Adiabatic evolution and dephasing

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Outline

The Landau-Zener model

An adiabatic theorem

Optimal parametrization

Linear response theory and geometry

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The Landau-Zener model

An adiabatic theorem

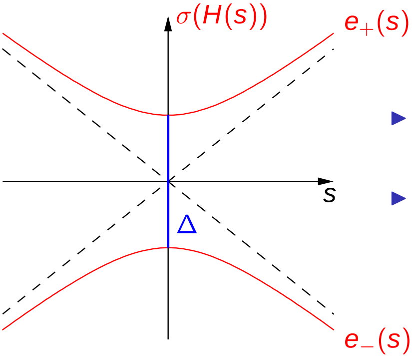
Optimal parametrization

Linear response theory and geometry

A motivating example: Landau-Zener tunnelling

The Hamiltonian case

$$H(s) = \frac{1}{2}\vec{x}\cdot\vec{\sigma} \quad \text{on } \mathbb{C}^2 \qquad (\vec{x}(s) = (s, 0, \Delta))$$
$$= \frac{|\vec{x}|}{2}P_+ - \frac{|\vec{x}|}{2}P_-$$



- eigenvalues
  $e_\pm(s) = \pm|\vec{x}(s)|/2$
- eigenprojections
  $P_\pm(s) \to (1 \pm \sigma_x)/2, (s \to \pm\infty)$

$H(s)$ is general form of single-parameter avoided crossing

Landau-Zener: Hamiltonian case (cont.)

- scaled time $s = \varepsilon t$:

$$\mathrm{i}\frac{d\psi}{dt} = H(\varepsilon t)\psi$$

or

$$\mathrm{i}\varepsilon\dot{\psi} = H(s)\psi \qquad (\dot{} = d/ds)$$

- initial state: spin down

$$(\psi(s), P_+(s)\psi(s)) \to 0 \qquad (s \to -\infty)$$

- tunnelling probability

$$(\psi(s), P_+(s)\psi(s)) \to T \qquad (s \to +\infty)$$

- Landau, Zener (1932)

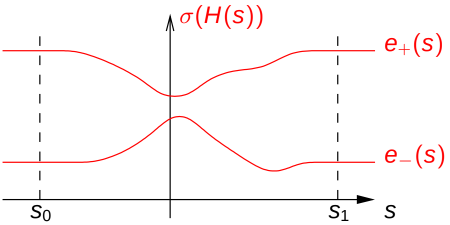
$$T = \mathrm{e}^{-\pi\Delta^2/2\varepsilon}$$

(exponentially small in $\varepsilon \to 0$).

Adiabatic tunnelling: Hamiltonian case

More generally, let

- $H(s)$ smooth
- $H(s)$ (or $P_\pm(s)$) constant near $s = \pm\infty$, e.g. at $s = s_0, s_1$.



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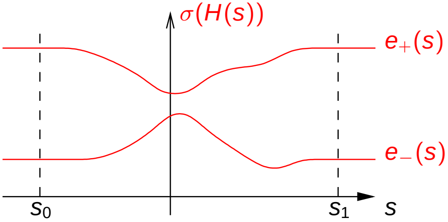

Then:

- $$T({\color{red}s}, s_0) = O(\varepsilon^2) \qquad (s \text{ generic})$$

  At intermediate times s, "down" state contains a coherent admixture $O(\varepsilon)$ of the "up" state.

- $$T({\color{red}s_1}, s_0) = O(\varepsilon^n) \qquad (n = 1, 2, \ldots)$$

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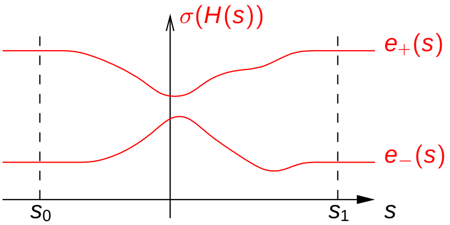

Then:

- $$T(s, s_0) = O(\varepsilon^2) \qquad (s \text{ generic})$$
  At intermediate times s, "down" state contains a coherent admixture $O(\varepsilon)$ of the "up" state.
- $$T(s_1, s_0) = O(\varepsilon^n) \qquad (n = 1, 2, \ldots)$$
  Essentially no memory is retained at the end: tunnelling is reversible.

