Spectral Transition for Random Quantum Walks on Trees

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Abstract

We define and analyze random quantum walks on homogeneous trees of degree $q \geq 3$. Such walks describe the discrete time evolution of a quantum particle with internal degree of freedom in $\mathbb{C}^q$ hopping on the neighboring sites of the tree in presence of static disorder. The one time step random unitary evolution operator on the Hilbert space of the particle depends on a unitary matrix $C \in U(q)$ which monitors the strength of the disorder. We prove for any q that there exist distinct open sets of matrices in $U(q)$ for which the random evolution is either pure point almost surely or absolutely continuous, thereby showing the existence of a spectral transition driven by $C \in U(q)$. For $q = 3$ and $q = 4$, we establish some properties of the spectral diagram which allows us to describe the spectral transition.

1 Introduction

Quantum walks in their various guise have become a popular research topic in the recent years due to the role they play in several different fields, [ADZ, Ke, M, V-A]. They are typically defined as discrete time quantum dynamical systems characterized by a unitary operator on the Hilbert space of a particle with internal degree of freedom on $\mathbb{Z}^d$, with the proviso that neighboring sites of the lattice only are coupled by the unitary operator. Quantum walks are used to approximate the dynamics of certain quantum systems in appropriate regimes. For example, the Chalker-Coddington model [CC, KOK] describes the dynamics of an electron in a two dimensional random background potential submitted to a large perpendicular magnetic field in terms of quantum walk. Also, atoms trapped in some time dependent optical lattices or ions caught in suitably tuned magnetic Paul traps display a dynamics which, in certain regimes, is experimentally well approximated by a simple quantum walk [K et al, Z et al]. Recent quantum optics experiments studying the propagation of polarized photons in networks of three dimensional waveguides acting as beam splitters and random phase shifters provide another implementation of random quantum walks on certain graphs [S et al].

In the quantum computing community, the algorithmic simplicity of quantum walks provides them with a distinguished role. They are used as tools assessing the probabilistic efficiency of elaborated quantum search algorithms to be implemented on quantum computers, in the same way classical random walks are used in theoretical computing. They

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also make up building blocks in the elaboration of such algorithms, see e.g. [S, MNRS]. Finally, quantum walks can be viewed as quantum extensions of classical random walks when supplemented with the probabilistic interpretation of quantum mechanics. As such, they became a topic in probability theory, displaying unusual transport properties, see e.g. [Ko2, CGMV].

Depending on the framework, several variants of quantum walks are considered: the underlying lattice can be replaced by general graphs [Ke, AAKV], completely positive maps can be used to extend the unitary setup [AAKV, Gu, APSS], the stationarity assumption can be relaxed allowing one to deal with genuinely time dependent walks [AVWW, J2, HJ] or the deterministic framework can be enlarged to accommodate random evolution operators from a set of unitary operators [CC, KLMW, J4]. The latter are called *random quantum walks* and they describe the motion of a quantum walker in a static random environment. The popularity of quantum walks in the situations described above certainly lies in the flexibility they provide in modeling and in their structure which allows for detailed, yet non trivial, mathematical analysis of their transport and spectral properties.

The present paper is devoted to the definition and analysis of random quantum walks describing the dynamics of a quantum particle with internal degree of freedom hopping on homogeneous trees of degree q, $q \geq 3$, in a static random environment. The internal degree of freedom, or coin state, lives in $\mathbb{C}^q$. The deterministic part of the walk is given by a so-called *coined quantum walk* defined as follows: the one time step unitary evolution $U(C)$ is obtained by the action of a unitary matrix $C \in U(q)$ on the coin state of the particle, followed by the action of a coin state conditioned shift S which moves the particle to its nearest neighbors on the tree. Then, static disorder is introduced in the model via *i.i.d. random phases* used to decorate the coin matrix C in such a way that the unitary coin state update becomes *site-dependent on the tree* and random. The coin matrix C is regarded as a parameter of the resulting random unitary operator $U_\omega(C)$, see the precise definition in the next section. Let us emphasize that our definition of quantum walks on infinite trees differs from those available in the literature, see *e.g.* [CHKS, D et al], in that the repeated action of the coin state conditioned shift S alone actually makes the quantum walker propagate on the tree.

We provide an analysis of the spectrum of the random evolution $U_\omega(C)$ as a function of C which, in analogy with the self-adjoint Anderson model, we consider as a $U(q)$ valued parameter monitoring the *strength of the disorder*. Randomness is expected to induce destructive interferences that lead in certain regimes to complete suppression of transport and to pure point spectrum, due to Anderson localization [Ki, St]. Accordingly, in a language borrowed from the analysis of the Anderson model, spectral and dynamical localization have been proven to hold for random quantum walks analogous to $U_\omega(C)$ defined on $\mathbb{Z}^d$, in a large disorder regime and at the band edges for arbitrary disorder strength for $d \geq 2$, and for any disorder when $d = 1$. See [J1, HJS1, HJS2, ABJ, JM, ASW, J3]. For random quantum walks on trees, spectral *delocalization* at weak disorder and spectral *localization* at large disorder are expected, by analogy with the self-adjoint case. For the Anderson model on the Bethe lattice, this spectral transition is a well known physical and mathematical fact, the detailed analysis of which is the object of ongoing investigations, see e.g. [A-CAT, Kl, AW1, AW2].

We show that a similar picture holds for random quantum walks on trees of degree q: the spectral properties of $U_\omega(C)$ depend crucially on the parameter $C \in U(q)$ which is proven, for any q, to determine regimes of spectral localization and spectral delocalization for $U_\omega(C)$. In other words, a spectral transition driven by C takes place. Moreover, for $q = 3, 4$, we discuss the salient features of the corresponding spectral diagram and describe the spectral transition.

Our first set of results state that for any $q \geq 3$, that there exist sets with non empty interiors $\mathcal{L} \subset U(q)$, respectively $\mathcal{D} \subset U(q)$, such that $C \in \mathcal{L}$ implies that $U_\omega(C)$ is pure point, almost surely, see Theorem 4.1 and Corollary 4.4 whereas $C \in \mathcal{D}$ implies that $U_\omega(C)$ is purely absolutely continuous for any realization of the static disorder, see Propositions 5.1 and 5.3. The set $\mathcal{D}$ consisting in *delocalizing coin matrices* characterizes the weak disorder regime, whereas the set $\mathcal{L}$ consisting in *localizing coin matrices* characterizing the large disorder regime. In Section 3, we further exhibit special families of coin matrices in $\mathcal{D}$ and in $\mathcal{L}$, see Lemmas 3.1 and 3.3 respectively, as well as coin matrices C which belong to the set $\mathcal{M} \subset U(q)$ of matrices giving rise to *mixed spectra* for the corresponding random quantum walk operator $U_\omega(C)$.

Our second set of results concerns the spectral transition. Since $U(q)$ is compact and connected, a spectral transition must take place somewhere along any continuous path in $U(q)$ between elements of $\mathcal{L}$ and $\mathcal{D}$. When $q = 3$ and $q = 4$, we consider in Section 6 specific families of coin matrices and study the spectral properties of the corresponding random quantum walk. These families give rise to continuous paths in $U(q)$ between elements of $\mathcal{L}$ and $\mathcal{D}$ along which we provide a complete description of the localization-delocalization transition. For $q = 3$, the transition between the two regimes takes place at a specific coin matrix in $\mathcal{M}$, which gives rise to mixed spectra. The corresponding spectral diagram is illustrated in Figure 5. Whereas for $q = 4$, the transition takes place away from $\mathcal{M}$ and the spectral diagram is somehow richer. See Figure 7 for an illustration.

The paper is organized as follows. The next section provides the definitions of our coined quantum walks on trees and of their random version, followed by a description of the spectral criteria suited to the present framework. Section 3 introduces special families of coin matrices, in particular permutation matrices which play a major role later on, and analyzes the corresponding random quantum walk operators. Our main general results about large disorder localization and weak disorder delocalization are stated in the next two sections. Localization is proven by means of the fractional moments method in Section 4, whereas delocalization is a consequence of dynamical spectral criteria described in Section 5. Finally, Section 6 is devoted to a detailed analysis of the spectral transition in the cases $q = 3, 4$.

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2 General Setup

2.1 Deterministic Quantum Walks

Let $\mathcal{T}_q$ be a homogeneous tree of degree $q \geq 3$. If q is even, we will consider $\mathcal{T}_q$ as the tree corresponding to the free group generated by

$$A_q = \{a_1, a_2, \dots, a_q\} \equiv \{a_1, a_2, \dots, a_{q/2}, a_1^{-1}, a_2^{-1}, \dots, a_{q/2}^{-1}\} \tag{1}$$

with $a_j a_j^{-1} = a_j^{-1} a_j = e$, e being the neutral element of the group, see Figure (1) for $q = 4$. If q is odd, $\mathcal{T}_q$ is considered as the tree generated by

$$A_q = \{a_1, a_2, \dots, a_q\} \text{ such that } a_j^2 = e. \tag{2}$$

We choose a vertex of $\mathcal{T}_q$ to be the root of the tree, denoted by e. Each vertex $x = x_1 x_2 \dots x_n$, $n \in \mathbb{N}$ of $\mathcal{T}_q$ is a reduced word made of finitely many letters from the alphabet A_q. An edge of $\mathcal{T}_q$ consists in a pair of vertices (x, y) such that $xy^{-1} \in A_q$. This last relation defines nearest neighbors in $\mathcal{T}_q$ so that the number of nearest neighbors of any vertex is q. Any pair of vertices x and y can be joined by a unique set of edges, or path in $\mathcal{T}_q$. The distance $|x|$ of a vertex $x = x_1 x_2 \dots x_n$ to the root is n and we denote by $d(x, y)$ the distance between two arbitrary vertices.

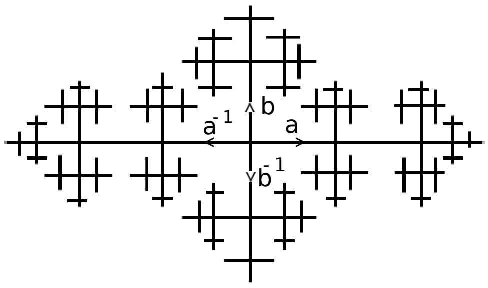

Figure 1: construction of $\mathcal{T}_4$

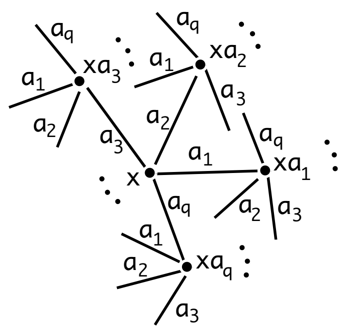

Figure 2: Construction of $\mathcal{T}_q$, with q odd.

