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Open Quantum Systems

Maison Jean Kuntzmann - 29 novembre au 02 décembre 2010

Lieb-Robinson Bounds and Construction of Dynamics

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Based on the paper in RMP 22(2010)207-331 by B.Nachtergaele, B.Schlein, R.Sims,Sh.Starr and VZ =[0]

1.Motivation: Two Ways to Infinite W^*-Dynamical Systems

• The traditional way is to first define the dynamics of anharmonic *quantum lattice* in **finite volume** (which can be done by standard means), and then studying the limit in which the volume tends to infinity [Amour,Levy-Bruhl,Nourrigat (2010)] =[1].

• We follow a *different* approach. The main difference is that we study the thermodynamic limit of *anharmonic perturbations* of an **infinite harmonic lattice** system described by an explicit W^*-dynamical system.

(a) It appears that controlling the continuity of the limiting dynamics is more straightforward in our approach and we are able to show that the resulting dynamics for the class of anharmonic lattices that we study is indeed *weakly continuous* and we obtain a W^*-dynamical system for the **infinite system.**

(b) Common to both approaches, ours [0] and [1], is the crucial role of an estimate the speed of *propagation* of *perturbations* in the system, commonly referred to as Lieb-Robinson bounds (1972).

• **Recall:** Let A and B be two observables of a spatially extended system, localized in regions X and Y, respectively, and τ_t denotes the time evolution of the system then, a *Lieb-Robinson bound* is an estimate of the form:

$$\|[\tau_t(A), B]\| \leq Ce^{-a(d(X,Y)-v|t|)} ,$$

where C, a, and v are positive constants and $d(X,Y)$ denotes the distance between X and Y.

(c) The Lieb-Robinson bounds for *anharmonic lattice systems* were recently proved in by [Nachtergaele,Raz,Schlein,Sims (2009)] =[2].

2.From Lieb-Robinson Bounds to Infinite Dynamics

• To each x in *lattice* Γ, we associate a Hilbert space $\mathcal{H}_x$. In many relevant systems, one considers $\mathcal{H}_x = L^2(\mathbb{R}, dq_x)$. Then $\mathcal{H}_\Lambda = \bigotimes_{x\in\Lambda}\mathcal{H}_x$ for finite subset $\Lambda \subset \Gamma$ and the local algebra of observables over Λ is $\mathcal{A}_\Lambda = \bigotimes_{x\in\Lambda}\mathcal{B}(\mathcal{H}_x)$ where $\mathcal{B}(\mathcal{H}_x)$ denotes the algebra of bounded linear operators on $\mathcal{H}_x$.

• Let local Hamiltonians $H^{\mathrm{loc}} := \{H_x\}_{x\in\Gamma}$, here H_x are *on-site* self-adjoint operators in $\mathcal{H}_x$, and interactions $\Phi(X) \in \mathcal{A}_X$. We consider self-adjoint Hamiltonians:

$$H_\Lambda = H_\Lambda^{\mathrm{loc}} + H_\Lambda^{\Phi} = \sum_{x\in\Lambda} H_x + \sum_{X\subset\Lambda}\Phi(X),$$

with domain $\bigotimes_{x\in\Lambda} D(H_x)$. They generate a dynamics $\{\tau_t^\Lambda\}$, which is the one parameter group of automorphisms defined by

$$\tau_t^\Lambda(A) = e^{itH_\Lambda}\, A\, e^{-itH_\Lambda} \quad \text{for any} \quad A \in \mathcal{A}_\Lambda.$$

- Consider the unitary propagator

$$\mathcal{U}_\Lambda(t,s) = e^{itH_\Lambda^{\mathrm{loc}}}\, e^{-i(t-s)H_\Lambda}\, e^{-isH_\Lambda^{\mathrm{loc}}}$$

and its associated *interaction-picture* evolution defined by

$$\tau_{t,\mathrm{int}}^{\Lambda}(A) = \mathcal{U}_\Lambda(0,t)\, A\, \mathcal{U}_\Lambda(t,0) \quad \text{for all } A \in \mathcal{A}_\Gamma\,.$$

