Evolution of entanglement of two qubits with a model of repeated interactions

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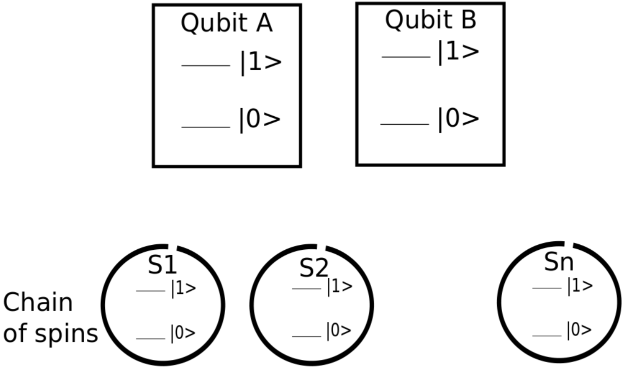
Model : 2 qubits in conctact with a heat bath consisting in an infinite chain of spins at thermal equilibrium at inverse temperature β.



More precisely :

- $h_0^{qubits} = \epsilon[\sigma_+\sigma_-^{(A)} + \sigma_+\sigma_-^{(B)}]$ is the free hamiltonian for qubits.
- $h_0^{chain} = \sum_{k=1} \epsilon a^\dagger a(k)$ is the free hamiltonian for the chain.
- During time $[(k-1)\tau, (k-1)\tau + \frac{\tau}{2}[$, qubit A interacts with spin Sk for $k \in \mathbb{N}$. The interaction is given by $W^{(A)} = \lambda \left[\sigma_+^{(A)} \otimes a(k) + \sigma_-^{(A)} \otimes a^\dagger(k)\right]$.
- Then, qubit B interacts with the same spin Sk during time $[(k-1)\tau + \frac{\tau}{2}, k\tau[$ with a similar interaction $W^{(B)}$.

Weak coupling limit : As Attal and Joye in Journal of Statistical Physics , **126** (2007), we take the limit :

- $T = \tau \frac{t}{\lambda^2}$ with $\frac{t}{\lambda^2} \in \mathbb{N}$ is the number of interactions and T is the macroscopic time scale
- $\frac{\tau}{2} = 1$
- $\lambda \to 0$

Initial state : We consider that at time $T = 0$, the system of two qubits is initially in an X-state $\rho(0)$, represented in the basis $\{|11>, |10>, |01>, |00>\}$ by :

$$\rho(0) = \begin{pmatrix} a & 0 & 0 & y \\ 0 & b & x & 0 \\ 0 & \bar{x} & c & 0 \\ \bar{y} & 0 & 0 & d \end{pmatrix} \qquad (1)$$

Furthermore, we suppose that the chain of spins is initially in a thermal state at inverse temperature β.

In the limit above and with $\beta = \infty$, we have asymptotically ($T = \tau \frac{t}{\lambda^2} \to \infty$) :

$$\rho(t) = \begin{pmatrix} a(t) & 0 & 0 & y(t) \\ 0 & b(t) & x(t) & 0 \\ 0 & \overline{x(t)} & c(t) & 0 \\ \overline{y(t)} & 0 & 0 & d(t) \end{pmatrix} \qquad (2)$$

with :

- $a(t) = e^{-2t}a(0)$
- $b(t) = [-e^{-2t} + e^{-t}]a(0) + e^{-t}b(0)$
- $c(t) = [-5e^{-2t} + (5 - 4t + t^2)e^{-t}]a(0) + t^2e^{-t}b(0) + e^{-t}c(0) + (-te^{-t}(x + \bar{x})(0))$
- $d(t) = [5e^{-2t} + (-6 + 4t - t^2)e^{-t} + 1]a(0) + [(-1 - t^2)e^{-t} + 1]b(0) + (1 - e^{-t})c(0) + d(0) + te^{-t}(x + \bar{x})(0)$
- $|y(t)| = e^{-t}|y(0)|$
- $|x(t)| = \left|(-2e^{-2t} + (2 - t)e^{-t})a(0) + (-te^{-t})b(0) + e^{-t}x(0)\right|$

In the case of $\beta = 0$, we have the same structure for the density matrix but :