Lindblad evolution

System coupled to Bath: Evolution of a mixed state $\rho = \rho_S$

$$\rho \mapsto \phi_t(\rho) = \mathrm{tr}_B\big(U_t(\rho \otimes \rho_B)U_t^*\big)$$

with joint unitary evolution U_t ($U_{t+s} = U_t U_s$)

Properties:

- $\mathrm{tr}\, \phi_t(\rho) = \mathrm{tr}\, \rho$
- ϕ_t completely positive
- $\phi_{t+s} = \phi_t \circ \phi_s$
  - approximately, if time scales of Bath $\ll$ time scales of System
  - exactly, if bath is white noise

Generator:

$$\mathcal{L} := \frac{d\phi_t}{dt}\Big|_{t=0}$$

Theorem (Lindblad, Sudarshan-Kossakowski-Gorini 1976)

The general form of the generator is

$$\mathcal{L}(\rho) = -\mathrm{i}[H, \rho] + \frac{1}{2}\sum_\alpha (2\Gamma_\alpha \rho \Gamma_\alpha^* - \Gamma_\alpha^* \Gamma_\alpha \rho - \rho \Gamma_\alpha^* \Gamma_\alpha)$$

Dephasing Lindbladians

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with

$$[\Gamma_\alpha, P_i] = 0 \qquad \text{for } H = \sum_i e_i P_i$$

Then $\mathcal{L}(P_i) = 0$, resp. $\phi_t(P_i) = P_i$: Like in the Hamiltonian case, eigenstates P_i are invariant.
Example: 2-level system

$$\mathcal{L}(\rho) = -\mathrm{i}[H, \rho] - \gamma(P_- \rho P_+ + P_+ \rho P_-) \qquad (\gamma \geq 0)$$

Evolution turns coherent into incoherent superpositions within a time $\sim \gamma^{-1}$. Is a model for measurement of H.
Application: Nuclear magnetic resonance

Dephasing 2-level Lindbladian

$$\mathcal{L}(s)(\rho) = -\mathrm{i}[H(s), \rho] - \gamma(s)(P_-(s)\rho P_+(s) + P_+(s)\rho P_-(s))$$

with

- $H(s) = \vec{x}(s) \cdot \vec{\sigma}/2$
- $\gamma(s) \geq 0$
- $\vec{x}(s)$ with $\dot{\vec{x}}(s) \to \dot{\vec{x}}(\pm\infty)$, ($s \to \pm\infty$).

Lindblad equation for $\rho = \rho(s)$

$$\varepsilon\dot{\rho} = \mathcal{L}(s)(\rho)$$

Result

$$T = \varepsilon \int_{-\infty}^{\infty} \frac{\gamma(s)}{\vec{x}(s)^2 + \gamma(s)^2} \operatorname{tr}\left(\dot{P}_-(s)^2\right) ds + O(\varepsilon^2)$$

Tunnelling has memory and is irreversible.

Dephasing Landau-Zener Lindbladian

$\vec{x}(s) = (s, 0, \Delta)$:

$$\mathrm{tr}\left(\dot{P}_-(s)^2\right) = \frac{\Delta^2}{2\vec{x}^4}$$

For $\gamma(s)$ constant:

$$T = \frac{\pi\varepsilon}{4\Delta^2} Q(\gamma/\Delta) + O(\varepsilon^2)$$

$$Q(x) = \frac{\pi}{2} \frac{x(2+\sqrt{1+x^2})}{\sqrt{1+x^2}(\sqrt{1+x^2}+1)^2}$$

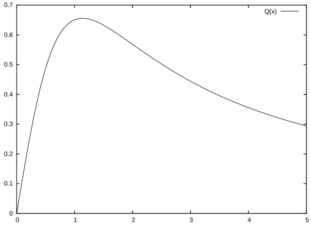

Figure: The function $Q(x)$. It has a maximum at $x = 1.13693$

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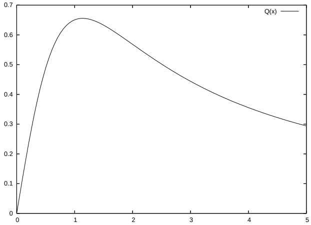
linear at small γ



