$\widetilde{E} = E^{\mathbb{N}^*}$, $\widetilde{\mathcal{E}}$ product σ-algebra, $\widetilde{\mu} = \mu^{\otimes \mathbb{N}^*}$

$$\begin{aligned} \widetilde{T} : \quad S \times \widetilde{E} &\longrightarrow S \times \widetilde{E} \\ (x, y = (y_n)_{n \in \mathbb{N}^*}) &\longmapsto (U(x, y_1), \theta(y)) \end{aligned}$$

where $\theta(y) = (y_{n+1})_{n \in \mathbb{N}^*}$ is the shift.

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Repeated Interactions

- T induced by $\widetilde{T}$ on applications

Theorem

For all m in $\mathbb{N}^*$, all x in S, and all f in $\mathcal{L}^\infty(S)$,

$$(L^m f)(x) = \int_{\widetilde{E}} T^m(f \otimes \mathbb{1})(x, y) d\widetilde{\mu}(y) \, .$$

- Initial state $X_0 = x$ of the system
  - State of the environment $y \Rightarrow X_{n+1}(y) = U(X_n(y), y_{n+1})$
  - State of the environment unknown $\Rightarrow (X_n)_{n \in \mathbb{N}}$ Markov chain

Repeated Interactions

For all k,

$$\widetilde{T}^k(x,y) = (X_k(y), \theta^k(y)), \text{ with } X_0 = x$$

Introduction of the time step h,
$U \to U^{(h)}$, $(X_n)_{n\in\mathbb{N}} \to (X^{(h)}_{nh})_{n\in\mathbb{N}}$

$$(\widetilde{T}^{(h)})^k(x,y) = (X^{(h)}_{kh}(y), (\theta^{(h)})^k(y)),$$

where $X_0^{(h)} = x$ et $\theta^{(h)}(y_{nh}) = (y_{(n+1)h})$.

Repeated Interactions

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Harmonic Interaction

- Particule of mass 1 interacting with an other one according to a harmonic interaction
- Hamiltonian for the whole system

$$H\left[\begin{pmatrix} Q_1 \\ P_1 \end{pmatrix}, \begin{pmatrix} Q_2 \\ P_2 \end{pmatrix}\right] = \underbrace{\frac{P_1^2}{2} + \frac{Q_1^2}{2}}_{H_1} + \underbrace{\frac{P_2^2}{2} + \frac{Q_2^2}{2}}_{H_2} \underbrace{-Q_2 Q_1}_{Interaction}$$

- Evolution of the particule 1

$$\begin{cases} Q_1(t) = & \frac{P_1(0)+P_2(0)}{2} t + \frac{Q_1(0)+Q_2(0)}{2} \\ & + \frac{Q_1(0)-Q_2(0)}{2} \cos(\sqrt{2}t) + \frac{P_1(0)-P_2(0)}{2\sqrt{2}} \sin(\sqrt{2}t) \\ P_1(t) = & \frac{P_1(0)+P_2(0)}{2} - \frac{Q_1(0)-Q_2(0)}{\sqrt{2}} \sin(\sqrt{2}t) \\ & + \frac{P_1(0)-P_2(0)}{2} \cos(\sqrt{2}t) \end{cases}$$

Harmonic Interaction - Repeated Interactions

- System $S = \mathbb{R}^2$, Environment $\widetilde{E}^{(h)} = (\mathbb{R}^2)^{h\mathbb{N}^*}$
- Evolution of the system

$$\begin{cases} Q_1((n+1)h) = & Q_1(nh) + hP_1(nh) \\ & +h^2 \dfrac{Q_2((n+1)h) - Q_1(nh)}{2} \\ & -h^3 \dfrac{(P_1(nh) - P_2((n+1)h))}{6} + \circ(h^3) \\ & \\ P_1((n+1)h) = & P_1(nh) + h(Q_2((n+1)h) - Q_1(nh)) \\ & +h^2 \dfrac{P_2((n+1)h) - P_1(nh)}{2} \\ & +h^3 \dfrac{Q_1(nh) - Q_2((n+1)h)}{3} + \circ(h^3) \end{cases}$$

Continuous Time

- Let x be in $\mathbb{R}^m$, (X_t^x) the solution of the stochastic differential equation (SDE)

$$dX_t = b(X_t)\,dt + \sigma(X_t)\,dW_t$$

where $X_0 = x$, (W_t) a d-dimensional Brownian Motion, $b : \mathbb{R}^m \longrightarrow \mathbb{R}^m$ and $\sigma : \mathbb{R}^m \longrightarrow \mathcal{M}_{m,d}(\mathbb{R})$ measurable

- Uniqueness and existence :
  - b, σ Lipschitz functions
  - b, σ linear growth condition