- Then for $n \leq m$ with $X \subset \Lambda_n \subset \Lambda_m$ one gets

$$\tau_{t,\mathrm{int}}^{\Lambda_m}(A) - \tau_{t,\mathrm{int}}^{\Lambda_n}(A) = \int_0^t \frac{d}{ds}\left\{\mathcal{U}_{\Lambda_m}(0,s)\,\mathcal{U}_{\Lambda_n}(s,t)\, A\, \mathcal{U}_{\Lambda_n}(t,s)\,\mathcal{U}_{\Lambda_m}(s,0)\right\} ds\,.$$

- Let $A \in \mathcal{A}_X$. Then with help of the operators:

$$\tilde{A}(t) = e^{-itH_{\Lambda_n}^{\mathrm{loc}}}\, A\, e^{itH_{\Lambda_n}^{\mathrm{loc}}} = e^{-itH_X^{\mathrm{loc}}}\, A\, e^{itH_X^{\mathrm{loc}}}$$

$$\tilde{B}(s) = e^{-isH_{\Lambda_n}^{\mathrm{loc}}}\left(H_{\Lambda_m}^{\mathrm{int}}(s) - H_{\Lambda_n}^{\mathrm{int}}(s)\right) e^{isH_{\Lambda_n}^{\mathrm{loc}}}$$

- One obtains the estimate:

$$\left\| \tau_{t,\mathrm{int}}^{\Lambda_m}(A) - \tau_{t,\mathrm{int}}^{\Lambda_n}(A) \right\| \le \int_0^t \left\| \left[\tau_{s-t}^{\Lambda_n}\left(\tilde{A}(t)\right), \tilde{B}(s) \right] \right\| \, ds \ .$$

- Application of the Lieb-Robinson bound [2] implies that the sequence $\{\tau_{t,\mathrm{int}}^{\Lambda_n}(A)\}$ is Cauchy in norm, uniformly for $t \in [-T,T]$:

$$\sup_{t\in[-T,T]} \left\| \tau_{t,\mathrm{int}}^{\Lambda_m}(A) - \tau_{t,\mathrm{int}}^{\Lambda_n}(A) \right\| \to 0 \qquad \text{as } n,m \to \infty.$$

- Since $\left(e^{it\sum_{x\in X} H_x}\, A\, e^{-it\sum_{x\in X} H_x}\right)$ is *localised* in X and

$$\tau_t^{\Lambda}(A) = \tau_{t,\mathrm{int}}^{\Lambda}\left(e^{itH_\Lambda^{\mathrm{loc}}}\, A\, e^{-itH_\Lambda^{\mathrm{loc}}}\right) = \tau_{t,\mathrm{int}}^{\Lambda}\left(e^{it\sum_{x\in X} H_x}\, A\, e^{-it\sum_{x\in X} H_x}\right) ,$$

an analogous statement then follows for $\{\tau_t^{\Lambda_n}(A)\}$.

• If all local Hamiltonians H_x are *bounded*, $\{\tau_t\}$ is strongly continuous. If the H_x are allowed to be densely defined unbounded self-adjoint operators, we only have weak continuity and the dynamics is more naturally defined on a von Neumann algebra.

• **Theorem.** Under the conditions stated above, for all $t \in \mathbb{R}$, $A \in \mathcal{A}_\Gamma$, the norm limit

$$\lim_{\Lambda \to \Gamma} \tau_t^\Lambda(A) = \tau_t(A)$$

exists in the sense of non-decreasing exhaustive sequences of finite volumes Λ and defines a group of $*-$automorphisms τ_t on the completion of $\mathcal{A}_\Gamma$. The convergence is uniform for t in a compact set.

