Engineering inverse power law decoherence of a qubit

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Strategy

Correct Theoretical Physics Strategy

At Mathematics Conference speak about Physics!
At Physics Conference speak about Mathematics!
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Dangerous strategy
At Mathematics Conference speak about Mathematics!
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Spontaneous emission and the Dicke model

Situation and strategy

**Situation**
- Analytical Exponential-like Decoherence processes for Lorentzian type distribution of field modes in Jaynes-Cummings model
- Oscillating decay and trapping for distribution of field modes with photonic band gap (PBG) edge near the resonant frequency of the two-level system

**Strategy**
- Delay the Decoherence process by engineering the reservoirs of field modes
- Search for inverse power laws in the exact dynamics
The Jaynes-Cummings model

The Hamiltonian of the whole system:

\[ H = H_S + H_E + H_I, \quad \hbar = 1 \]

\[ H_S = \omega_0 \sigma_+ \sigma_-, \quad H_E = \sum_{k=1}^{\infty} \omega_k a_k^\dagger a_k, \quad H_I = \sum_{k=1}^{\infty} \left( g_k \sigma_+ \otimes a_k + g_k^* \sigma_- \otimes a_k^\dagger \right) \]

The operators acting on the Hilbert space of the qubit:

\[ \sigma_+ |0\rangle = |1\rangle, \quad \sigma_+ |1\rangle = 0, \quad \sigma_- = \sigma_+^\dagger \]

The operators acting on the Hilbert space of the field modes:

\[ a_k^\dagger |\cdots, n_k, \cdots\rangle_E = \sqrt{n_k + 1} |\cdots, n_k + 1, \cdots\rangle_E \]

\[ N = \sigma_+ \sigma_- + \sum_{k=1}^{\infty} a_k^\dagger a_k, \quad [H, N] = [H_I, N] = 0 \]
Initial condition and time evolution

Initial unentangled condition between the qubit and the vacuum state of the external environment:

\[ |\Psi(0)\rangle = (c_0 |0\rangle + c_1(0) |1\rangle) \otimes |0\rangle_E \]

Exact time evolution

\[ |\Psi(t)\rangle = c_0 |0\rangle \otimes |0\rangle_E + c_1(t) |1\rangle \otimes |0\rangle_E + \sum_{k=1}^{\infty} d_k(t) |0\rangle \otimes |k\rangle_E \]

where

\[ |k\rangle_E = a_k^\dagger |0\rangle_E, \quad k = 1, 2, \ldots \]
The equations of the exact dynamics: Ansatz

Interaction picture

\[ |\Psi(t)\rangle_I = e^{i(H_S + H_E)t} |\Psi(t)\rangle \]

\[ = c_0 |0\rangle \otimes |0\rangle_E + C_1(t) |1\rangle \otimes |0\rangle_E + \sum_{k=1}^{\infty} \Lambda_k(t) |0\rangle \otimes |k\rangle_E \]

where

\[ \Lambda_k(t) = e^{i\omega_k t} d_k(t), \quad k = 1, 2, \ldots \]
The equations of the exact dynamics: Convolution equation

Equations for the coefficients:

\[
\dot{C}_1(t) = -i \sum_{k=1}^{\infty} g_k e^{i(\omega_0 - \omega_k) t} \Lambda_k(t),
\]

\[
\dot{\Lambda}_k(t) = -i g_k^* e^{-i(\omega_0 - \omega_k) t} C_1(t)
\]

Closed equation for \(C_1(t)\)

\[
\dot{C}_1(t) = -(f \ast C_1)(t),
\]

where two-point correlation function of the reservoir of field modes

\[
f(t - t') = \sum_{k=1}^{\infty} |g_k|^2 e^{-i(\omega_k - \omega_0)(t-t')}
\]
The correlation function and the spectral density

Two-point correlation function of the reservoir of field modes

\[ f(t - t') = \sum_{k=1}^{\infty} |g_k|^2 e^{-i(\omega_k - \omega_0)(t - t')} \]