When q is odd, the edges going away from a vertex x are labeled as in Figure 2: Given the order $A_q = \{a_1, a_2, \dots, a_q\}$, the sequence of nearest neighbors of x, xa_j, for $j = 1, \dots, q$ are ordered around x in the positive orientation. Then, the nearest neighbors of each xa_j are arranged in the same order in such a way that the edge between xa_j and x corresponds to a_j. Identifying $\mathcal{T}_q$ with its set of vertices, the configuration Hilbert space of the walker is defined as

$$l^2(\mathcal{T}_q) = \Big\{ \psi = \sum_{x \in \mathcal{T}_q} \psi_x |x\rangle \text{ s.t. } \psi_x \in \mathbb{C}, \sum_{x \in \mathcal{T}_q} |\psi_x|^2 < \infty \Big\}, \tag{3}$$

where $|x\rangle$ denotes the element of the canonical basis of $l^2(\mathcal{T}_q)$ which sits at vertex x. The coin Hilbert space of our quantum walker on $\mathcal{T}_q$ is $\mathbb{C}^q$. It allows us to label the elements of the canonical basis of $\mathbb{C}^q$ by means of the letters of the alphabet A_q as $\{|a_j\rangle \in \mathbb{C}^q\}_{j=1,\dots,q}$. Altogether, the total Hilbert space is

$$\mathcal{K}_q = l^2(\mathcal{T}_q) \otimes \mathbb{C}^q \text{ with canonical basis } \{x \otimes a \equiv |x\rangle \otimes |a\rangle, \quad x \in \mathcal{T}_q, a \in A_q\}. \tag{4}$$

Remark 2.1 *The dimension q of the coin space is the smallest choice allowed by the condition that our quantum walk operator couples nearest neighbors on $\mathcal{T}_q$ only.*

The dynamics we consider is defined as the composition of a unitary update of the coin variables in $\mathbb{C}^q$ followed by a coin state dependent shift on the tree. Let $C \in U(q)$, the set of $q \times q$ unitary matrices. The unitary update operator given by $\mathbb{I} \otimes C$ acts on the canonical basis of $\mathcal{K}_q$ as

$$(\mathbb{I} \otimes C) x \otimes a = |x\rangle \otimes |Ca\rangle = \sum_{b \in A_q} C_{ba}\, x \otimes b, \tag{5}$$

where $\{C_{ba}\}_{(b,a)\in A_q^2}$ denote the matrix elements of C.

The definition of the coin state dependent shift S depends on the parity of q.

When q is even, the shift operator S on $\mathcal{K}_q$ it is given by

$$S\, x \otimes a = (xa) \otimes a, \quad x \in \mathcal{T}_q, a \in A_q, \tag{6}$$

with inverse s.t. $S^{-1}\, x \otimes a = (xa^{-1}) \otimes a$. It follows that $S^* = S^{-1}$. Introducing for all $a \in A_q$ the unitary operator S_a acting on $l^2(\mathcal{T}_q)$ as

$$S_a |x\rangle = |xa\rangle, \quad \forall x \in \mathcal{T}_q \tag{7}$$

we can write equivalently

$$S = \sum\nolimits_{a \in A_q} S_a \otimes |a\rangle\langle a| = \sum_{\substack{a \in A_q \\ x \in \mathcal{T}_q}} |xa\rangle\langle x| \otimes |a\rangle\langle a|. \tag{8}$$

That this expression can be considered as a direct sum of shifts stems from the following lemma. Let $\mathcal{H}_x^a$ denotes the S_a-cyclic subspace generated by $|x\rangle$,

$$\mathcal{H}_x^a = \overline{\text{span}}\, \{S_a^n |x\rangle,\ n \in \mathbb{Z}\} \subset l^2(\mathcal{T}_q). \tag{9}$$

By convention, the notation $\overline{\text{span}}$ means the closure of the span of the vectors considered.

Lemma 2.2 *The subspace $\mathcal{H}_x^a$ is isomorphic to $l^2(\mathbb{Z})$ and $S_a|_{\mathcal{H}_x^a}$ is unitarily equivalent to the shift on $l^2(\mathbb{Z})$.*

We define the one step unitary evolution operator on $\mathcal{H} = l^2(\mathcal{T}_q) \otimes \mathbb{C}^q$ for q even by

$$U(C) = S(\mathbb{I} \otimes C), \tag{10}$$

where we consider $C \in U(q)$ as a parameter. We have explicitly

$$U(C) x \otimes a = \sum_{b \in A_q} C_{ba}\, (xb) \otimes b. \tag{11}$$

When q is odd, we construct a shift operator S on $\mathcal{K}_q = l^2(\mathcal{T}_q) \otimes \mathbb{C}^q$ as a direct sum similar to (8) as follows. We start by defining a family of shifts on $l^2(\mathcal{T}_q)$. Let x_e, respectively x_o, denote vertices of even, respectively odd length. Such vertices will be called odd sites, respectively even sites in the sequel. For $a \neq b \in A_q$, we define S_{ab} on $l^2(\mathcal{T}_q)$ by

$$S_{ab} = \sum_{x_e \in \mathcal{T}_q} |x_e a\rangle\langle x_e| + \sum_{x_o \in \mathcal{T}_q} |x_o b\rangle\langle x_o|. \tag{12}$$

One has $S_{ab}^* = S_{ab}^{-1} = S_{ba}$ and, for any $a \neq b, c \neq d \in A_q$,

$$S_{ab}S_{cd}|x_e\rangle \;=\; |x_e cb\rangle, \quad S_{ab}S_{cd}|x_o\rangle = |x_o da\rangle. \tag{13}$$

The name shift stems from the following property. For each $x \in \mathcal{T}_q$, consider $\mathcal{H}_x^{ab}$ the S_{ab}-cyclic subspace generated by $|x\rangle$,

$$\mathcal{H}_x^{ab} = \overline{\text{span}}\left\{S_{ab}^n|x\rangle,\ n \in \mathbb{Z}\right\} \subset l^2(\mathcal{T}_q). \tag{14}$$

Lemma 2.3 *The subspace $\mathcal{H}_x^{ab}$ is isomorphic to $l^2(\mathbb{Z})$ and $S_{ab}|_{\mathcal{H}_x^{ab}}$ is unitarily equivalent to the shift on $l^2(\mathbb{Z})$.*

Remark 2.4 *i) The sites corresponding to $\mathcal{H}_e^{ab}$ are shown in Figure 3 for $\mathcal{T}_3$ (see also Section 6.1 for the notation).*
ii) Also, $\mathcal{H}_x^{ab} = \mathcal{H}_{xa}^{ab} = \mathcal{H}_{xb}^{ab}$, for all $x \in \mathcal{T}_q$.

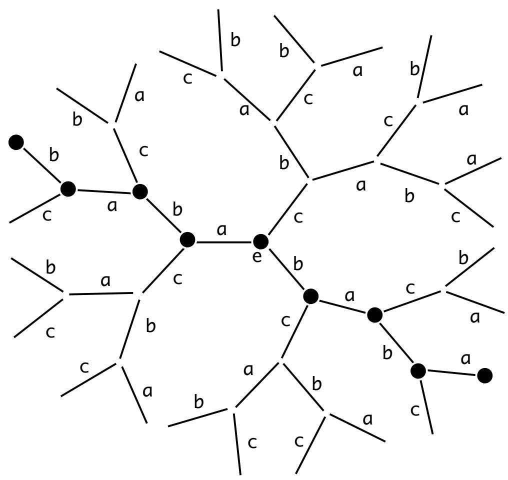

Figure 3: The sites $\{S_{ab}^n e\}_{n\in\mathbb{Z}}$, $q = 3$.

Proof: Let us assume that $|x|$ is even. Then, for $n \geq 0$, $S_{ab}^{2n}x = xabab\ldots ab$ whereas $S_{ab}^{2n+1}x = xabab\ldots aba$. If $n < 0$, one uses $S_{ab}^n = S_{ba}^{|n|}$. Hence, one gets

$$\mathcal{H}_x^{ab} = \overline{\text{span}}\left\{\ldots xbaba, xbab, xba, xb, x, xa, xab, xaba, xabab\ldots\right\}. \tag{15}$$

This subspace is equivalent to $l^2(\mathbb{Z})$ and S_{ab} is equivalent to the shift on $l^2(\mathbb{Z})$. If $|x|$ is odd, the same result is true, *mutatis mutandis*. ∎

To define S, we make use of the q shifts

$$S_{a_1a_2}, S_{a_2a_3}, \cdots, S_{a_{q-1}a_q}, S_{a_qa_1} \tag{16}$$

only. Let $A_q = \{a_1, a_2, \ldots, a_q\}$, and let us denote by $\{|a_j\rangle\}_{j=1,2,\ldots,q}$ the elements of the canonical basis of $\mathbb{C}^q$. We define $S : \mathcal{K}_q \to \mathcal{K}_q$ by

$$S \;=\; \sum_{1\le j\le q} S_{a_{j+1}a_{j+2}} \otimes |a_j\rangle\langle a_j|, \quad \text{with } a_{q+1} = a_1. \tag{17}$$

As above, the one step unitary evolution operator on $\mathcal{K}_q = l^2(T_q) \otimes \mathbb{C}^q$ for q odd is defined by

$$U(C) = S(\mathbb{I} \otimes C), \tag{18}$$

where $C \in U(q)$ is considered as a parameter. Equivalently, with $C = (C_{a_ja_k})_{j,k}$, and $a_{q+1} = a_1$,

$$\begin{aligned} U(C)x_e \otimes a_j &= \sum_{1\le k\le q} C_{a_ka_j}\,(x_e a_{k+1}) \otimes a_k \\ U(C)x_o \otimes a_j &= \sum_{1\le k\le q} C_{a_ka_j}\,(x_o a_{k+2}) \otimes a_k. \end{aligned} \tag{19}$$

In order to have a unified notation for both cases q even and odd, we denote the canonical basis of the coin state space by $\{|\tau\rangle\}_{\tau\in I_q}$, where $I_q = \{1, 2, \ldots, q\}$.

A natural generalization consists in considering families of coin matrices $\mathcal{C} = \{C(x) \in U(q)\}_{x\in\mathcal{T}_q}$, indexed by the vertices $x \in \mathcal{T}_q$ instead of a single coin matrix C. Then a quantum walk with site dependent coin matrices is defined through the formula, slightly abusing notations,

$$U(\mathcal{C})\, x \otimes \tau = S(\mathbb{I} \otimes C(x))x \otimes \tau, \quad \forall x \in \mathcal{T}_q, \tau \in I_q. \tag{20}$$

Equivalently, for q even, respectively q odd,

$$U(\mathcal{C}) \;=\; \sum_{\substack{a\in A_q\\ x\in\mathcal{T}_q}} |xa\rangle\langle x| \otimes |a\rangle\langle a|C(x), \quad \text{respectively} \tag{21}$$

$$U(\mathcal{C}) \;=\; \sum_{1\le j\le q}\left(\sum_{x_e\in\mathcal{T}_q} |x_e a_{j+1}\rangle\langle x_e| \otimes |a_j\rangle\langle a_j|C(x_e) + \sum_{x_e\in\mathcal{T}_q} |x_o a_{j+2}\rangle\langle x_o| \otimes |a_j\rangle\langle a_j|C(x_o)\right).$$

The interpretation is that the quantum walker at site $x \in \mathcal{T}_q$ undergoes a coin state update by means of the coin matrix sitting at site $x \in \mathcal{T}_q$, before jumping to its neighbors via the shift operator. We introduce and study below certain families of site dependent random coin matrices of this sort.

2.2 Random Quantum Walks

We address the properties of quantum walks when the evolution operator depends on random phases which turn the deterministic coin matrices into site dependent random coin matrices of the following form.