- $a(t) = s_1(t)a(0) + s_3(t)[b(0) + c(0) + (x + \bar{x})(0)] + s_2(t)d(0)$
- $b(t) = s_3(t)a(0) + s_1(t)b(0) - s_3(t)(x + \bar{x})(0)] + s_2(t)c(0) + s_3 d(0)$
- $c(t) = s_3(t)a(0) + s_2(t)b(0) - s_3(t)(x + \bar{x})(0)] + s_1(t)c(0) + s_3 d(0)$
- $d(t) = s_2(t)a(0) + s_3(t)[b(0) + c(0) + (x + \bar{x})(0)] + s_1 d(0)$
- $|x(t)| = |s_3(t)[a(0) + d(0) - (b(0) + c(0))] + s_1(t)x(0) + s_2(t)\bar{x}(0)|$
- $|y(t)| = e^{-t}|y(0)|$

with :

- $s_1(t) = \frac{1}{3} + \frac{1}{2}e^{-t} + \frac{1}{6}e^{-3t}$
- $s_2(t) = \frac{1}{3} - \frac{1}{2}e^{-t} + \frac{1}{6}e^{-3t}$
- $s_3(t) = \frac{1}{6}(1 - e^{-3t})$

Concurrence for an X-state : We measure entanglement via the concurrence C of Wooters. In the case of an X-state, we have a beutiful formula for the concurrence :

$$C(\rho) = 2\max\{0, |x| - \sqrt{ad}, |y| - \sqrt{bc}\} \tag{3}$$

So it is easy to calculate...
we have done it and we find the following results :

for $\beta = \infty$

We can vary the parameter t and see the evolution of $C(t)$:

Proposition

For $\beta = \infty$ and for an inital X-state

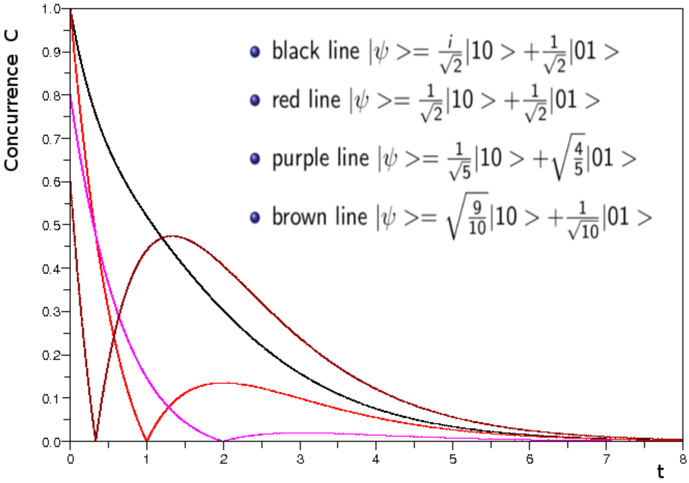
- *if $a + b > 0$ then $C(t) \underset{t \to \infty}{\sim} 2(a+b)te^{-t}$.*
- *Else $C(t) \underset{t \to \infty}{\sim} 2\max\{0, |x|, |y|\}e^{-t}$ (here $(x, y) \neq (0, 0)$).*

A curious phenomenon :

Proposition

We suppose that initially $\rho(0) = |\psi >< \psi|$ *where* $|\psi >= \alpha|10 > +\beta|01 >$ *then*

- *if* $\arg\alpha = \arg\beta$ *then* $C(\frac{\beta}{\alpha}) = 0$ *and* $C(t) > 0$ *for* $t \neq \frac{\beta}{\alpha}$*.*
- *else* $C(t) > 0$ *for all* t*.*



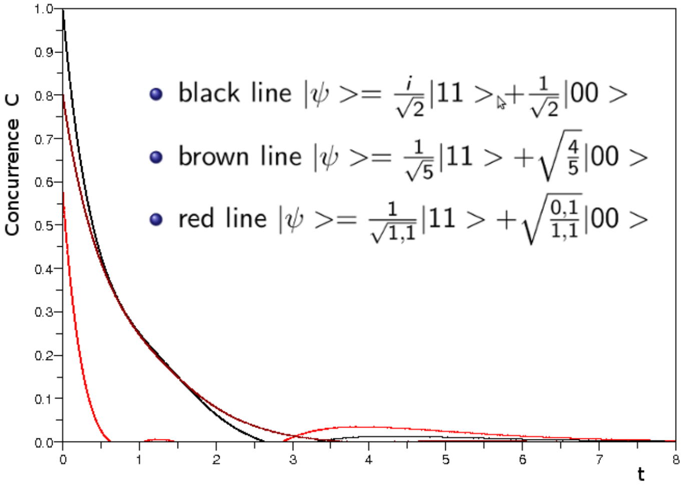
Concurrence C
t
black line $|\psi> = \frac{i}{\sqrt{2}}|10> + \frac{1}{\sqrt{2}}|01>$
red line $|\psi> = \frac{1}{\sqrt{2}}|10> + \frac{1}{\sqrt{2}}|01>$
purple line $|\psi> = \frac{1}{\sqrt{5}}|10> + \sqrt{\frac{4}{5}}|01>$
brown line $|\psi> = \sqrt{\frac{9}{10}}|10> + \frac{1}{\sqrt{10}}|01>$