3.Dynamics of a Harmonic Lattice

• Consider a system of *harmonic* oscillators restricted to cubic subsets $\Lambda_L = (-L, L]^d \subset \mathbb{Z}^d$, with harmonic *n.n.* couplings:

$$H_L^h = \sum_{x \in \Lambda_L} p_x^2 + \omega^2 q_x^2 + \sum_{j=1}^{d} \lambda_j \, (q_x - q_{x+e_j})^2$$

in the Hilbert space $\mathcal{H}_{\Lambda_L} = \bigotimes_{x \in \Lambda_L} L^2(\mathbb{R}, dq_x)$. Here p_x, q_x are **single site** *momentum* and *position* operators satisfying the CCR:

$$[p_x, p_y] = [q_x, q_y] = 0 \quad \text{and} \quad [q_x, p_y] = i\delta_{x,y},$$

valid for all $x, y \in \Lambda_L$, the numbers $\lambda_j \geq 0$ and $\omega \geq 0$ are the parameters of the system, and the Hamiltonian is assumed to have *periodic boundary conditions*. It is well-known that Hamiltonians of this form can be *diagonalized* in the *Fourier space*.

• Using this diagonalization, one can determine the action of the dynamics corresponding to H_L^h on the Weyl algebra $\mathcal{W}(\mathcal{D} = \ell^2(\Lambda_L))$. In fact, by setting

$$W(f) = \exp\left[i \sum_{x \in \Lambda_L} \mathrm{Re}[f(x)] q_x + \mathrm{Im}[f(x)] p_x \right],$$

for each $f \in \ell^2(\Lambda_L)$, and symplectic form $\sigma(f, g) = \mathrm{Im}[\langle f, g \rangle]$.

• Limiting harmonic dynamics is *quasi-free* on $\mathcal{W}(\mathcal{D})$: it is a one-parameter group of *-automorphisms τ_t (Bogoliubov transformations) $\tau_t(W(f)) = W(T_t f), f \in \mathcal{D}$ where $T_t : \mathcal{D} \to \mathcal{D}$ is a group of real-linear, symplectic transformations, $\sigma(T_t f, T_t g) = \sigma(f, g)$.

• As $\|W(f) - W(g)\| = 2$ for all $f \neq g \in \mathcal{D}$, one should not expect τ_t to be strongly continuous.

• In the present context, it suffices to regard a W^*-*dynamical system* as a pair $\{\mathcal{M}, \alpha_t\}$ where $\mathcal{M}$ is a von Neumann algebra and α_t is a weakly continuous, one parameter group of $*$-automorphisms of $\mathcal{M}$.

• For the harmonic systems a specific W^*-dynamical system arises as follows. Let ρ be a state on $\mathcal{W}$ and denote by $(\mathcal{H}_\rho, \pi_\rho, \Omega_\rho)$ the corresponding GNS representation. Assume that ρ is both regular and τ_t-invariant. For the algebra $\mathcal{M}$, take the weak-closure of $\pi_\rho(\mathcal{W})$ in $\mathcal{L}(\mathcal{H}_\rho)$ and let α_t be the weakly continuous, one parameter group of $*$-automorphisms of $\mathcal{M}$ obtained by lifting τ_t to $\mathcal{M}$.

• Lieb-Robinson bounds for harmonic lattice and $f, g \in \ell^2(\Gamma)$:

$$\|[\tau_t(W(f)), W(g)]\| \leq c_a e^{v_a|t|} \sum_{x,y} |f(x)|\,|g(y)|\, F_a\left(d(x,y)\right) .$$

4.On-Site and Multiple-Site Anharmonisities

• Our *first* Lien-Robinson estimate involves perturbations defined as finite sums of *on-site terms*. To each site $x \in \Gamma$, we will associate an element $P_x \in \mathcal{W}(\mathcal{D})$. For $\Lambda \subset \Gamma$ we set $P^\Lambda = \sum_{x\in\Lambda} P_x$, and note that $(P^\Lambda)^* = P^\Lambda \in \mathcal{W}(\mathcal{D})$. We will denote by $\tau_t^{(\Lambda)}$ the dynamics that results from applying *Dyson expansion* to the W^*-dynamical system $\{\mathcal{M}, \tau_t^0\}$ and P^Λ.