Consider $\Omega = \mathbb{T}^{\mathcal{T}_q \times I_q}$, $\mathbb{T}$ being the torus, as a probability space with σ algebra generated by the cylinder sets and measure $\mathbb{P} = \otimes_{\substack{x\in\mathcal{T}_q\\ \tau\in I_q}} d\nu$ where $d\nu$ is a probability measure on $\mathbb{T}$. Let $\{\omega_x^\tau\}_{x\in\mathcal{T}_q,\tau\in I_q}$ be a set of i.i.d. random variables on the torus $\mathbb{T}$ with common distribution $d\nu$. We will note $\Omega \ni \omega = \{\omega_x^\tau\}_{x\in\mathcal{T}_q,\tau\in I_q}$ and we will always assume that

$$d\nu(\theta) = l(\theta)d\theta, \quad \text{where} \ \ l \in L^\infty(\mathbb{T}) \tag{22}$$

the support of which has non-empty interior. We define a random diagonal unitary operator on $\mathcal{K}_q$ by

$$\mathbb{D}_\omega x \otimes \tau = e^{i\omega_x^\tau} x \otimes \tau, \ \ \forall (x,\tau) \in \mathcal{T}_q \times I_q. \tag{23}$$

The random version of our quantum walks is characterized by the unitary operator

$$U_\omega(C) = \mathbb{D}_\omega U(C) \ \ \text{on } \mathcal{K}_q. \tag{24}$$

This definition amounts to replacing the constant matrix $C \in \mathbb{C}^q$ by a family of site dependent random matrices $C_\omega(x) \in \mathbb{C}^q$, $x \in \mathcal{T}_q$, acting as in (20). Indeed, for q even we have

$$C_{ab} \mapsto C_\omega(x)_{ab} = e^{i\omega_{xa}^a} C_{ab}, \tag{25}$$

whereas for q odd we get

$$C_{a_j a_k} \mapsto C_{a_j a_k}(x_e) = e^{i\omega_{x_e a_{j+1}}^{a_j}} C_{a_j a_k}, \quad C_{a_j a_k} \mapsto C_{a_j a_k}(x_o) = e^{i\omega_{x_o a_{j+2}}^{a_j}} C_{a_j a_k}. \tag{26}$$

These site dependent random phases are responsible for the manifestation of Anderson localization in certain regimes that we study below.

Regarding ergodic properties of these random operators, we have the following. Let $z \in \mathcal{T}_q$ and let T_z denote the isometric simply transitive map map $\mathcal{T}_q \to \mathcal{T}_q$ such that $T_z x = zx$. We use the same symbol T_z to denote the measure preserving map $T_z : \Omega \to \Omega$ defined by $T_z\omega = \{\omega_{zx}^\tau\}_{x\in\mathcal{T}_q,\tau\in I_q}$ and the unitary operator on $\mathcal{K}_q$ given by $T_z x \otimes \tau = zx \otimes \tau$. One checks rightaway that $T_z^{-1} = T_{z^{-1}} = T_z^*$ on $\mathcal{K}_q$ and $T_z^* \mathbb{D}_\omega T_z = \mathbb{D}_{T_z\omega}$. Moreover, for all $a, b \in A_q$

$$\begin{aligned} T_z^* S_a T_z &= S_a \ \ \text{if } q \text{ is even} \\ T_z^* S_{ab} T_z &= S_{ab} \ \ \text{if } q \text{ is odd and } |z| \text{ is even,} \\ T_z^* S_{ab} T_z &= S_{ba} \ \ \text{if } q \text{ is odd and } |z| \text{ is odd.} \end{aligned} \tag{27}$$

Consequently, for any z such that $|z|$ is even, and any q we have

$$T_z^* U_\omega(C) T_z = U_{T_z\omega}(C) \tag{28}$$

and the same holds for any function of $U_\omega(C)$

The random unitary operator $U_\omega(C)$ on $\mathcal{K}_q$ depends parametrically and continuously on the coin matrix C. Indeed for any coin matrices $C, C' \in \mathbb{C}^q$,

$$\|U_\omega(C) - U_\omega(C')\|_{\mathcal{K}^q} = \|\mathbb{I} \otimes (C - C')\|_{\mathcal{K}^q} = \|C - C'\|_{\mathbb{C}^q}. \tag{29}$$

As we will see, the spectral properties of $U_\omega(C)$ are dependent on $C \in U(q)$, and it is our goal to explore the corresponding spectral diagram.

2.3 Spectral Criteria

We shall repeatedly make use of the following general spectral criteria. Let U be unitary operator on a Hilbert space $\mathcal{H}$. The spectral measure $d\mu_\phi$ on the torus $\mathbb{T}$ associated with a vector $\phi \in \mathcal{H}$ decomposes as $d\mu_\phi = d\mu_\phi^{pp} + d\mu_\phi^{ac} + d\mu_\phi^{sc}$ into its pure point, absolutely continuous and singular continuous components. The corresponding supplementary orthogonal spectral subspaces are denoted by $\mathcal{H}^\#(U)$, with $\# \in \{pp, ac, sc\}$. The Fourier coefficients of the spectral measure read

$$\hat{\mu}_\phi(n) = \overline{\hat{\mu}_\phi(-n)} = \langle \phi | U^n \phi \rangle = \int_{\mathbb{T}} e^{i\theta n} d\mu_\phi(\theta), \quad \forall n \in \mathbb{Z}. \tag{30}$$

Then, Wiener Theorem says that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N} |\langle \phi | U^n \phi \rangle|^2 = \sum_{\theta \in \mathbb{T}} (\mu_\phi^{pp}\{\theta\})^2, \tag{31}$$

whereas the absolutely continuous spectral subspace of U, $\mathcal{H}^{ac}(U)$, is given by

$$\mathcal{H}^{ac}(U) = \overline{\Big\{ \phi \mid \sum_{n\in\mathbb{N}} |\langle \phi | U^n \phi \rangle|^2 < \infty \Big\}}. \tag{32}$$

Given $\{P_r\}_{r\in\mathbb{N}}$ a family of finite rank orthogonal projectors such that $\lim_{r\to\infty} P_r = \mathbb{I}$ in the strong sense, one has the following. The vector $\phi \in \mathcal{H}^c(U) = \mathcal{H}^{ac}(U) \oplus \mathcal{H}^{sc}(U)$, the continuous spectral subspace of U, if and only if for any $r \geq 0$

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N} \|P_r U^n \phi\| = 0, \tag{33}$$

whereas $\phi \in \mathcal{H}^{pp}(U)$, if and only if for any $r \geq 0$

$$\lim_{r\to\infty} \sup_{n\in\mathbb{N}} \|(\mathbb{I} - P_r) U^n \phi\| = 0. \tag{34}$$

It is possible to replace U by U^{-1} in the previous formulae.

When the criteria (32) is applied to vectors from an orthonormal basis of $\mathcal{H}$, $\{e_j\}_{j\in I}$, I being a discrete set of indices, one expands to get

$$\langle e_{k_0} | U^n e_{k_n} \rangle = \sum_{(k_1, k_2, \ldots, k_{n-1}) \in I^{n-1}} \langle e_{k_0} | U e_{k_1} \rangle \langle e_{k_1} | U e_{k_2} \rangle \ldots \langle e_{k_{n-1}} | U e_{k_n} \rangle, \tag{35}$$

and we consider each sequence $\{k_0, k_1, k_2, \ldots, k_n\} \in I^{n+1}$ as a path with complex weight given by the product of matrix elements of U in the summand above. For operators U whose matrix representation has a band structure, the sum (35) is finite.

We note here a simple consequence of these criteria. Consider $U(C)$ on $\mathcal{K}_q$ given by (10) and (18). The diagonal elements of $U^{2n+1}(C)$ in the canonical orthonormal basis (4) are all zero, for $n \in \mathbb{N}$.

Lemma 2.5 *Let $\mathbb{U} = \{z \in \mathbb{C} \text{ s.t. } |z| = 1\}$ and $x \otimes \tau \in \mathcal{K}_q$. If there exist $\varphi \in \mathbb{R},\ 0 < \rho < 1$ so that*

$$\langle x \otimes \tau | U^{2n}(C) x \otimes \tau \rangle = (\rho e^{i\varphi})^n, \quad \text{for all } n \in \mathbb{N}, \tag{36}$$

then, the corresponding spectral measure $d\mu_{x\otimes\tau}$ is given by

$$d\mu_{x\otimes\tau}(\theta) = \frac{1-\rho^2}{1+\rho^2 - 2\rho\cos(2\theta - \varphi)} \frac{d\theta}{2\pi} \tag{37}$$

and $\sigma(U(C)) = \sigma_{ac}(U(C)) = \mathbb{U}$.

Remark 2.6 *In case $C = \mathbb{I}$, $\langle x \otimes \tau | S^{2n} x \otimes \tau \rangle = \delta_{0,n}$, for all $x \otimes \tau \in \mathcal{K}_q$. Hence $d\mu_{x\otimes\tau} = \frac{d\theta}{2\pi}$ and $\sigma(S) = \sigma_{ac}(S) = \mathbb{U}$.*

3 Special Cases

3.1 Propagating Matrices

We introduce here families of coin matrices which, by construction, induce absolutely continuous spectrum. We call them propagating matrices and the set of such matrices defined by the following Lemma is denoted by $\mathcal{P}$.

Lemma 3.1 *For q even, if $C = (C_{ab})_{(a,b)\in A_q^2} \in U(q)$ satisfies $C_{aa^{-1}} = 0$ for all $a \in A_q$, then $U_\omega(C)$ is purely absolutely continuous and $\sigma(U_\omega(C)) = \mathbb{U}$.*
For q odd, if $C = (C_{a_j a_k})_{(a,b)\in A_q^2} \in U(q)$ satisfies $C_{a_j a_{j\pm 1}} = 0$, for all $j = 1, 2, \ldots, q$, with $a_{q+1} = a_1$, then $U_\omega(C)$ is purely absolutely continuous and $\sigma(U_\omega(C)) = \mathbb{U}$.

Remark 3.2 *These properties extend to families of unitary matrices $\mathcal{C} = \{C(x)\}_{x\in\mathcal{T}_q}$, of the sort considered in (20), provided $C(x)$ satisfies the hypotheses for each $x \in \mathcal{T}_q$.*

Proof: The propagating matrices are designed so that, in the language of (35), it is impossible to come back to a site of $\mathcal{T}_q$ already visited, irrespectively of the coin variable. Hence criteria (32) applies immediately to all basis vectors which shows that $\mathcal{P} \subset \mathcal{D}$. ∎

3.2 Matrices Reducing to one D Problems

For $q \geq 4$ even, there are families of coin matrices C^R which reduce the analysis of $U_\omega(C^R)$ to a direct sum of $q/2$ one dimensional random quantum walks, in the sense that they take place on $l^2(\mathbb{Z}) \otimes \mathbb{C}^2$. Such walks are known, [JM], to give rise to dynamical localization almost surely, whatever the underlying deterministic coin matrix is, except in the diagonal case. These coin matrices are called reducing matrices and they from a set denoted by $\mathcal{R}$.