An other :

Proposition

We suppose that initially $\rho(0) = |\psi >< \psi|$ *where* $|\psi >= \alpha|11 > +\beta|00 >$ *then there exists two constants* $c_1 \simeq 1.093$ *and* $c_2 \simeq 1.106$ *such as :*

- *if* $c_1 \leq \frac{1}{|\alpha|^2} \leq c_2$ *then there exists two steps such as* $C(t) = 0$.
- *else there exists only one step such as* $C(t) = 0$.


black line $|\psi>= \frac{i}{\sqrt{2}}|11> + \frac{1}{\sqrt{2}}|00>$
brown line $|\psi>= \frac{1}{\sqrt{5}}|11> + \sqrt{\frac{4}{5}}|00>$
red line $|\psi>= \frac{1}{\sqrt{1,1}}|11> + \sqrt{\frac{0,1}{1,1}}|00>$
Concurrence C
t
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
0
1
2
3
4
5
6
7
8

For $\beta = 0$

Proposition

For $\beta = 0$ and for an inital X-state

- *if* $(b+c)-(x+\bar{x}) < 1$*, there exists* $t_0 < \infty$ *such as* $C(t) = 0$ *for all* $t \geq t_0$.
- *if* $(b+c)-(x+\bar{x}) > 1$ *then* $C(t \to \infty) = (b+c)-(x+\bar{x}) - 1 > 0$.
- *if* $\left\{ \begin{array}{c} (b+c)-(x+\bar{x})=1 \\ a=d \quad \text{and} \quad \Im(\text{x})=0 \end{array} \right.$ *there exists* $t_0 < \infty$ *such as* $C(t) = 0$ *for all* $t \geq t_0$.
- *if* $\left\{ \begin{array}{c} (b+c)-(x+\bar{x})=1 \\ a\neq d \quad \text{or} \quad \Im\text{x}\neq 0 \end{array} \right.$ *then* $C(t) \underset{t\to\infty}{\sim} 3[(\Im x)^2 + \frac{1}{2}(a-d)^2]e^{-2t}$.

A bit more on $\mathcal{E}_{\mathcal{X}}$, the set of X-states :

Proposition

$\mathcal{E}_{\mathcal{X}}$ is a convex compact set of 7 real dimensions.

Due to Krein-Milman theorem, we know that $\mathcal{E}_{\mathcal{X}}$ is the convex hull of its extreme points.

Proposition

The extreme points of $\mathcal{E}_{\mathcal{X}}$ are the Bell states. Id est the states :

- $|\psi >< \psi|$ *for* $|\psi >= \alpha|11 > +\delta|00 >$ *(*$\alpha, \delta \in \mathbb{C}$*)*
- $|\phi >< \phi|$ *for* $|\phi >= \beta|10 > +\gamma|01 >$ *(*$\beta, \gamma \in \mathbb{C}$*)*

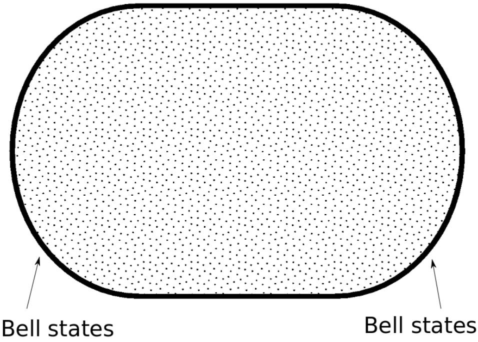


Figure: A view of $\mathcal{E}_X$

Proposition

The concurrence function, $C : \begin{smallmatrix} \mathcal{E}_{\mathcal{X}} \to [0,1] \\ \rho \mapsto C(\rho) \end{smallmatrix}$ *is convex.*

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- W.K. Wootters, Phys. Rev. Lett. **80**, 2245 (1998)
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