$\exists K > 0 / \forall X \in \mathbb{R}^m$,

$$|b(X)| \leq K(1 + |X|), \quad \|\sigma(X)\| \leq K(1 + |X|)$$

Continuous Time

- Aim : Find an environment and a dynamical system (a semigroup) which allow to « dilate »the solution of the SDE
- Environment : $(\Omega, \mathcal{F}, \mathbb{P})$ Wiener space associated to (W_t),

$$\Omega = \Big\{\omega \text{ continuous function from } \mathbb{R}_+ \text{ to } \mathbb{R}^d \text{ such that } \omega(0) = 0\Big\}$$

- For all t, for all ω, $W_t(\omega) = \omega(t)$
- Shift θ_t on Ω,

$$\theta_t(\omega)(s) = \omega(t+s) - \omega(t)$$

Continuous Time

Theorem

The family $(T_t)_{t\in\mathbb{R}_+}$ of applications from $\mathbb{R}^m \times \Omega$ to $\mathbb{R}^m \times \Omega$ defined by

$$T_t(x,\omega) = (X_t^x(\omega), \theta_t(\omega))$$

is a continuous time dynamical system.

Question : Can we obtain a continuous time dynamical system as limit of discrete dynamics ?

Problems : - Dynamical systems defined on different spaces
- Dynamics in discrete and continuous time
- Convergence ?

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Environment

- $S = \mathbb{R}^m$
- $E^{h\mathbb{N}^*} = (\mathbb{R}^d)^{h\mathbb{N}^*}$,

  - « Injection » : $\varphi_I^{(h)} : (\mathbb{R}^d)^{h\mathbb{N}^*} \longrightarrow \Omega$

$$\varphi_I^{(h)}(y)(t) = \sum_{n=0}^{\lfloor t/h \rfloor} y_{nh} + \frac{t - \lfloor t/h \rfloor h}{h} y_{(\lfloor t/h \rfloor + 1)h}$$

  - « Projection » : $\varphi_P^{(h)} : \Omega \longrightarrow (\mathbb{R}^d)^{h\mathbb{N}^*}$

$$\varphi_P^{(h)}(\omega) = (W_{nh}(\omega) - W_{(n-1)h}(\omega))_{n \in \mathbb{N}^*}$$

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Environment

- Rem : $\varphi_P^{(h)} \circ \varphi_I^{(h)} = Id \Rightarrow \Omega^{(h)} = \varphi_I^{(h)}((\mathbb{R}^d)^{h\mathbb{N}^*}) \cong (\mathbb{R}^d)^{h\mathbb{N}^*}$
- Is the space $\Omega^{(h)}$ suitable ?

Ω is endowed with its canonical metric D defined by

$$D(\omega, \omega') = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\sup_{0 \le t \le n} |\omega(t) - \omega'(t)|}{1 + \sup_{0 \le t \le n} |\omega(t) - \omega'(t)|}$$

Lemma

For all ω in Ω,

$$\lim_{h \to 0} D(\omega, \varphi_I^{(h)} \circ \varphi_P^{(h)}(\omega)) = 0 \,.$$

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Construction of Continuous Dynamics on the same space

- Construction of dynamical system on $\mathbb{R}^m \times \Omega$

$\Phi_I^{(h)} = (Id, \varphi_I^{(h)})$ et $\Phi_P^{(h)} = (Id, \varphi_P^{(h)})$

$$\bar{T}^{(h)} = \Phi_I^{(h)} \circ \widetilde{T}^{(h)} \circ \Phi_P^{(h)}$$

- Continuity of dynamics

$$\bar{T}_t^{(h)} = (\bar{T}^{(h)})^{\lfloor t/h \rfloor} + \frac{t - \lfloor t/h \rfloor h}{h} \left\{ (\bar{T}^{(h)})^{(\lfloor t/h \rfloor + 1)} - (\bar{T}^{(h)})^{\lfloor t/h \rfloor} \right\}$$

- For all initial state x, $\bar{T}_t^{(h)}(x, \omega) = (\bar{X}_t^{(h)}(\varphi_P^{(h)}(\omega)), \bar{\theta}_t^{(h)}(\omega))$, where $\bar{X}_0^{(h)} = x$

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Harmonic Interaction

- State of environment are sampled from $(W_{nh} - W_{(n-1)h})$, where (W_t) is a 2-dimensional Brownian motion.

- Reinforcement of interactions
  Values of $\begin{pmatrix} Q_2 \\ P_2 \end{pmatrix}$ sampled from $\frac{1}{h}(W_{nh} - W_{(n-1)h})$.