Lemma 3.3 *Let q be even and $\mathcal{R} \subset U(q)$ be the set of matrices C^R such that for all $a, b \in A_q$ with $a \neq b$,*

$$C^R_{a\ b} = C^R_{a^{-1}\ b} = C^R_{a\ b^{-1}} = C^R_{a^{-1}\ b^{-1}} = 0. \tag{38}$$

If all other matrix elements are non-zero, $\sigma(U_\omega(C^R))$ is pure point, almost surely.

Proof: Eq. (38) means that the q subspaces $\mathcal{H}^a$ characterized by coin state components in $\overline{\text{span}}\,\{|a\rangle, |a^{-1}\rangle\}$ are invariant under $U_\omega(C^R)$. For any $x \in \mathcal{T}_q$, let $x_a \in \mathcal{T}_q$ be obtained by stripping from $x = x_1 x_2 \cdots x_n$ the last consecutive symbols of the form a or a^{-1}. Then,

$$U_\omega(C^R)|_{\mathcal{H}_a} = \oplus_{x_a \in \mathcal{T}_q} U_\omega(C^a), \tag{39}$$

where all $U_\omega(C^a)$ are independent one dimensional random quantum walk with common coin matrix $C^a \in U(2)$

$$C^a = \begin{pmatrix} C^R_{a\ a} & C^R_{a\ a^{-1}} \\ C^R_{a^{-1}\ a} & C^R_{a^{-1} a^{-1}} \end{pmatrix}, \tag{40}$$

and iid random phases which carry the dependence on x_a. The results of [JM] show that $\sigma(U_\omega(C^a))$ is pure point, almost surely, which proves the result. ∎

Remark 3.4 *i) The condition that C^a has non zero elements excludes C^a diagonal, for which we know that delocalization occurs, and C^a off-diagonal, for which we know deterministic localization takes place. This shows $\mathcal{R} \cap \mathcal{L} \neq \emptyset$.*
ii) Matrices that are propagating and reducing are diagonal: $\mathcal{R} \cap \mathcal{P} = \{\Phi = \text{diag}(e^{i\varphi_j})\}$.
iii) If C^R is not diagonal and one of the blocks (40) at least in the decomposition (39) is diagonal, $U_\omega(C^a)$ has mixed spectrum almost surely, which shows $\mathcal{R} \cap \mathcal{M} \neq \emptyset$. The intersection consists of such matrices only.

3.3 Permutation Matrices

We consider here coin matrices given by permutation matrices, which lead to an explicit spectral analysis. We do not attempt to analyze all cases for arbitrary q, but instead, for each of the cases q odd and q even, we limit ourselves to the analysis of two permutations around which we shall perturb later on. Section 6 presents a more complete analysis for $q = 3$ and $q = 4$.

Let $\pi \in \mathfrak{S}_q$ that we view as acting on A_q. Then $C_\pi = \sum_{\tau \in A_q} |\pi(\tau)\rangle\langle\tau|$ is the corresponding permutation matrix. We will generalize this set of special matrices by allowing the matrix elements of C_π to carry phases. To this end we introduce

$$\Phi = \text{diag}\ (e^{i\varphi_1}, e^{i\varphi_2}, \cdots, e^{i\varphi_q}) \in U(q) \tag{41}$$

and C^Φ_π by

$$C^\Phi_\pi = \Phi C_\pi = \sum_{\tau \in A_q} e^{i\varphi_{\pi(\tau)}} |\pi(\tau)\rangle\langle\tau|, \tag{42}$$

that we call a decorated permutation matrix.

Among the decorated permutation matrices, those C^Φ_π which give rise to pure point spectrum for $U_\omega(C^\Phi_\pi)$ with finite dimensional cyclic subspaces for all ω play a special role.

Such coin matrices are said to *fully localize* the quantum walker. The set of fully localizing permutation matrices will be denoted by $\Lambda \in U(q)$. We exhibit an element of Λ for any q in the following lemma.

Lemma 3.5 *Let q be odd and let $C^{\Phi}_{(12\cdots q)}$ be the decorated permutation matrix corresponding to $(12\cdots q) \in \mathfrak{S}_q$. Then $U_\omega(C^{\Phi}_{(12\cdots q)})$ is pure point and admits*

$$\begin{aligned}\mathcal{H}_{x_o} = \overline{span} \quad \{ \quad & x_o \otimes a_1, x_o a_4 \otimes a_2, x_o \otimes a_3, \cdots x_o \otimes a_{q-2}, x_o a_1 \otimes a_{q-1}, x_o \otimes a_q \\ & x_o a_3 \otimes a_1, x_o \otimes a_2, \cdots x_o a_q \otimes a_{q-2}, x_o \otimes a_{q-1}, x_o a_2 \otimes a_q\}, \end{aligned} \tag{43}$$

for any $x_o \in \mathcal{T}_q$ with $|x_o|$ odd, as cyclic subspaces. Moreover,

$$\sigma(U_\omega(C^{\Phi}_{(12\cdots q)})|_{\mathcal{H}_{x_o}}) = e^{i\theta_\omega^{x_o}/(2q)} e^{i\varphi}\{1, e^{i2\pi/2q}, \cdots, e^{i2\pi(2q-1)/2q}\}, \tag{44}$$

where $\varphi = \frac{1}{q}\sum_{j=1}^{q} \varphi_j$ and

$$\theta_\omega^{x_o} = \sum_{j=1}^{q} (\omega^{a_{j+1}}_{x_o a_{j+3}} + \omega^{a_j}_{x_o}) \tag{45}$$

are distributed according to the $2q$-fold convolution $d\nu * d\nu * \ldots d\nu$ and $\{\theta_\omega^{x_o}\}_{x_o \in \mathcal{T}_q}$ are i.i.d.

On the other hand, for q even the operator $U_\omega(C^{\Phi}_{(1\ q/2+1)(2\ q/2+2)\cdots(q/2\ q)})$ is pure point with cyclic subspaces given by

$$\mathcal{H}_{x_o} = \overline{\text{span}} \bigcup_{j \in I_q} \{x_o \otimes a_j, x_o a_{j+q/2} \otimes a_{j+q/2}\} \equiv \bigoplus\nolimits_{j \in I_q} \mathcal{H}_{x_o \otimes a_j}, \tag{46}$$

for odd $x_o \in \mathcal{T}_q$. Moreover,

$$\sigma(U_\omega(C^{\Phi}_{(1q/2+1)(2q/2+2)\cdots(q/2 2q)})|_{\mathcal{H}_{x_o \otimes a_j}}) = e^{i\tilde{\theta}_\omega^{x_o}} e^{i\tilde{\varphi}}\{1, e^{i\pi}\}, \tag{47}$$

where $\tilde{\varphi} = \frac{1}{2}(\varphi_j + \varphi_{q/2+j})$ and $\tilde{\theta}_\omega^{x} = \frac{1}{2}(\omega^{a_{j+q/2}}_{x_o a_j^{-1}} + \omega^{a_j}_{x_o})$.

Remark 3.6 *i) There exist other permutations which give rise to fully localizing permutation coin matrices, see the analysis below of the cases $q = 3$ and $q = 4$. They all share similar properties to those above.*
ii) Note that $\mathcal{H}_{x_o}$ contains all basis vectors $\{x_o \otimes a_j\}_{j=1,2,\ldots,q}$ and $\mathcal{H}_{x_o} \perp \mathcal{H}_{x'_o}$, for $x_o \neq x'_o$.

Proof: Since this is a deterministic result, we can assume without loss that $\Phi = \mathbb{I}$. One checks by explicit computation that the list of vectors in (43) correspond to the successive images of any of them by $U(C_{(12\cdots q)})$, so that $U(C_{(12\cdots q)})^{2d}|_{\mathcal{H}_{x_o}} = \mathbb{I}_{\mathcal{H}_{x_o}}$. The addition of phases via the diagonal operator $\mathbb{D}_\omega$ preserves invariance of $\mathcal{H}_{x_o}$ and turns the previous identity into $U_\omega(C_{(12\cdots q)})^{2d}|_{\mathcal{H}_{x_o}} = e^{\theta_\omega^{x_o}}\mathbb{I}_{\mathcal{H}_{x_o}}$, from which we get the spectrum of this restriction. We conclude by observing that $\oplus_{x_o \in \mathcal{T}_q}\mathcal{H}_{x_o} = \mathcal{H}$. A similar calculations give the required result for q even. ■

Examples of permutation matrices which give rise to absolutely continuous spectrum for the corresponding random quantum walk include $C_{(1)(2)\cdots(q)} = \mathbb{I}$ for all q, and $C_{(12\cdots q)}$ for q even, both belonging to $\mathcal{P}$. We'll come back to these cases below.

3.4 Boundary conditions

Making use of Lemma 3.5, we can define boundary conditions for q odd and even which preserve unitarity and restrain the motion of the walker.

Let q be odd, $C \in U(q)$ be given and note $\pi_o = (12\dots q) \in \mathfrak{S}_q$. Let $C^{\Phi}_{\pi_o}$ be the decorated permutation matrix associated with π_o and let $x_o \in \mathcal{T}_q$ be an odd site. We consider a site-dependent family of matrices $\mathcal{C}_{x_o} = \{C(x) \in U(q)\}_{x\in\mathcal{T}_q}$ defined by

$$C(x) = \begin{cases} C^{\Phi}_{\pi_o} & \text{if } d(x,x_o) \le 1 \\ C & \text{otherwise.} \end{cases} \tag{48}$$

Lemma 3.7 *For q odd, the operator $U(\mathcal{C}_{x_o})$ defined by (20) admits $\mathcal{H}_{x_o}$ defined by (43) as 2q-dimensional invariant subspace. Moreover, there exist q infinite dimensional invariant subspaces under $U(\mathcal{C}_{x_o})$ denoted by $\mathcal{H}^j_{x_o}$, $j \in I_q$ and given by*

$$\mathcal{H}^j_{x_o} = \overline{\text{span}}\,\{x_o a_j \otimes a_k,\}_{k\in I_q\setminus\{j-2\}} \cup \{x_o a_j y \otimes a_k,\}_{k\in I_q, |a_j y|\ge 2}. \tag{49}$$

Proof: First note that in the neighborhood of x_o, the coin matrices are such that $U(\mathcal{C}_{x_o})$ acts as $U(C^{\Phi}_{\pi_0})$ so that Lemma 3.5 applies. Then, one observes that since all vectors of the form $x_o \otimes a_j$, $j \in I_q$ belong to the invariant subspace $\mathcal{H}_{x_o}$, it is impossible to connect vectors from $\mathcal{H}^j_{x_o}$ to $\mathcal{H}^k_{x_o}$, if $j \neq k$, with $U(\mathcal{C}_{x_o})$ which links nearest neighbors on $\mathcal{T}_q$ only. ■

Now for q even, let $\pi_e = (1\ \frac{q}{2}+1)(2\ \frac{q}{2}+2)\cdots(q/2\ q)$ and let $x_o \in \mathcal{T}_q$ be an odd site. We consider as before the site-dependent family of matrices $\mathcal{C}_{x_o} = \{C(x) \in U(q)\}_{x\in\mathcal{T}_q}$ defined by

$$C(x) = \begin{cases} C^{\Phi}_{\pi_e} & \text{if } d(x,x_o) \le 1 \\ C & \text{otherwise.} \end{cases} \tag{50}$$

Using the same method as before we get the following version of lemma 3.7

Lemma 3.8 *For q even, the operator $U(\mathcal{C}_{x_o})$ defined by (20) admits $\mathcal{H}_{x_o}$ defined by (46) as 2q-dimensional invariant subspace. Moreover, there exist q infinite dimensional invariant subspaces under $U(\mathcal{C}_{x_o})$ denoted by $\mathcal{H}^j_{x_o}$, $j \in I_q$ and given by*

$$\mathcal{H}^j_{x_o} = \overline{\mathit{span}}\,\{x_o a_j \otimes a_k\}_{k\in I_q\setminus\{j\}} \cup \{x_o a_j y \otimes a_k\}_{k\in I_q, |a_j y|\ge 2}. \tag{51}$$

Remark 3.9 *i) The same result with the same proof holds for $U_\omega(\mathcal{C}_{x_o}) = \mathbb{D}_\omega U(\mathcal{C}_{x_o})$. The restriction $U(\mathcal{C}_{x_o})|_{\mathcal{H}^j_{x_o}}$ can be viewed as a quantum walk on a rooted tree with root $x_o a_j$ going forward in the direction a_j, with coin space of dimension q over each site of this rooted tree, except over $x_o a_j$ where the coin space is of dimension $q-1$.*
ii) Boundary conditions of the same sort and with similar properties can be constructed by making use of any other fully localizing permutation matrices. See [J3] for the lattice case.