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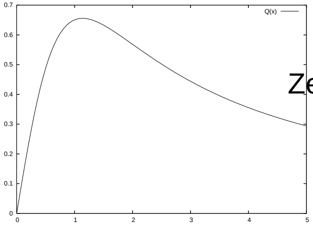

Zeno effect at large γ

Figure: The function $Q(x)$. It has a maximum at $x = 1.13693$

Outline

A question

Recall:

- Hamiltonian case $\longrightarrow$ reversible tunnelling, oblivion
- Deph. Lindbladian case $\longrightarrow$ irreversible tunnelling, memory

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Question: Is there a common point of view making this evident?

The scheme

Setup:

- V linear space, finite-dimensional.
- $L(s) : V \to V, x \mapsto L(s)x$ linear in $x \in V$, smooth in $0 \le s \le 1$.

Assumptions:

- 0 is an eigenvalue of $L(s)$, isolated uniformly in s.
- $V = \ker L \oplus \operatorname{ran} L$. In particular:
  - L is invertible on $\operatorname{ran} L$: L^{-1}
  - $1 = P + Q$ (projections), $x = a + b$ (decomposition)

Evolution equation for $x = x(s)$: $\varepsilon \dot{x} = L(s)x$
Parallel transport $T(s, s') : V \to V$ with

$$\frac{\partial}{\partial s} T(s, s') = [\dot{P}(s), P(s)] T(s, s'), \qquad T(s', s') = 1$$

implying $P(s)T(s, s') = T(s, s')P(s')$

The theorem

i) $\varepsilon\dot{x} = L(s)x$ admits solutions of the form

$$x(s) = \sum_{n=0}^{N} \varepsilon^n (a_n(s) + b_n(s)) + \varepsilon^{N+1} r_N(\varepsilon, s)$$

with

- $a_n(s) \in \ker L(s)$, $b_n(s) \in \operatorname{ran} L(s)$
- $a_n(0) \in \ker L(0)$, $r_N(\varepsilon, 0) \in V$ arbitrary

ii) Coefficients ($n = 0, 1, \ldots$):

- $b_0(s) = 0$
- $a_n(s) = T(s,0)a_n(0) + \int_0^s T(s,s')\dot{P}(s')b_n(s')ds'$
- $b_{n+1}(s) = L(s)^{-1}(\dot{P}(s)a_n(s) + Q(s)\dot{b}_n(s))$

iii) If $L(s)$ generates a contraction semigroup, then $r_N(\varepsilon, s)$ is uniformly bounded in ε and in s, if so at $s = 0$

A corollary

Recall:

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(Note: $b_0 \rightsquigarrow a_0 \rightsquigarrow b_1 \rightsquigarrow a_1 \ldots$)

Corollary If $P(s)$ is constant near $s = s_0$, then

$$b_n(s_0) = 0\,, \qquad (n = 0, 1, 2, \ldots)$$

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Next: One result, different applications.

Appl. to Quantum Mechanics: Hamiltonian case

$$V = \mathcal{H}\,, \qquad x = \psi$$

$$\mathrm{i}\varepsilon\dot{\psi} = H(s)\psi$$

$$e(s) : \text{isolated, simple eigenvalue of } H(s)$$

Set $\tilde{\psi}(s) = \psi(s)\exp(\mathrm{i}\varepsilon^{-1}\int^{s} e(s')ds')$ and rewrite

$$\varepsilon\dot{\tilde{\psi}} = -\mathrm{i}(H(s) - e(s))\tilde{\psi} \equiv L(s)\tilde{\psi}$$

with 0 isolated, simple eigenvalue of $L(s)$.

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Tunnelling out of $e(s)$ is motion out of $\ker L(s)$. Hence reversible.