For a continuous distribution of modes \( \eta(\omega) \)

\[ f(\tau) = \int_0^{\infty} J(\omega) e^{-i(\omega - \omega_0)\tau} d\omega, \]

where

- spectral density function

\[ J(\omega) = \eta(\omega) |g(\omega)|^2 \]

- frequency dependent coupling constant \( g(\omega) \)
Reduced density matrix

By tracing over the degrees of freedom of the reservoir:

\[ \rho_{1,1}(t) = 1 - \rho_{0,0} = \rho_{1,1}(0) |G(t)|^2, \]
\[ \rho_{1,0}(t) = \rho_{0,1}(t) = \rho_{1,0}(0) e^{-i\omega_0 t} G(t) \]

The term \( G(t) \) fulfills

\[ \dot{G}(t) = -(f \ast G)(t), \]

with

\[ G(0) = 1 \]
The Lorentzian spectral density and Exponential-like relaxations (1)


- Lorentzian spectral density function

\[ \tilde{J}_L(\omega) = \frac{1}{2\pi} \frac{\gamma \lambda^2}{(\omega - \omega_0)^2 + \lambda^2} \]

- Reservoir correlation function:

\[ \tilde{f}_L(\tau) = \int_{-\infty}^{\infty} \tilde{J}_L(\omega) e^{-i(\omega-\omega_0)\tau} d\omega = \frac{\gamma \lambda}{2} e^{-\lambda|\tau|} \]

where

- \( \lambda > 0 \): spectral width of the coupling
- \( \gamma > 0 \): relaxation rate
The Lorentzian spectral density and Exponential-like relaxations (2)

The exact dynamics of the qubit:

\[ \rho_{1,1}(t) = 1 - \rho_{0,0}(t) = \rho_{1,1}(0) |G_L(t)|^2 \]
\[ \rho_{1,0}(t) = \rho_{0,1}^*(t) = \rho_{1,0}(0) e^{-i\omega_0 t} G_L(t) \]

The weak and strong coupling regimes:

\[ G_L(t) = e^{-\lambda t/2} \left( \cosh \left( \frac{d}{2} t \right) + \frac{\lambda}{\hat{d}} \sinh \left( \frac{d}{2} t \right) \right), \quad \lambda > 2\gamma \]
\[ G_L(t) = e^{-\lambda t/2} \left( \cos \left( \frac{\hat{d}}{2} t \right) + \frac{\lambda}{\hat{d}} \sin \left( \frac{\hat{d}}{2} t \right) \right), \quad \lambda < 2\gamma \]

\[ \hat{d} = \sqrt{2\gamma \lambda - \lambda^2}, \quad d = \sqrt{\lambda^2 - 2\gamma \lambda} \]
Lorentzian type spectral densities

Other known solutions for:

\[
\tilde{J}_{L_+} (\omega) = \frac{W_1 \Gamma_1}{(\omega - \omega_r^{(1)})^2 + (\Gamma_1/2)^2} + \frac{W_2 \Gamma_2}{(\omega - \omega_r^{(2)})^2 + (\Gamma_2/2)^2}
\]

\[
\tilde{J}_{L'} (\omega) = \frac{4\Gamma^3/\sqrt{2}}{(\omega - \omega_r)^4 + \Gamma^4}
\]

\[
\tilde{J}_{L_-} (\omega) = \frac{W_1 \Gamma_1}{(\omega - \omega_r)^2 + (\Gamma_1/2)^2} - \frac{W_2 \Gamma_2}{(\omega - \omega_r)^2 + (\Gamma_2/2)^2}, \quad (PBG)
\]

(Exponential-like decay and trapping)
Literature

The Fox $H$-function

Very general function defined by:

$$
H_{p,q}^{m,n} \left[ \begin{array}{c}
(a_1, \alpha_1), \ldots, (a_p, \alpha_p) \\
(b_1, \beta_1), \ldots, (b_q, \beta_q)
\end{array} \right] = \frac{1}{2\pi i} \int_C \frac{\prod_{j=1}^m \Gamma(b_j + \beta_j s) \prod_{m=1}^n \Gamma(1 - a_i - \alpha_i s) z^{-s}}{\prod_{l=n+1}^p \Gamma(a_l + \alpha_l s) \prod_{j=m+1}^q \Gamma(1 - b_j - \beta_j s)} \, ds
$$