3.5 Finite Volume Restrictions

As a preparation for the analysis of strong disorder localization below, we define finite volume restrictions of our random unitary operators. Adding up boundary conditions of the type described above, we define here, for q odd and even, restrictions of $U_\omega(C)$ to finite dimensional subspaces associated with balls $\Lambda_L(x_e) \subset \mathcal{T}_q$ of odd radius $L \in 2\mathbb{N}+1$, centered at even sites $x_e \in \mathcal{T}_q$.

Let $C \in U(q)$ be given, $\pi_o = (12\dots q)$, $\pi_e = (1\ \frac{q+2}{2})(2\ \frac{q+4}{2})\cdots(\frac{q}{2}\ q)$, and $C^\Phi_{\pi_0}$, $C^\Phi_{\pi_e}$ be the corresponding decorated permutation matrices. Let $L \in 2\mathbb{N}+1$ and $x_e \in \mathcal{T}_q$ be an even site. We consider the site-dependent family of coin matrices $\mathcal{C}_{L,x_e} = \{C(x) \in U(q)\}_{x\in\mathcal{T}_q}$ defined by

$$C(x) = \begin{cases} C^\Phi_{\tilde{\pi}} & \text{if } d(x, x_e) \in \{L-1, L, L+1\} \\ C & \text{otherwise.} \end{cases} \tag{52}$$

where $\tilde{\pi} = \pi_0$ for odd q and $\tilde{\pi} = \pi_e$ for even q.

By construction, all odd sites a distance L away from x_e and all their nearest neighbors carry a matrix $C^\Phi_{\pi_0}$, whereas the other sites carry a matrix C. We emphasize the role of $C \in U(q)$ as a parameter in the operator $U(\mathcal{C}_{L,x_e})$ defined by (20) and its randomized version by using the notations

$$U^{L,x_e}(C) = U(\mathcal{C}_{L,x_e}), \quad U^{L,x_e}_\omega(C) = \mathbb{D}_\omega U^{L,x_e}(C). \tag{53}$$

We have

Lemma 3.10 *For any $\omega \in \Omega$, the operator $U^{L,x_e}_\omega(C)$ given by (53) admits the subspace $\mathcal{H}_{\Lambda_L(x_e)}$ of dimension $\frac{2q}{q-2}((q-1)^{L+1} - 1)$ defined by*

$$\mathcal{H}_{\Lambda_L(x_e)} = \bigoplus_{\substack{x_o\in\mathcal{T}_q \\ d(x_o,x_e)\le L}} \mathcal{H}_{x_o}, \tag{54}$$

and $\mathcal{H}^\perp_{\Lambda_L(x_e)}$, as invariant subspaces. For odd x_o, the subspaces $\mathcal{H}_{x_o}$ are given by (43) and (46) for odd and even q respectively. Moreover, $\|U^{L,x_e}_\omega(C) - U_\omega(C)\| \le \|C - C^\Phi_{\tilde{\pi}}\|_{\mathbb{C}^q}$.

Remark 3.11 *Finite volume restrictions of the same sort can be constructed on the basis of any fully localizing permutation matrix.*

Proof: By Lemma 3.7, and 3.8, the subspace $\bigoplus_{\substack{x_o\in\mathcal{T}_q \\ d(x_o,x_e)=L}} \mathcal{H}_{x_o}$ is invariant. Since sites x, y with $d(x, x_e) \le L-1$ and $d(y, x_e) \ge L+1$ are at least a distance 2 apart from each other, $\langle x \otimes a_k | U_\omega(\mathcal{C}_{L,x_e}) y \otimes a_k\rangle = 0$, for all $j, k \in I_q$, which shows that $\mathcal{H}_{\Lambda_L(x_e)}$ is invariant. The dimension of $\mathcal{H}_{\Lambda_L(x_e)}$ is determined by Lemmas 3.7, 3.8 and by the number of sites x such that $|x| = l$ which is $q(q-1)^{l-1}$, so that $\dim \mathcal{H}_{x_o} = 2q^2(q-1)^{l-1}$ if $d(x_o, x_e) = l$. Summing over all l odds up to L gives the result. The last estimate is straightforward from (21). ■

We eventually define the finite volume unitary operator associated to the ball $\Lambda_L(x_e)$ as the restriction

$$U^{\Lambda_L(x_e)}_\omega(C) = U^{L,x_e}_\omega|_{\mathcal{H}_{\Lambda_L(x_e)}} \quad \text{and} \quad U^{\Lambda^C_L(x_e)}_\omega(C) = U^{L,x_e}_\omega|_{\mathcal{H}_{\Lambda^\perp_L(x_e)}}. \tag{55}$$

As in Lemma 3.10, we have for any $C, C' \in U(q)$,

$$\|U^{\Lambda_L(x_e)}_\omega(C) - U^{\Lambda_L(x_e)}_\omega(C')\| \le \|C - C'\|_{\mathbb{C}^q}. \tag{56}$$

If $C = C^{\Phi}_{\tilde{\pi}}$, by construction and Lemma (3.5) for q odd

$$\sigma(U_\omega^{\Lambda_L(x_e)}(C_{\pi_0})) = \bigcup_{\substack{x_o\in\mathcal{T}_q,\, d(x_o,x_e)\leq L \\ k=0,1,\dots,2q-1}} \{e^{i(\theta_\omega^{x_o}+k2\pi)/(2q)}\}, \tag{57}$$

while for q even

$$\sigma(U_\omega^{\Lambda_L(x_e)}(C_{\pi_e})) = \bigcup_{\substack{x_o\in\mathcal{T}_q,\, d(x_o,x_e)\leq L \\ k=0,1,}} \{e^{i(\theta_\omega^{x_o}+k\pi)}\}. \tag{58}$$

4 Strong Disorder Localization

Adapting the analysis of random quantum walks on the lattice $\mathbb{Z}^d$, we prove localization of $U_\omega(C)$ in regimes where the coin matrix C is close enough to Λ, the set of permutation matrices which fully localize the quantum walker. By analogy with the Anderson model, we call this regime the strong disorder regime.

The strategy to prove localization of random quantum walks on $\mathcal{T}_q$ is the same as the one used on the lattice in [J3] making use of the fractional moments method of Aizenman and Molchanov [AM] adapted to the unitary framework in [HJS2]. We consider the finite volume restriction (55) of the operator $U_\omega(C)$ and estimate the fractional moments of the resolvent of this finite volume restriction. Then we take the limit $L\to\infty$ to get suitable estimates on the fractional moments of the full resolvent. However, the behavior in L of the size of the boundary of the ball of radius L being exponential on $\mathcal{T}_q$ rather than algebraic in L on the lattice, we need to adapt some aspects of the argument. In particular, we need to prove that the fractional moment estimates have an arbitrarily large exponential decay.

In the following, the symbol c denotes unessential constants, that may vary from line to line. The Green function of $U_\omega(C)$ is denoted by

$$G_{a_j,a_k,\omega}(x,y;C,z) = \langle x\otimes a_j|\,(U_\omega(C)-z)^{-1}y\otimes a_k\rangle \tag{59}$$

and the finite volume Green function is denoted by $G^{\Lambda_L(x_e)}_{a_j,a_k,\omega}(x,y;z)$, with $U_\omega^{\Lambda_L(x_e)(C)}$ in place of $U_\omega(C)$. The result we are aiming for is the fractional moments estimate

Theorem 4.1 *Let $\pi_o\in\mathfrak{S}_q$ be such that $C^{\Phi}_{\pi_o}\in\Lambda\subset U(q)$. For all $0<s<1/3$, and all $\gamma>0$, there exist $K(s,\gamma)<\infty$ and $\epsilon(s,\gamma)>0$ such that for all $C\in U(q)$ with $\|C-C^{\Phi}_{\pi_0}\|\leq\epsilon(s,\gamma)$, all $x,y\in\mathcal{T}_q$ with $d(x,y)>2$, all $z\notin\mathbb{U}$, and all $j,k\in I_q$,*

$$\mathbb{E}(|G_{a_j,a_k,\omega}(x,y;C,z)|^s)\leq K(s,\gamma)e^{-\gamma d(x,y)}. \tag{60}$$

The estimate also holds for $\gamma=0$, without restriction on C or $d(x,y)$.

Proof: While are results hold for any q and any permutation matrix in Λ, to fix the ideas and without loss, we provide a detailed proof for q odd and for $\pi_o=(12\dots q)$ only. The case q even is somehow simpler whereas the modifications induced by different choices of fully localizing permutations are dealt with along the lines of [J3].

The first step towards (60) is the following estimate on the fractional moments of the finite volume Green function, which we prove in the Appendix.

Proposition 4.2 *For all* $0 < s < 1$, *all* $p' > 1/(1-s)$ *fixed, there exist* $C(s)$ *and* $c_0(s) < \infty$ *so that for all* $\alpha > 0$, *all* $C \in U(q)$ *such that* $\|C - C^{\Phi}_{\pi_0}\| \le c_0(s)e^{-L\alpha(1/s+2/p')}(q-1)^{-2L}$, *the estimate*

$$\mathbb{E}(|G^{\Lambda_L(x_e)}_{a_j,a_k,\omega}(x,y;C,z)|^s) \le C(s)e^{-\alpha L}, \tag{61}$$

holds for all $L \ge 3$, *all* $z \notin \mathbb{U}$, *all* $x \otimes a_j, y \otimes a_k \in \mathcal{H}^{\Lambda_L(x_e)}$ *with* $d(x,y) > 2$.
The estimate also holds for $\alpha = 0$, *without restriction on* C *or* $d(x,y)$.

The second step consists in making the link between estimates on finite and infinite volume Green functions. This is achieved along the lines of [HJS2], via geometric resolvent identities, decoupling estimates and iteration, taking inspiration from [AENSS] for the self adjoint case. This step requires dealing with the metric peculiarities of the tree.