Appl. to QM: Dephasing Lindbladian case

$$V = \{\text{operators on } \mathcal{H}\}\,, \qquad x = \rho\,, \qquad L(s) = \mathcal{L}(s)$$

For simplicity $\dim \mathcal{H} = 2$, hence $\dim V = 4$.

$$\mathcal{L}(\rho) = -\mathrm{i}[H, \rho] - \gamma(P_- \rho P_+ + P_+ \rho P_-)$$

with $\gamma \geq 0$ and $H|\psi_i\rangle = e_i|\psi_i\rangle$, ($i = \pm$)
Basis of V:

$$E_{ij} = |\psi_i\rangle\langle\psi_j|$$

In particular, $P_i = E_{ii}$.

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$$\mathcal{L}(P_i) = 0$$

$$\mathcal{L}(E_{+-}) = (-\mathrm{i}(e_+ - e_-) - \gamma)E_{+-} \equiv \lambda_{+-}E_{+-}\,, \qquad \lambda_{-+} = \bar{\lambda}_{+-}$$

$$\ker \mathcal{L} = \mathrm{span}(P_+, P_-)\,, \qquad \mathrm{ran}\, \mathcal{L} = \mathrm{span}(E_{+-}, E_{-+})$$

Tunnelling $T(s, 0) = \mathrm{tr}(P_+(s)\rho(s))$ for $\rho(0) = P_-(0)$ is motion within $\ker \mathcal{L}$. Hence irreversible.

Dephasing Lindbladian case: Quantitative result

Recall: In the general scheme, solution of $\varepsilon\dot{x} = L(s)x$ of the form

$$x(s) = a_0(s) + \varepsilon(a_1(s) + b_1(s)) + O(\varepsilon^2)$$
$$a_n(s) = T(s,0)a_n(0) + \int_0^s T(s,s')\dot{P}(s')b_n(s')ds'$$
$$b_{n+1}(s) = L(s)^{-1}(\dot{P}(s)a_n(s) + Q(s)\dot{b}_n(s))$$

In the application $x = \rho$, $a_0(0) = P_-(0)$, $a_1(0) = 0$ one obtains

$$a_0(s) = P_-(s)$$
$$a_1(s) = (-P_-(s) + P_+(s)) \int_0^s \alpha(s')ds' \qquad \text{(loss \& gain)}$$
$$\alpha(s) = -(\lambda_{+-}(s)^{-1} + \lambda_{-+}(s)^{-1})\,\mathrm{tr}\big(P_+(s)\dot{P}_-(s)^2P_+(s)\big)$$
$$-(\lambda_{+-}^{-1} + \lambda_{-+}^{-1}) = \frac{2\gamma}{(e_+ - e_-)^2 + \gamma^2}$$

as claimed.

Outline

Optimal parametrization: Statement of the problem

Family of 2-level Lindbladians parametrized by $0 \le q \le 1$

$$\mathcal{L}(q)(\rho) = -\mathrm{i}[H(q), \rho] - \gamma(q)(P_-(q)\rho P_+(q) + P_+(q)\rho P_-(q))$$

Allotted time $1/\varepsilon$ to get from $q = 0$ to $q = 1$:

$$q = q(s) = q(\varepsilon t)$$

with $q : [0,1] \to [0,1], s \mapsto q; q(0) = 0, q(1) = 1.$

Tunnelling $T[q] = \mathrm{tr}(P_+(s)\rho(s))_{s=1}$ for $\rho(0) = P_-(0)$

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Question: Given $\varepsilon > 0$. Which are the parametrizations q minimizing $T[q]$?
Aside: In the Hamiltonian case, for small ε, minimizers (with $T[q] = 0$) are ubiquitous.