- $0 \leq m \leq q$, $0 \leq n \leq p$
- $\alpha_j, \beta_j > 0$; $a_j, b_j$ complex numbers such that no pole of $\Gamma(b_j + \beta_j s)$ for $j = 1, 2, \ldots, m$ coincides with any pole of of $\Gamma(1 - a_j + \alpha_j s)$ for $j = 1, 2, \ldots, n$.
- $C$ is a contour in the complex $s$-plane from $\omega - i\infty$ to $\omega + i\infty$ such that $(b_j + k)/\beta_j$ and $(a_j - 1 - k)/\alpha_j$ lie to the right and left of $C$, respectively.
References

Special cases

The generalized Bessel-Maitland function

\[ J_{\mu,\nu}^\lambda(z) = H_{1,1}^{1,3} \left[ \frac{z^2}{4} \right] \begin{pmatrix} \frac{\lambda + \frac{\nu}{2}}{2}, 1 \\ \frac{\lambda + \frac{\nu}{2}}{2}, 1, \frac{\nu}{2}, 1, \mu \left( \lambda + \frac{\nu}{2} - \lambda - \nu, \mu \right) \end{pmatrix} \]

The Wright generalized hypergeometric functions

\[ p \psi_q \left[ \begin{array}{c} z \\ (a_p, A_p) \\ (b_q, B_q) \end{array} \right] = H_{1,p}^{p,q+1} \left[ -z \right] \begin{pmatrix} 1 - a_1, A_1 \\ \vdots \\ 1 - a_p, A_p \\ 0, 1, 1 - b_1, b_1, \ldots 1 - b_q, b_q \end{pmatrix} \]
More special cases

The Meijer $G$-function

\[
G_{p,q}^{m,n} \left[ z \left| \begin{array}{c}
(a_1, \ldots, a_p) \\
(b_1, \ldots, b_q)
\end{array} \right. \right] = H_{m,n}^{p,q} \left[ -z \left| \begin{array}{c}
(a_1, 1) \ldots (a_p, 1) \\
(b_1, 1), \ldots (b_q, 1)
\end{array} \right. \right]
\]

The Generalized Mittag-Leffler function

\[
E_\gamma^{\alpha,\beta}(-z) = \frac{1}{\Gamma(\gamma)} H_{1,2}^{1,1} \left[ z \left| \begin{array}{c}
(1 - \gamma, 1) \\
(0, 1), (1 - \beta, \alpha)
\end{array} \right. \right]
\]
Even more special cases

**The MacRobert’s $E$-function**

\[
E(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; z) = H^p,1_{q+1,p} \left[ z \ \bigg| \ \begin{array}{c}
(1, 1), (\beta_1, 1), \ldots, (\beta_q, 1) \\
(\alpha_1, 1), \ldots, (\alpha_p, 1)
\end{array} \right]
\]

**The Whittaker function**

\[
W_{k,m}(z) = z^{-\rho} e^{z/2} H^{1,2}_{2,0} \left[ \frac{z^2}{4} \ \bigg| \ \begin{array}{c}
(\rho - k + 1, 1) \\
(\rho + m + \frac{1}{2}, 1)
\end{array} \right]
\]
Structured photonic band gap reservoirs

- Discontinuity in the distribution of frequency modes
- New phenomena in atom-cavity interactions (oscillatory relaxation)
Special reservoir with structured photonic band gap

Continuous spectral density

\[ J_\alpha(\omega) = \frac{2A(\omega - \omega_0)^\alpha \Theta(\omega - \omega_0)}{a^2 + (\omega - \omega_0)^2}, \quad A > 0, \quad a > 0, \quad 1 > \alpha > 0 \]