• **Theorem:** There exist positive numbers c_a and v_a, for which the estimate

$$\left\| \left[\tau_t^{(\Lambda)} \left(W(f) \right), W(g) \right] \right\| \leq c_a e^{(v_a + c_a \kappa C_a)|t|} \sum_{x,y} |f(x)| \, |g(y)| F_a \left(d(x,y) \right)$$

holds for all $t \in \mathbb{R}$ and for any functions $f, g \in \mathcal{D}$.

• Multiple-site anharmonisity:
For any finite subset $\Lambda \subset \Gamma$, we will set $P^{\Lambda} = \sum_{X \subset \Lambda} P_X$ where the sum is over all subsets of Λ. Here we will again let $\tau_t^{(\Lambda)}$ denote the dynamics resulting from *Dyson expansion* applied to the W^*-dynamical system $\{\mathcal{M}, \tau_t^0\}$ and the perturbation P^{Λ}.
• **Theorem:** There exist positive numbers c_a and v_a for which one has *second* Lien-Robinson estimate

$$\left\| \left[\tau_t^{(\Lambda)} \left(W(f) \right), W(g) \right] \right\| \leq c_a e^{(v_a + c_a \kappa_a C_a^2)|t|} \sum_{x,y} |f(x)| \, |g(y)| F_a \left(d(x,y) \right) \; ,$$

for all $t \in \mathbb{R}$ and for any functions $f, g \in \mathcal{D}$.

•**Proposition (Dyson expansion):** Let $\{\mathcal{M}, \alpha_t\}$ be a W^*-dynamical system and let δ denote the infinitesimal generator of α_t. Given any $P = P^* \in \mathcal{M}$, set δ_P to be the bounded derivation with domain $D(\delta_P) = \mathcal{M}$ satisfying $\delta_P(A) = i[P, A]$ for all $A \in \mathcal{M}$. It follows that $\delta + \delta_P$ generates a one-parameter group of $*$-automorphisms α^P of $\mathcal{M}$ which is the unique solution of the integral equation

$$\alpha_t^P(A) = \alpha_t(A) + i \int_0^t \alpha_s^P \left([P, \alpha_{t-s}(A)]\right) ds\,.$$

In addition, the estimate $\left\|\alpha_t^P(A) - \alpha_t(A)\right\| \leq \left(e^{|t|\|P\|} - 1\right) \|A\|$ holds for all $t \in \mathbb{R}$ and $A \in \mathcal{M}$.

•Since the initial dynamics α_t is weakly continuous, one can show that the perturbed dynamics is also weakly continuous. Hence, for $P = P^* \in \mathcal{M}$ the pair $\{\mathcal{M}, \alpha_t^P\}$ is also a W^*-dynamical system.

• **Theorem (Existence of the dynamics):**
Let τ_t^0 be a harmonic dynamics defined on $\mathcal{W}(\ell^1(\Gamma))$. Let $\{\Lambda_n\}$ denote a non-decreasing, exhaustive sequence of finite subsets of Γ. Consider a family of perturbations P^{Λ_n}. Then, for each $f \in \ell^1(\Gamma)$ and $t \in \mathbb{R}$ fixed, the limit

$$\lim_{n \to \infty} \tau_t^{(\Lambda_n)}(W(f))$$

exists in norm. The limiting dynamics, which we denote by τ_t, is weakly continuous.

• By an $\epsilon/3$ argument, weak continuity follows since we know that it holds for the finite volume dynamics. This completes the proof of Theorem.

THANK YOU !