Optimal parametrization: The functional

$$\mathcal{L}(q)(\rho) = -\mathrm{i}[H(q), \rho] - \gamma(q)(P_-(q)\rho P_+(q) + P_+(q)\rho P_-(q))$$

To leading order in ε

$$T[q] = \varepsilon \int_0^1 \frac{\gamma(s)}{\vec{x}(s)^2 + \gamma(s)^2} \operatorname{tr}\big(\dot{P}_-(s)^2\big) ds$$

with $f(s) := f(q(s))$ for $f = \vec{x}, \gamma$ and

$$\dot{P}_-(s) = P'_-(q(s))\dot{q}(s) , \quad (\dot{} = \frac{d}{ds} , \quad ' = \frac{d}{dq})$$

Functional of Lagrangian type

$$T[q] = \int_0^1 L(q(s), \dot{q}(s), s) ds$$

$$L(q, \dot{q}, \cancel{s}) = \varepsilon \frac{\gamma(q)}{\vec{x}(q)^2 + \gamma(q)^2} \operatorname{tr}\big(P'_-(q)^2\big)\dot{q}^2$$

(weighted Fubini-Study metric)

Optimal parametrization: Results

$$T[q] = \int_0^1 L(q(s), \dot{q}(s), s)ds$$

$$L(q, \dot{q}, \cancel{s}) = \varepsilon \frac{\gamma(q)}{\vec{x}(q)^2 + \gamma(q)^2} \operatorname{tr}\left(P'_-(q)^2\right)\dot{q}^2$$

Minimizing parametrization has conserved “energy”

$$\frac{\partial L}{\partial \dot{q}}\dot{q} - L = L$$

Theorem The parametrization minimizes tunnelling iff it has constant tunnelling rate.

In particular: velocity $\dot{q}$ is

- large, where gap $|\vec{x}(q)|$ is large
- small, where projection $P_-(q)$ changes rapidly

Outline

A particular Lindbladian

A dephasing Lindbladian determined by the Hamiltonian:

- $$\alpha = i, \quad \Gamma_i^* \Gamma_i = \gamma_i P_i \qquad \text{for } H = \sum_i e_i P_i$$
- No energy scale beyond the spectrum $0 = e_0 < e_1 < \ldots$: $\gamma_i = \gamma e_i$, $(\gamma \geq 0)$

Resulting in:

$$\mathcal{L}(\rho) = -\mathrm{i}[H, \rho] + \gamma \sum_i e_i (P_i \rho P_i - P_i \rho - \rho P_i)$$

Adiabatic evolution (recall)

$$\rho(s) = a_0(s) + \varepsilon(a_1(s) + b_1(s)) + \dots$$

with

$$a_0(s) = P_0(s)$$

$$a_1(s) = -\sum_{j\neq 0} T_j(s)(P_0(s) - P_j(s))$$

(loss & gain; cumulated tunneling $T_j(s) \propto \gamma$)

$$b_1(s) = \sum_{j\neq 0} \lambda_{j0}^{-1} P_j \dot{P}_0 + \text{h.c.}$$

$$(\lambda_{j0} = -e_j(\mathrm{i} + \gamma))$$

Linear response (setting)

Family of Hamiltonians $H(\varphi)$ with control parameters $\varphi = (\varphi^1, \dots, \varphi^n) \in M$

Geometric data associated to ground state projection $P(\varphi) = P_0(\varphi)$

- adiabatic curvature 2-form ω

$$\omega_{\mu\nu} = -\mathrm{i}\,\mathrm{tr}(P[\partial_\mu P, \partial_\nu P])$$

  (satisfies $d\omega = 0$, hence a symplectic form if non-degenerate)

- Fubini-Study metric g

$$g_{\mu\nu} = \mathrm{tr}(\partial_\mu P)(\partial_\nu P)$$

with $\partial_\mu = \partial \cdot / \partial \varphi^\mu$

Linear response (results)

Observables $F_\mu = \partial_\mu H$, conjugate to φ^μ. (Examples: force and displacement, torque and angle, current and flux.)

For slowly time-dependent controls $\varphi(\varepsilon t)$

$$\langle F_\mu \rangle = \mathrm{tr}(\rho(s)\partial_\mu H)$$
$$= \varepsilon \frac{\gamma}{1+\gamma^2} \sum_{j\neq 0} T_j(s)\partial_\mu e_j + \varepsilon \frac{1}{1+\gamma^2} \sum_\nu (\omega_{\mu\nu} + \gamma g_{\mu\nu})\dot{\varphi}^\nu + O(\varepsilon^2)$$

Remarks: No contribution from $P_0(s)$:

$$\langle F_\mu \rangle_0 = \mathrm{tr}(P_0 \partial_\mu H) = \partial_\mu \mathrm{tr}(P_0 H) = \partial_\mu e_0 = 0$$

Similarly $\partial_\mu e_j = 0$ if e_j independent of φ.