- PBG edge in the qubit transition frequency
- sub-ohmic at low frequencies \( \omega \sim \omega_0 \)
- inverse power law for \( \omega \gg \omega_0 \) (similar to Lorentz)

\[
J_\alpha(\omega) \approx 2A/a^2(\omega - \omega_0)^\alpha \text{ for } \omega \rightarrow \omega_0^+
\]

\[
J_\alpha(\omega) \approx 2A\omega^{\alpha-2}, \text{ for } \omega \rightarrow +\infty
\]
Lorentzian type and PBG spectral densities

Figure: Various forms of spectral densities. The curve \((LP)\) represents \(\tilde{J}_{L+}(\omega)\), the sum of two Lorentzians; \((LM)\) is \(\tilde{J}_{L-}(\omega)\), the difference of two Lorentzians with PBG in the resonance frequency; \((L4)\) represents \(\tilde{J}_{L'}(\omega)\) while \((E)\) represents \(J_E(\omega)\) with a PBG.
Exact dynamics of the qubit

Exact density matrix evolution:

$$\rho_{1,1}(t) = \rho_{1,1}(0) \left| G_\alpha(t) \right|^2, \quad \rho_{1,0}(t) = \rho_{0,1}^*(t) = \rho_{1,0}(0) e^{-i\omega_0 t} G_\alpha(t)$$

Exact result

$$G_\alpha(t) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(-1)^n z_\alpha^k z_0^{n-k} t^{3n-\alpha k}}{k!(n-k)!}$$

$$\times \left( H_{1,2}^{1,1} \left[ z_1 t^2 \left| (-n, 1) \right. \right. \\ \left. \left. (0, 1), (\alpha k - 3n, 2) \right] \right)$$

$$- a^2 t^2 H_{1,2}^{1,1} \left[ z_1 t^2 \left| (-n, 1) \right. \right. \\ \left. \left. (0, 1), (\alpha k - 3n - 2, 2) \right] \right)$$
The special case $\alpha = 1/2$ and the Eulerian dynamics (1)

$$J_E(\omega) = \frac{2A(\omega - \omega_0)^{1/2} \Theta(\omega - \omega_0)}{a^2 + (\omega - \omega_0)^2}$$

Exact dynamics (linear combination of Euler Incomplete Gamma functions)

$$\rho_{1,1}(t) = 1 - \rho_{0,0}(t) = \rho_{1,1}(0) \left| G_E(t) \right|^2$$
$$\rho_{1,0}(t) = \rho^*_{0,1}(t) = \rho_{1,0}(0) e^{-\omega_0 t} G_E(t)$$
The special case $\alpha = 1/2$ and the Eulerian dynamics (2)

$$G_E(t) = \frac{1}{\sqrt{\pi}} \sum_{l=1}^{4} R(z_l) z_l e^{z_l^2 t} \Gamma \left(1/2, z_l^2 t\right)$$

where

$$R(z) = \frac{(1 - \iota) \left(a^{1/2} + z\right) \left(\iota a^{1/2} + z\right)}{2z \left((1 + \iota) a + 3a^{1/2}z + 2(1 - \iota)z^2\right)}$$

- $z_1, z_2, z_3, z_4$ roots of

$$Q(z_l) = \pi \sqrt{2/a} A + \iota az_l^2 + (1 + \iota) a^{1/2} z_l^3 + z_l^4 = 0, \quad l = 1, 2, 3, 4$$

Longtime behaviour

Asymptotic expansion identifies

- time scale $\tau$
- Decoherence factor $D$

such that for time scales $t \ll \tau$

$$G(t) \approx Dt^{-3/2}, \quad \text{for} \quad t \to \infty$$

Asymptotic form of $\rho \left( t \to \infty \right)$

$$\rho_{1,1}(t) \approx \rho_{1,1}(0)|D|^2t^{-3}$$
$$\rho_{1,0}(t) \approx \rho_{1,0}(0)\exp(-i\omega_0 t)t^{-3/2}$$
Lorentzian vs Eulerian relaxation