We simplify the notation by dropping the symbols C, ω, x_e and $z \notin \mathbb{U}$. We set T^L by

$$U = U^L + T^L = U^{\Lambda_L} \oplus U^{\Lambda_L^c} + T^L, \tag{62}$$

see Lemma 3.10, and we keep track of the dependence in $t = \|T^L\|$, where $t \le c\|C - C^{\Phi}_{\pi_0}\|$, uniformly in L and ω. We note

$$G^L = (U^L - z)^{-1} = (U^{\Lambda_L} \oplus U^{\Lambda_L^c} - z)^{-1} = (U^{\Lambda_L} - z)^{-1} \oplus (U^{\Lambda_L^c} - z)^{-1} \tag{63}$$

and $G^L_{a_j,a_k}(x,y)$ the corresponding Green function. We prove the following Proposition in Appendix

Proposition 4.3 *For every* $s \in (0, 1/3)$ *there exists a constant* $c_1(s) < \infty$ *depending on* s *(and* q*), such that*

$$\begin{aligned}\mathbb{E}\left(|G_{a_j,a_k}(x,y)|^s\right) &\le c_1(s)t^{2s}(1 + c_1(s)t^s(q-1)^L) \\ &\times \sum_{\substack{u \in \mathcal{H} \\ |d(u_1,x_e)-L| \le 2}} \mathbb{E}\left(|G^L_{a_j,u_2}(x,u_1)|^s\right) \sum_{\substack{x' \in \mathcal{H} \\ |d(x'_1,x_e)-(L+3)| \le 2}} \mathbb{E}\left(|G_{x'_2,a_k}(x'_1,y)|^s\right)\end{aligned} \tag{64}$$

uniformly in $z \notin \mathbb{U}$ *with* $1/2 < |z| < 2$, $L \in \mathbb{N}$ *and* $x, y \in \mathcal{T}_q$ *with* $d(x,x_e) \le L$ *and* $d(y,x_e) > L+5$, *with the notation* $u = u_1 \otimes u_2 \in \mathcal{H}$, $u_1 \in \mathcal{T}_q$, $u_2 \in A_q$

From this point, one uses an iterative argument to eventually reach the sought for estimate (4.1), taking care of the dependence in (s,α) of the different parameters, considering q as fixed.

We first note that for $L \ge 3$, and for some $C_q(s)$

$$\sum_{\substack{u \in \mathcal{H} \\ |d(u_1,x_e)-L| \le 2}} \mathbb{E}\left(|G^L_{a_j,u_2}(x,u_1)|^s\right) \le C_q(s)(q-1)^L e^{-\alpha L}, \tag{65}$$

if $d(x,x_e) \le L$ and that our hypothesis on the perturbation $\|C - C^{\Phi}_{\pi_0}\|$ with $t \le c\|C - C^{\Phi}_{\pi_0}\|$ implies for any $p' > 1/(1-s)$ and some $c_q(s) < \infty$,

$$t^s(q-1)^L \le c_q(s)e^{-L\beta(\alpha)}, \text{ with } \beta(\alpha) = \alpha(1 + 2s/p') - \ln(q-1)(1-2s). \tag{66}$$

Hence, given $0 < s < 1/3$ and $p' > 1/(1-s)$, there exists $\alpha_0(s) > 0$ (depending on q and p') such that, for all $x, y \in \mathcal{T}_q$ with $d(x, x_e) \le L$, $d(y, x_e) > L + 5$, and $\alpha \ge \alpha_0(s) > 0$ we have $\beta(\alpha) > 0$ and for some (q and p' dependent) $c(s) < \infty$

$$\begin{aligned} \mathbb{E}\left(|G_{a_j,a_k}(x,y)|^s\right) &\le c(s) e^{-L(\alpha(3+4s/p')-(1-4s)\ln(q-1))}(1 + c(s)e^{-\beta(\alpha)L}) \\ &\times \sup_{\substack{x' \in \mathcal{H} \\ |d(x_1', x_e)-(L+3)| \le 2}} \mathbb{E}\left(|G_{x_2',a_k}(x_1', y)|^s\right) \\ &\le c_0(s) e^{-L\delta(\alpha)} \sup_{\substack{x' \in \mathcal{H} \\ |d(x_1', x_e)-(L+3)| \le 2}} \mathbb{E}\left(|G_{x_2',a_k}(x_1', y)|^s\right), \end{aligned} \tag{67}$$

with

$$\delta(\alpha) = \alpha(3 + 4s/p') - (1-4s)\ln(q-1) \ge \alpha_0(s)(3 + 4s/p') - (1-4s)\ln(q-1) > 0. \tag{68}$$

Eventually, we fix $L_0(s) = L_0(s, \alpha_0(s))$ odd and large enough so that

$$c_0(s) e^{-L_0(s)\delta(\alpha_0(s))} < 1. \tag{69}$$

Thus, with

$$b(s, \alpha) = c_0(s) e^{-L_0(s)\delta(\alpha)}, \tag{70}$$

we get for any $\alpha \ge \alpha_0(s)$, $b(s, \alpha) \le b(s, \alpha_0(s)) < 1$. This determines the size of the perturbation via

$$\|C - C^{\Phi}_{\pi_0}\| \le \Delta(s, \alpha) := c_0(s) e^{-L_0(s)\alpha(1/s+2/p')}(q-1)^{-2L_0(s)} \le \Delta(s, \alpha_0(s)). \tag{71}$$

Then, by ergodicity, see (28),

$$\max_{a_j,a_k \in A_q} \mathbb{E}\left(|G_{a_j,a_k}(x,y)|^s\right) = \max_{a_j,a_k \in A_q} \mathbb{E}\left(|G_{a_j,a_k}(x',y')|^s\right) \tag{72}$$

for all $x' = zx, y' = zy \in \mathcal{T}_q$ with $|z|$ even, where $d(x', y') = d(x, y)$. Thus, in the right hand side of (67), we can shift the arguments of the Green function so that x_1' is equal or close to the center of the ball $\Lambda(x_e)$ and provided $d(x_1', y) \ge L + 5$ one can iterate (67). Doing this along a sequence of points forming a path of length of order $d(x, y) = n_y L_0$, we eventually get that

$$\mathbb{E}\left(|G_{a_j,a_k}(x,y)|^s\right) \le c b^{n_y}(s, \alpha) \le c e^{-\gamma(\alpha) d(x,y)}, \tag{73}$$

where, for α large enough,

$$\gamma(\alpha) = \delta(\alpha) - \ln(c_0)/L_0(s) = \alpha(3 + 4s/p') - (1-4s)\ln(q-1) - \ln(c_0)/L_0(s) > 0. \tag{74}$$

Since $\gamma(\alpha)$ is invertible and can be made arbitrarily large by increasing α, we get the result by defining $\epsilon(s, \gamma) = \Delta(s, \alpha^{-1}(\gamma))$. ■

Corollary 4.4 *Under the hypotheses of Proposition 4.1, and for $\gamma > 0$ large enough,*

$$\sigma(U_\omega(C)) = \sigma_{pp}(U_\omega(C)) \text{ almost surely.} \tag{75}$$

Proof: The result follows from the criteria (33) applied to all basis vectors with P_r the projector on the span of $\{x \otimes a, | a \in A_q, |x| \le r\}$, along the lines of Proposition 3.1 in [HJS2]. Taking the decay rate γ large enough allows us to compensate for the exponential growth in r of $\dim P_r \le c(q-1)^r$ on trees. ■

5 Weak Disorder Delocalization

The situation in which the coin matrix C in $U_\omega(C)$ is close to a decorated permutation matrix which for which the corresponding quantum walk is absolutely continuous is called this a weak disorder situation. We prove that on any tree $\mathcal{T}_q$, there exists special decorated permutation matrices such that the spectrum of $U_\omega(C)$ is purely absolutely continuous, provided C is close enough to these permutation matrices. This result can be viewed as an analog in the realm of random quantum walks on trees of the result [Kl] that Klein proved for the Anderson model on trees. We call these permutations matrices *fully delocalizing* and the set they form will be denoted by $\mathcal{S}$.

5.1 Delocalization close to $C = \Phi$ for q odd

Let $\pi = \text{Id} = (1)(2)\dots(q)$ be the identity permutation in $\mathfrak{S}_q$ so that $C^\Phi_{\text{Id}} = \Phi$. We prove here the following perturbative result

Proposition 5.1 *Let $q \geq 3$ be odd and $\epsilon = 1/(4q^2(q-1))$. Then, for any $\Phi = diag(e^{i\varphi_j}) \in U(q)$, $\|C - \Phi\| \leq \epsilon$ implies for any $\omega \in \Omega$*

$$\sigma(U_\omega(C)) = \sigma_{ac}(U_\omega(C)). \tag{76}$$

Remark 5.2 *As the proof shows, the result is deterministic and extends to families of unitary matrices $\mathcal{C} = \{C(x)\}_{x\in\mathcal{T}_q}$ of the sort considered in (20), provided $C(x)$ satisfies the hypothesis for each $x \in \mathcal{T}_q$.*

Proof: We consider $C = \Phi + E$, where $E \in M_q(\mathbb{C})$ is such that $\|E\| \leq \epsilon$ and $\Phi + E \in U(q)$. The argument consists in showing that there exist $C, \kappa > 0$ such that for all $x \in \mathcal{T}_q$, $\tau \in A_q$

$$|\langle x \otimes \tau | U_\omega^{2n}(C) x \otimes \tau\rangle| \leq C(\kappa\epsilon)^n / n^{3/2}. \tag{77}$$

This implies that $x \otimes \tau \in \mathcal{H}^{ac}(U_\omega(C))$ if $\epsilon \leq 1/\kappa$, according to (32). We introduce $0 < \gamma \leq 1$ such that $|e^{i\varphi_a} + E_{a,a}| \leq \gamma$, for all $a \in A_q$. Separating the part on $l^2(\mathcal{T}_q)$ from that on $\mathbb{C}^q$ of the basis vectors $y \otimes \sigma$, each path contributing to (77) in the decomposition corresponding to (35) has a trace on $\mathcal{T}_q$ of the form

$$x a_{i_1} a_{i_2} a_{i_3} a_{i_4} \dots a_{i_{2n}}, \text{ where } a_{i_j} \in A_q \text{ and } a_{i_1} a_{i_2}, \dots a_{i_{2n}} = e. \tag{78}$$

The corresponding sequence of coin variables depends on the parity of x: for $|x|$ even

$$\tau a_{i_1-1} a_{i_2-2} a_{i_3-1} a_{i_4-2} \dots a_{i_{2n}-2}, \text{ where } \tau, a_{i_j} \in A_q \text{ and } a_{i_{2n}-2} = \tau, \tag{79}$$

whereas for $|x|$ odd

$$\tau a_{i_1-2} a_{i_2-1} a_{i_3-2} a_{i_4-1} \dots a_{i_{2n-1}}, \text{ where } \tau, a_{i_j} \in A_q \text{ and } a_{i_{2n}-1} = \tau. \tag{80}$$

The weight of these paths is bounded above in modulus by $\epsilon^{2n-j}\gamma^j$, for some $0 \leq j \leq 2n$ counting the number of diagonal elements of C, see (21). We show that $j \leq n$.