- Understanding of this factor $\frac{1}{h}$:
  * $\frac{1}{\sqrt{h}}$ to obtain state of the environment independent of h

    (physically, sampled from $\frac{e^{-H_2}}{Z}$)
  * $\frac{1}{\sqrt{h}}$ real renormalization of interactions

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Harmonic Interaction

Evolution of the system given by the Markov chain (X_{nh}) defined by

$$X(nh) = U^{(h)}(X((n-1)h), Y(nh))$$

where

$$U^{(h)}(X, Y) = X + \sigma(X)Y + hb(X) + h\eta^{(h)}(X, Y)$$

with

$$b\begin{pmatrix} Q_1 \\ P_1 \end{pmatrix} = \begin{pmatrix} P_1 \\ -Q_1 \end{pmatrix}, \quad \sigma\begin{pmatrix} Q_1 \\ P_1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

et

$$\eta^{(h)}\left[\begin{pmatrix} Q_1 \\ P_1 \end{pmatrix}, \begin{pmatrix} Q_2 \\ P_2 \end{pmatrix}\right] = \frac{1}{2}\begin{pmatrix} Q_2 \\ P_2 \end{pmatrix} - \frac{h}{2}\begin{pmatrix} Q_1 - P_2/3 \\ P_1 + 2Q_2/3 \end{pmatrix} + \circ(h)$$

Convergence of Shift

- D metric on Ω

$$D(\omega,\omega') = \sum_{n=1}^{\infty} \frac{1}{2^n} \quad \frac{\sup\limits_{0\leq t\leq n} |\omega(t)-\omega'(t)|}{1+\sup\limits_{0\leq t\leq n} |\omega(t)-\omega'(t)|}$$

Theorem (*J. D.*)

Let ω be a function in Ω. For all $t \in \mathbb{R}_+$,

$$\lim_{h\to 0} \quad D\left(\theta_t(\omega), \bar{\theta}_t^{(h)}(\omega)\right) = 0\,.$$

Convergence of Processes

Theorem (*J. D.*)

Suppose that there exist :
- b, σ Lipschitz and linearly bounded applications
- $\eta^{(h)}$ where, for a $\alpha \in [0, +\infty]$, $\left|\eta^{(h)}(x, y)\right| \leq K(h^\alpha |x| + |y|)$

such that,

$$U^{(h)}(x, y) = x + \sigma(x)y + hb(x) + h\eta^{(h)}(x, y) \, .$$

Then,for all x_0 in $\mathbb{R}^m$, and all $\tau > 0$, the process $(\bar{X}_t^h)$, starting in x_0, converges to $(X_t^{x_0})$ when h tends to 0 in L^{2p}, for all $p \geq 1$, and almost surely on $[0, \tau]$, where $(X_t^{x_0})$ is the solution of the SDE

$$dX_t^{x_0} = b(X_t^{x_0}) \, dt + \sigma(X_t^{x_0}) \, dW_t \, ,$$

starting in $X_t^{x_0} = x_0$.

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Harmonic Interaction

- Theorem, with $\alpha = 1 \Rightarrow$ For all initial conditions Q_0, P_0 and for all $\tau > 0$, the limit evolution on $[0, \tau]$ is given by the solution of the SDE

$$dX_t = \begin{pmatrix} X_t^2 \\ -X_t^1 \end{pmatrix} dt + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} dW_t \,,$$

with the notation $X_t = \begin{pmatrix} X_t^1 \\ X_t^2 \end{pmatrix}$ and the initial condition $X_0 = \begin{pmatrix} Q_0 \\ P_0 \end{pmatrix}$.

Sketch of Proof

- Stochastic numerical analysis

  - Markov chain $(X^h_{nh})_{n\in\mathbb{N}}$ defined by

$$X^h_{(n+1)h} = X^h_{nh} + hb(X^h_{nh}) + \sigma(X^h_{nh})(W_{(n+1)h} - W_{nh}) + h\eta^{(h)}(X^h_{nh}, W_{(n+1)h} - W_{nh}),$$

  with $X^h_0 = X_0$.

  - Linear Interpolation

$$X^h_t = X^h_{\lfloor t/h \rfloor h} + \frac{t - \lfloor t/h \rfloor h}{h}\left\{X^h_{(\lfloor t/h \rfloor + 1)h} - X^h_{\lfloor t/h \rfloor h}\right\}$$

Sketch of Proof

- If $p \geq 1$, $\mathbb{E}(|X_0|^{2p}) < \infty$,

Lemma

For all $t \in [0, \tau]$, the solution X_t of the SDE verifies the inequality

$$\mathbb{E}(|X_t|^{2p}) \leq (1 + \mathbb{E}(|X_0|^{2p}))e^{Ct}.$$

Lemma

For all $t \in [0, \tau]$, the process X_t^h verifies the inequality

$$\mathbb{E}(\left|X_t^h\right|^{2p}) \leq C_0(1 + \mathbb{E}(\left|X_0^h\right|^{2p}))e^{C_1 t}.$$

Sketch of Proof

- Upper bound on $\epsilon_t = X_t - X_t^h$ for all t :
  - Control over ϵ_t according to $\epsilon_{\lfloor t/h \rfloor h}$
  - Bound the evolution of the sequence $(\epsilon_{nh})_{n \in \mathbb{N}}$

Theorem (*J. D.*)

Under the previous conditions,

$$\mathbb{E}\left(\sup_{t \in [0,\tau]} \left| X_t - X_t^h \right|^{2p} \right) \leq C\left(h^{2p\alpha} + h^p(-\log h)^p\right)$$

- If $p > 1$ and $2p\alpha > 1 \Rightarrow$ almost sure convergence.