Figure: The time evolution of coherent term, $|\rho_{1,0}(t)|$, for a reservoir, described by either $\tilde{J}_L(\omega)$, both in strong coupling regime (red line) and weak coupling regime (yellow line), or $J_E(\omega)$ (blue line) spectral density function, respectively.
Exponential vs inverse power law

Figure: The relaxation of coherent term, $|\rho_{1,0}(t)|$, over long time scales, $t \gg 1$, $\tau \simeq 0.974$, $\tau_B = 1$ in strong coupling regime, $\tau_B = 0.05$ in weak coupling regime, of the reduced density matrix of a qubit, interacting with a reservoir, described by either $\tilde{J}_L(\omega)$, both in strong coupling regime (red line) and weak coupling regime (yellow line), or $J_E(\omega)$ (blue line) spectral density function.
The general case

\[ J_\alpha(\omega) = \frac{2A(\omega - \omega_0)^\alpha \Theta(\omega - \omega_0)}{a^2 + (\omega - \omega_0)^2}, \quad A > 0, \quad a > 0, \quad 1 > \alpha > 0 \]

Time scale for inverse power law behaviour:

\[ \tau_\alpha = \max \left\{ 1, \left| \frac{3}{z_0} \right|^{1/3}, \frac{3}{z_0}, 3 \left| \frac{z_\alpha}{z_0} \right|^{1/\alpha}, 3 \left| \frac{z_1}{z_0} \right| \right\} \]

where

\[ z_0 = i\pi A a^\alpha \cos \left( \frac{\pi \alpha}{2} \right) \]
\[ z_\alpha = -2i\pi A e^{-i\pi \alpha/2} \csc \left( \frac{\pi \alpha}{2} \right) \]
\[ z_1 = \pi A a^{\alpha-1} \sec \left( \frac{\pi \alpha}{2} \right) - a^2 \]
Towards $1/t$ qubit decoherence

Time scales $t \gg \tau_\alpha$:

$$G_\alpha(t) \sim -D_\alpha \ t^{-1-\alpha}, \quad t \to +\infty, \quad 1 > \alpha > 0$$

where

$$D_\alpha = \frac{2^{-\alpha} a^{2(1-\alpha)} e^{-\pi \alpha/2} \csc(\pi \alpha) \sec^2(\pi \alpha/2)}{\pi A \Gamma(1 - \alpha)}$$

Exact dynamics of the qubit over long time scales

$$\rho_{1,1}(t) = 1 - \rho_{0,0}(t) \sim \rho_{1,1}(0) \ |D_\alpha|^2 \ t^{-2-2\alpha}$$

$$\rho_{1,0}(t) = \rho_{0,1}^*(t) \sim \rho_{1,0}(0) \ D_\alpha \ e^{-i \omega_0 t} \ t^{-1-\alpha}$$

Spontaneous emission of an excited atom

Total Hamiltonian: $H = H_A + H_E + H_I$

$$H_A = \omega_0 |1 \rangle_a a \langle 1|,$$
$$H_E = \sum_{k=1}^{\infty} \omega_k b_k^\dagger b_k,$$
$$H_I = \gamma \sum_{k=1}^{\infty} g_k \left( b_k^\dagger \otimes |0 \rangle_a a \langle 1| - b_k \otimes |1 \rangle_a a \langle 0| \right).$$

Initial state of the system

$$|\psi(0)\rangle = |1 \rangle_a \otimes |0 \rangle_E$$
Time evolution of the population

The case $\alpha = 1/2$

$$P(t) = \frac{1}{\pi} \left| \sum_{l=1}^{4} \chi_l R(\chi_l) e^{\chi_l^2 t} \Gamma \left(1/2, \chi_l^2 t \right) \right|^2.$$
Time evolution of the population (2)

The general case

\[ P_\alpha(t) = \left| \sum_{n=0}^{\infty} \sum_{k=0}^{n} (-1)^n \frac{z_\alpha^n z_0^{n-k} t^{3n-\alpha k}}{k! (n-k)!} \left( H_{1,2}^{1,1} \left| z_1 t^2 \right| \begin{array}{c} (-n, 1) \\ (0, 1), (\alpha k - 3n, 2) \end{array} \right) \right|^2 \]

\[-a^2 t^2 H_{1,2}^{1,1} \left| z_1 t^2 \right| \begin{array}{c} (-n, 1) \\ (0, 1), (\alpha k - 3n - 2, 2) \end{array} \right) \right|^2 \]