Generalized conductances

$$\delta \langle F_\mu \rangle \equiv \langle F_\mu \rangle - \langle F_\mu \rangle_0 = f_{\mu\nu} \dot{\phi}^\nu$$

Hence:

$$f = (1 + \gamma^2)^{-1} (\gamma g + \omega)$$

Decomposition into dissipative (symmetric) and reactive (antisymmetric) parts

$$f_{\mu\nu} = f_{(\mu,\nu)} + f_{[\mu,\nu]}$$

Hence

$$f_{(\mu,\nu)} = \frac{\gamma}{1+\gamma^2} g_{\mu\nu} \qquad f_{[\mu,\nu]} = \frac{1}{1+\gamma^2} \omega_{\mu\nu}$$

both affected by dephasing.

Kähler structure

A manifold M with metric g and symplectic form ω is almost Kähler if $J := g^{-1}\omega$ (mapping vectors to vectors) is an almost complex structure:

$$J^2 = -1$$

Equivalently,

$$\omega^{-1}g = -g^{-1}\omega \qquad (*)$$

M is Kähler if, in addition, M is a complex manifold w.r.t. J.

Examples: 1) $\mathbb{C}P^{n-1}$ (the rays of an n-dimensional Hilbert space) is Kähler.
2) Manifold $M \ni \varphi$ of controls: g, ω are pull-backs by way of $P : M \to \mathbb{C}P^{n-1}$. Iff (*) holds, M is Kähler.

Generalized resistances

$$\dot{\phi}^{\nu} = (f^{-1})^{\mu\nu} \delta \langle F_{\nu} \rangle$$

If M is Kähler, then

$$f^{-1} = \gamma g^{-1} + \omega^{-1}$$

and the reactive resistance is immune to dephasing γ.

Indeed

$$f = (\gamma^2 + 1)^{-1}(\gamma g + \omega)$$

and

$$(\gamma g^{-1} + \omega^{-1})(\gamma g + \omega) = \gamma^2 + 1 + \gamma(g^{-1}\omega + \omega^{-1}g) = \gamma^2 + 1$$

Examples

The Hamiltonians of these examples have spectrum independent of controls.

1) Harmonic oscillator

$$H(\zeta, \mu) = \frac{\omega}{2}((p-\mu)^2 + (x-\zeta)^2 - 1)$$

with ground states $P(\zeta, \mu)$ (coherent states): $M = \mathbb{C} \ni \zeta + \mathrm{i}\mu$

2) Spin 1/2

$$H(\hat{e}) = \hat{e} \cdot \vec{\sigma} + 1 \quad (\hat{e} \in S^2)$$

with ground state $P(\hat{e})$ (spin down $|-\hat{e}\rangle$): $M = S^2 \ni \hat{e}$ (Riemann sphere)

3) Let $\tau = \tau_1 + \mathrm{i}\tau_2 \in \mathbb{C}$ define the torus $\mathbb{T} = \mathbb{R}^2/(\mathbb{Z} + \tau\mathbb{Z})$. Landau Hamiltonian $H(\varphi_1, \varphi_2)$ on $\mathbb{T}$ with boundary conditions φ_1, φ_2 and flux 2π. Then $M = \mathbb{R}^2 \ni (\varphi_1, \varphi_2)$ with complex structure τ. Reactive resistance is Hall resistance.

Summary

- Dephasing Lindbladians describe open systems with several invariant states.
- Tunnelling between them if Lindbladian is changed adiabatically.
- Tunnelling has memory, unlike for Hamiltonian dynamics
- Analog of Landau-Zener formula for Hamiltonian 2-level systems
- General adiabatic theorem encompassing Lindbladian and Hamiltonian dynamics
- Optimal parametrization: No unique minimizers in Hamiltonian case. Unique minimizers in Lindbladian case, characterized by constant tunnelling rate.
- Linear response theory for single-scale Lindbladians. Reactive resistance immune to dephasing if ground states define a Kähler geometry