In the list of matrix elements that constitute the weight of the path, there are $k \geq 0$ sequences of consecutive diagonal elements of length m_i, $i = 1, 2, \dots, k$ so that there are

$r = 2n - \sum_{i=1}^k m_i = 2n - j$ off diagonal elements. Each of the m_i diagonal elements correspond to a sequence of the form (79) or (80) which form an irreducible word by definition. Moreover, different such sequences cannot reduce one another and they must be separated by elements associated to off-diagonal elements. Since the irreducible words can only be reduced by the r letters corresponding to off diagonal elements, the total length of the reduced word made of $2n$ letters is bounded below by $\sum_{i=1}^k m_i - r = 2(j-n)$. Hence the requirement $j \leq n$.

Finally, for any $q \geq 3$, $\mathcal{N}_q(2n)$, the number of paths of length $2n$ from x to x in $\mathcal{T}_q$, is given for large n by

$$\mathcal{N}_q(2n) = \tilde{C}(q)\frac{(4(q-1))^n}{n^{3/2}}\left(1 + O(n^{-1/2})\right), \tag{81}$$

for some finite constant $\tilde{C}(q)$, see e.g. [W]. Taking into account the q coin variables at each step, the number of contributing paths of the form (78) is less than $C\kappa^n/n^{3/2}$, with $\kappa = 4q^2(q-1)$, which proves (77). ■

5.2 Delocalization close to $C^{\Phi}_{(12\cdots q)}$ for q even

A similar argument allows us to prove a delocalization result for $q > 2$ even.

Consider the permutation $(12\cdots q)$ and the corresponding decorated permutation matrix $C^{\Phi}_{(12\cdots q)}$. This matrix is a particular propagating matrix and gives rise to cyclic subspaces whose trace on $\mathcal{T}_q$ can be viewed as spirals, see Figure 6 for the case $q = 4$.

Proposition 5.3 *Let $q > 2$ be even and $\epsilon = 1/(4q^2(q-1))$. Then, for any $\Phi = diag(e^{i\varphi_j}) \in U(q)$, $\|C - C^{\Phi}_{(12\cdots q)}\| \leq \epsilon$ implies for any $\omega \in \Omega$*

$$\sigma(U_\omega(C)) = \sigma_{ac}(U_\omega(C)). \tag{82}$$

Remark 5.4 *i) As the proof shows, the result extends to any coin matrix of the from C^{Φ}_{π}, where $\pi \in \mathfrak{S}_q$ gives rise to cyclic subspaces whose trace on $\mathcal{T}_q$ does not contain any consecutive sequence of the form aa^{-1}, $a \in A_q$. Hence, all such matrices belong to $\mathcal{S}$.*
ii) Again, the result is deterministic and extends to families of unitary matrices $\mathcal{C} = \{C(x)\}_{x\in\mathcal{T}_q}$ of the sort considered in (20), provided $C(x)$ satisfies the hypothesis for each $x \in \mathcal{T}_q$.

Proof: This case is similar to Proposition 5.1 and actually simpler. As above, let $\{m_i\}_{i=1,2,\ldots,k}$ denote the list of successive coin matrix elements that are of order one, i.e. close to elements of $C^{\Phi}_{(12\cdots q)}$. They corresponds to strings of consecutive letters that form words which cannot be reduced, or pieces of spirals, in the image used above. Only the $r = 2n - \sum_{i=1}^k m_i = 2n - j$ letters associated with coin matrix elements that are of order ϵ can be used to reduce the total word of $2n$ symbols. As above, for a path from any x back to x in $2n$ steps, we need to have $j \leq n$ which allows us to apply criterion (32) in the same way, using (81) again. ■

Remark 5.5 *Although the decorated permutation matrix Φ, corresponding to the identity yields ac spectrum, $\Phi \notin \mathcal{S}$ for q even, see See Lemma 3.3.*

6 Spectral Diagrams for the Cases $q = 3$ and $q = 4$

This section is devoted to a partial analysis of the spectral diagram for random quantum walks on the trees $\mathcal{T}_3$ and $\mathcal{T}_4$. We exhibit certain families of coin matrices for which $\sigma(U_\omega(C))$ is pure point almost surely or purely absolutely continuous for any ω. In particular, we show that the neighborhood of certain permutation matrices contains coin matrices which induce mixed spectra, pure point or purely absolutely continuous spectra.

6.1 $q = 3$

For the case $q = 3$, we sketch a more complete spectral diagram in Section 6.1.5. It is based on the definition of one parameter families of coin matrices that interpolate between the six different permutation matrices for which we determine the nature of the spectrum of the associated random quantum walk operator $U_\omega(C)$. The corresponding picture is given in Figure 5.

Let us first simplify the notation: the alphabet is denoted by $A_3 = \{a, b, c\}$ and the orthonormal basis of the coin Hilbert space is denoted by $\{|a\rangle, |b\rangle, |c\rangle\}$. We write the coin dependent shift S on $\mathcal{K}_3 = \mathcal{T}_3 \otimes \mathbb{C}^3$ as

$$S = S_{bc} \otimes |a\rangle\langle a| + S_{ca} \otimes |b\rangle\langle b| + S_{ab} \otimes |c\rangle\langle c|. \tag{83}$$

The coin matrices C are written as 3×3 matrices in the basis ordered as above. In particular, the action of $U_\omega(C) = \mathbb{D}_\omega U(C)$ reads for any $\tau \in \{a, b, c\}$

$$\begin{aligned} U_\omega(C) x_e \otimes \tau &= e^{i\omega^a_{x_e b}} C_{a\tau}\ x_e b \otimes a + e^{i\omega^b_{x_e c}} C_{b\tau}\ x_e c \otimes b + e^{i\omega^c_{x_e a}} C_{c\tau}\ x_e a \otimes c \\ U_\omega(C) x_o \otimes \tau &= e^{i\omega^a_{x_o c}} C_{a\tau}\ x_o c \otimes a + e^{i\omega^b_{x_o a}} C_{b\tau}\ x_o a \otimes b + e^{i\omega^c_{x_o b}} C_{c\tau}\ x_o b \otimes c \end{aligned} \tag{84}$$

6.1.1 Permutation Coin Matrices

In order not to burden the notation, we refrain from decorating the permutation matrices by phases Φ in this section. We shall simply comment wherever necessary on the modifications required to generalize the statement made to the case of decorated permutation matrices.

The six different permutations of $\{a, b, c\}$ give rise to coin matrices inducing walks $U_\omega(C)$ with the following spectral properties, for any deterministic choice of diagonal $\mathbb{D}_\omega$.

The permutations matrices which give rise to fully localized walks consists in

$$\Lambda = \{C_{(abc)}, C_{(acb)}\} \tag{85}$$

with respective six-dimensional cyclic subspaces

$$\begin{aligned} &\overline{\text{span}}\,\{x_o \otimes a, x_o a \otimes b, x_o \otimes c, x_o c \otimes a, x_o \otimes b, x_o b \otimes c\}, \\ &\overline{\text{span}}\,\{x_e \otimes a, x_e a \otimes c, x_e \otimes b, x_e b \otimes a, x_e \otimes c, x_e c \otimes b\} \end{aligned} \tag{86}$$

labelled by $x_o \in \mathcal{T}_3$ respectively $x_e \in \mathcal{T}_3$. The identity matrix belongs to the set $\mathcal{S}$,

$$\mathcal{S} = \{C_{(a)(b)(c)}\}, \tag{87}$$

and yields absolutely continuous spectrum. The other three matrices give rise to walks with mixed spectra and belong to the set $\mathcal{M}$:

$$\{C_{(a)(bc)}, C_{(b)(ac)}, C_{(c)(ab)}\} \subset \mathcal{M}. \tag{88}$$

It is enough to consider the first matrix of the list (88), the other cases being similar. The coin matrix $C_{(a)(bc)}$ is such that $U_\omega(C_{(a)(bc)})$ leaves the subspace $l^2(\mathcal{T}_3) \otimes |a\rangle$ invariant, which gives rise to a shift essentially driven by S_{bc} on the corresponding cyclic subspaces $\overline{\text{span}}\{\dots x_e cb \otimes a, x_e c \otimes a, x_e \otimes a, x_e b \otimes a, x_e bc \otimes a, \dots\}$ labelled by $x_e \in \mathcal{T}_3$. Moreover, $U_\omega(C_{(a)(bc)})$ gives rise to another shift on the cyclic subspaces $\overline{\text{span}}\{\dots x_o bc \otimes c, x_e b \otimes b, x_e \otimes c, x_e c \otimes b, x_e cb \otimes c, \dots\}$ labelled by $x_o \in \mathcal{T}_3$ with alternating coin state, essentially driven this time by $S_{cb} = S_{bc}^*$. Finally, for all $x_e \in \mathcal{T}_3$, the two-dimensional subspace $\overline{\text{span}}\{x_e \otimes b, x_e a \otimes c\}$ is invariant under $U_\omega(C_{(a)(bc)})$. Therefore,

$$\sigma(U_\omega(C_{(a)(bc)})) = \sigma_{pp}(U_\omega(C_{(a)(bc)})) \cup \sigma_{ac}(U_\omega(C_{(a)(bc)})) = \mathbb{U}. \tag{89}$$

Remark 6.1 *All spectral conclusions hold if the permutation matrices considered C_π are replaced by decorated permutation matrices C_π^Φ.*

6.1.2 Propagating Coin Matrices

We first note that the set of propagating coin matrices $\mathcal{P}$ is reduced to the set of diagonal unitary matrix. Indeed, forbidding the successive compositions

$$S_{ca}S_{ab}x_e = x_e aa, \quad S_{ab}S_{ca}x_e = x_o aa \tag{90}$$

we get that the elements of the coin matrix C satisfy $C_{bc} = C_{cb} = 0$. Dealing with the cyclic permutations of symbols $\{a, b, c\}$ in the same manner, we get that $\mathcal{P} = \{\Phi = \text{diag}(e^{i\varphi_j})\}$ for $q = 3$. However, matrices in a neighborhood of the identity matrix yield delocalization, as we saw in Proposition 5.1.

6.1.3 Delocalizing Coin Matrices

We introduce here three one-parameter families of coin matrices $\{C_j^d(r)\}_{0<r<1}^{j\in\{1,2,3\}}$ which give rise to absolutely continuous operators $U_\omega(C_j^d(r))$, for any choice of phases $\mathbb{D}_\omega$.

For $0 \le r \le 1$ and $t = \sqrt{1-r^2}$, set

$$C_1^d(r) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r & t \\ 0 & -t & r \end{pmatrix}, C_2^d(r) = \begin{pmatrix} r & 0 & t \\ 0 & 1 & 0 \\ -t & 0 & r \end{pmatrix}, C_3^d(r) = \begin{pmatrix} r & t & 0 \\ -t & r & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{91}$$

If $r = 1$, all matrices reduce to $\mathbb{I} = C_{(a)(b)(c)}$ and for $r = 0$ they are correspond up to phases to the permutation matrices $C_{(a)(bc)}, C_{(b)(ac)}, C_{(c)(ab)}$.