For \( t \gg \tau_\alpha \)

\[ P_\alpha(t) \sim \zeta_\alpha t^{-2(1+\alpha)}, \quad t \to +\infty, \quad 1 > \alpha > 0, \]

where

\[ \zeta_\alpha = \frac{4 \alpha^2 a^{4(1-\alpha)} \csc^2 (\pi \alpha) \sec^4 (\pi \alpha/2)}{\pi^2 A^2 (\Gamma (1 - \alpha))^2}. \]
Spontaneous emission of an excited TLA in the presence of N-1 TLAs in the ground state

The Dicke model

\[ H_N = \sum_{k=1}^{\infty} (\omega_k - \omega_0) b_k^\dagger b_k + i \sum_{k=1}^{\infty} g_k \left( J_{1,0} b_k^\dagger - J_{0,1} b_k \right) \]

where

- \( J_{l,m} = \sum_{n=1}^{N} |l\rangle \langle n| |n\rangle \langle m|, \quad l, m = 0, 1, \)
- \( J^2 = J_3^2 + (J_{2,1} J_{1,2} + J_{1,2} J_{2,1}) / 2 \)
- \( J_3 = (J_{2,2} - J_{1,1}) / 2 \)

The superradiant states (initial condition): \( |J, M = 1 - J\rangle \)

The exact decay

\[ P_{N, \alpha}(t) = \left| \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(-N)^n z^k z_0^{n-k} t^{3n-\alpha k}}{k!(n-k)!} \right| \]

\[ \times \left( H_{1,2}^{1,1} \left| z_{N,1} t^2 \right| (-n, 1) \right) \left( (0, 1), (\alpha k - 3n, 2) \right) \]

\[ - a^2 t^2 H_{1,2}^{1,1} \left| z_{N,1} t^2 \right| (-n, 1) \left( (0, 1), (\alpha k - 3n - 2, 2) \right) \right) \right| ^2 \]

\[ z_{N,1} = \pi A N a^{\alpha-1} \sec \left( \frac{\pi \alpha}{2} \right) - a^2, \quad z_{N,0} = N z_0, \quad z_{N, \alpha} = N z_\alpha \]
Time scales and critical number of atoms for inverse power laws

The long time scale:

\[ t \gg \tau_{N, \alpha}, \quad \tau_{N, \alpha} = \max \left\{ 1, \left| \frac{3}{Z_{N, 0}} \right|^{1/3}, 3 \left| \frac{z_{\alpha}}{z_0} \right|^{1/\alpha}, 3 \left| \frac{Z_N, 1}{Z_{N, 0}} \right| \right\} \]

\[ P_{N, \alpha}(t) \sim \zeta_{N, \alpha} t^{-2(1+\alpha)}, \quad 1 > \alpha > 0 \]

\[ \zeta_{N, \alpha} = \frac{4 \alpha^2 a^{4(1-\alpha)} \csc^2 (\pi \alpha) \sec^4 (\pi \alpha/2)}{\pi^2 A^2 N^2 (\Gamma (1 - \alpha))^2} \]

\[ N \gg N_{\alpha}^{(*)} \Rightarrow \zeta_{N, \alpha} \ll 1, \quad N_{\alpha}^{(*)} = \left[ \frac{2 \alpha a^{2(1-\alpha)} \csc (\pi \alpha) \sec^2 (\pi \alpha/2)}{\pi A \Gamma (1 - \alpha)} \right] \]

Parameters: $a = 20$, $A = 1/3$

- K6: $N = 2$
- K5: $N = 7$
- K4: $N = 30$
- K3: $N = 50$
- K2: $N = 90$
- K1: $N = 1000$

Suppression of trapping for large $N$

Critical number: $N_{1/2}^* = 21$
Thank you for your attention!

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