Remark 6.2 *A parameterization of the form $(r, t) = (\cos(\varphi), \sin(\varphi))$, $\varphi \in [0, 2\pi)$, could be more natural, but, for notational simplicity later on, we stick to the convention above.*

Lemma 6.3 *For any* $0 < r \leq 1$, $j \in \{1, 2, 3\}$ *and any deterministic choice of* $\mathbb{D}_\omega$

$$\sigma(U_\omega(C_j^d(r))) = \sigma_{ac}(U_\omega(C_j^d(r))) = \mathbb{U}. \tag{92}$$

Proof: The case $r = 1$ corresponds to the identity and was dealt with already. It is enough to consider $C_2^d(r)$, the other cases being similar. First, the invariant subspace characterized by a coin variable $|b\rangle$ gives rise to a shift S_{ca} which is absolutely continuous with spectrum $\mathbb{U}$. We now consider the restriction to coin variables $|a\rangle, |c\rangle$. For $0 < r < 1$, observe with (84) that $C_2^d(r)$ makes the walker jump on $\mathcal{T}_3$ from x_e to $x_e a$ and $x_e b$ and from x_o to $x_o b$ and $x_o c$ only. Therefore, as soon as a path contains a step $x_e a$ or $x_o c$, it is impossible to get back to x_e or x_o. Thus, we get for any $x_e, x_o \in \mathcal{T}_3$ and any $n \in \mathbb{N}$

$$|\langle x_e \otimes c | U_\omega^{2n}(C_2^d(r)) x_e \otimes c\rangle| = |\langle x_o \otimes a | U_\omega^{2n}(C_2^d(r)) x_o \otimes a\rangle| = t^{2|n|}, \tag{93}$$

whereas all corresponding scalar products with other basis vectors yield δ_{0n}. Since $t < 1$, criterion (32) yields the result. ■

Remark 6.4 *As the proof shows, the results holds for arbitrary site dependent alterations of the matrix elements of* $C_j^d(r)$ *by phases which preserve unitarity. This is true in particular if* $C_j^d(r) \mapsto \Phi C_j^d(r)$, *where, possibly,* Φ *can be* r *dependent. Note also that different values of the parameter* $0 < r(x) \leq 1$ *at different sites* $x \in \mathcal{T}_3$ *are allowed provided* $\inf_x r(x) \geq r_0 > 0$.

6.1.4 Localizing Coin Matrices

We introduce here other families of one-parameter coin matrices $\{C_j^l(r)\}_{0<r<1}^{j\in\{1,2,\dots,6\}}$ which give rise to pure point random operators $U_\omega(C_j^l(r))$, almost surely.

Consider for $0 \leq r \leq 1$ and $t = \sqrt{1 - r^2}$,

$$C_1^l(r) = \begin{pmatrix} 0 & r & t \\ 1 & 0 & 0 \\ 0 & -t & r \end{pmatrix}, C_2^l(r) = \begin{pmatrix} 0 & 1 & 0 \\ r & 0 & t \\ -t & 0 & r \end{pmatrix}, C_3^l(r) = \begin{pmatrix} 0 & 0 & 1 \\ -t & r & 0 \\ r & t & 0 \end{pmatrix},$$
$$C_4^l(r) = \begin{pmatrix} 0 & t & r \\ 0 & r & -t \\ 1 & 0 & 0 \end{pmatrix}, C_5^l(r) = \begin{pmatrix} r & 0 & -t \\ t & 0 & r \\ 0 & 1 & 0 \end{pmatrix}, C_6^l(r) = \begin{pmatrix} r & -t & 0 \\ 0 & 0 & 1 \\ t & r & 0 \end{pmatrix}. \tag{94}$$

These matrices are obtained from the first one by permuting the order of the basis vector. Note that for $r = 1$ these matrices reduce by pairs to one of the permutation matrices $C_{(a)(bc)}, C_{(b)(ac)}, C_{(c)(ab)}$, whereas for $r = 0$, they are correspond, up to phases, to the permutation matrices $C_{(abc)}$, respectively $C_{(acb)}$, for odd, respectively even indices.

This section is devoted to prove the following

Proposition 6.5 *For all* $0 < r < 1$, *and* $j \in \{1, 2, \dots, 6\}$ *we have almost surely*

$$\sigma(U_\omega(C_j^l(r))) = \sigma_{pp}(U_\omega(C_j^l(r))). \tag{95}$$

Remark 6.6 *The same result holds if* $C_j^l(r) \mapsto \Phi C_j^l(r)$, *where* $\Phi = diag\ (e^{i\varphi_1}, e^{i\varphi_2}, e^{i\varphi_3})$, *where* φ_j *can possibly depend on* r.

Without loss, we can consider the matrix $C_1^l(r)$ only. The strategy to prove Proposition 6.5 is as follows. The shape of the matrices $C_j^l(r)$ is such that the one step evolution operator $U_\omega(C_j^l(r))$ admits cyclic subspaces in each of which it acts as a one-dimensional random unitary operator. Then transfer matrix methods allows us to prove localization for all values of $0 < r < 1$. We first determine the cyclic subspaces of $U_\omega(C_1^l(r))$.

Lemma 6.7 *The $U_\omega(C_1^l(r))$-cyclic subspaces $\mathcal{H}_{x_e\otimes a}$ generated by the vectors $x_e\otimes a$, $x_e \in \mathcal{T}_3$ an even site, are given by*

$$\mathcal{H}_{x_e\otimes a} = \overline{span} \quad \{ \quad \dots, x_eca\otimes b, x_eca\otimes c, x_e\otimes a, x_ec\otimes a, x_ec\otimes b, x_ec\otimes c, \\ x_ecb\otimes b, x_ecb\otimes c, x_ecbac\otimes a, x_ecba\otimes a, x_ecba\otimes b, x_ecba\otimes c, \cdots \}. \tag{96}$$

Their direct sum over x_e spans $\mathcal{K}_3$, taking into account the identities

$$\mathcal{H}_{x_e\otimes a} = \mathcal{H}_{x_ecabc\otimes a} = \mathcal{H}_{x_ecbac\otimes a}, \quad \forall x \in \mathcal{T}_3. \tag{97}$$

Remark 6.8 *Graphically, and without taking care of the coin states, the sites of $\mathcal{T}_3$ involved in (96) are depicted in figure 4.*

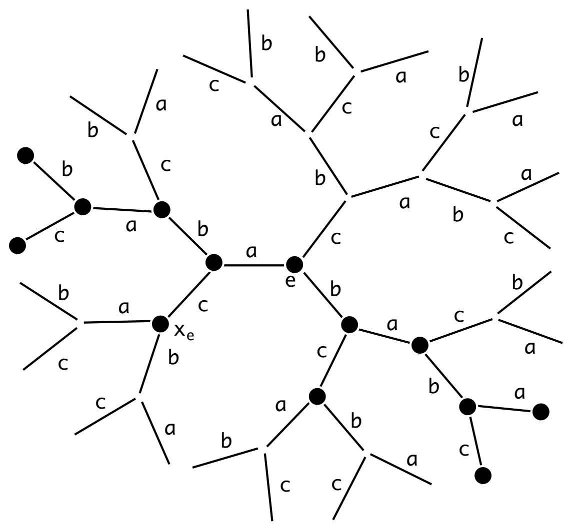

Figure 4: Sites from the cyclic subspace $\mathcal{H}_{x_e\otimes a}$.

Proof: One looks at the effect of powers of $U_\omega(C_1^l(r))$ on vectors related to the even site $y_e \in \mathcal{T}_3$: First note that $|\langle y_o\otimes\tau|U_\omega(C_1^l(r))y_e\otimes a\rangle| = \delta_{y_o,y_ec}\delta_{\tau,b}$, which means that $y_e\otimes a$ is sent to $y_ec\otimes b$ by $U_\omega(C_1^l(r))$. On the other hand, $\langle y_e\otimes a|U_\omega(C_1^l(r))y_o\otimes\tau\rangle$ equals zero, unless $y_o = y_ec$ and $\tau \in \{b, c\}$. Hence the vector $y_e \otimes a$ is never connected to $y_ea\otimes\tau$ or $y_eb\otimes\tau$, for any $\tau \in A_3$. Similarly, if $\tau \in \{b, c\}$,$\langle y_ec\otimes\sigma|U_\omega(C_1^l(r))y_e\otimes\tau\rangle = 0$, for all $\sigma \in A_3$, and the same is true for $\langle y_ec\otimes\sigma|U_\omega^*(C_1^l(r))y_e\otimes\tau\rangle$. In other words, the vectors $y_e\otimes\tau$, with $\tau \in \{b, c\}$, are never connected to $y_ec\otimes\sigma$, for any $\sigma \in A_3$. This is enough to reach the first conclusion of the lemma, while the second conclusion follows immediately. ∎

We now consider the restriction of $U_\omega(C_1^l(r))$ to one of its cyclic subspaces $\mathcal{H}_{x_e\otimes a}$. While the order provided in (96) allows for an easier identification of the periodicity, we use the following order to get a matrix representation for $U_\omega(C_1^l(r))$:

$$\begin{aligned}\{\dots, x_eca\otimes c, x_eca\otimes b, x_e\otimes a, x_ec\otimes c, x_ec\otimes a, x_ec\otimes b, x_ecb\otimes c, x_ecb\otimes b,\\ x_ecbac\otimes a, x_ecba\otimes c, x_ecba\otimes a, x_ecba\otimes b, \cdots\}\end{aligned} \tag{98}$$

To simplify the notation, we use labels e_j, $j\in\mathbb{Z}$, to denote these vectors in such a way that the set (98) corresponds to

$$\{\dots, e_{-1}, e_0, e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}, \cdots\}, \tag{99}$$

and the cyclic subspace is equivalent to $l^2(\mathbb{Z})$. Therefore, $U(C_1^l(r))|_{\mathcal{H}_{x_e\otimes a}}$ is equivalent to a seven-diagonal unitary matrix $V(C)$ in this basis

$$U(C_1^l(r))|_{\mathcal{H}_{x_e\otimes a}} \simeq V(C) = \begin{pmatrix} \ddots & 0 & & & & & & & & \\ 0 & 0 & & & 1 & & & & & \\ 0 & 0 & 0 & t & 0 & r & & & & \\ r & -t & 0 & 0 & 0 & 0 & & & & \\ & & 0 & 0 & 0 & 0 & t & r & & \\ & & 1 & 0 & 0 & 0 & 0 & 0 & & \\ & & & r & & -t & 0 & 0 & & \\ & & & & & & 0 & 0 & & \\ & & & & & & 0 & 0 & 0 & t \\ & & & & & & r & -t & 0 & \ddots \end{pmatrix}, \tag{100}$$

where the dots mark the diagonal, the first listed column is the image of e_{-1} and the periodicity is six along the diagonal. The matrix representation of $U_\omega(C_1^l(r))|_{\mathcal{H}_{x_e\otimes a}}$, denoted by $V_\omega(C)$, is obtained from (100) by multiplying the row labelled by $j\in\mathbb{Z}$ by a phase $e^{i\omega_j}$, where $\{\omega_j\}_{j\in\mathbb{Z}}$ are distributed according to $d\nu$.

The spectral analysis of the above one-dimensional unitary random operator on $l^2(\mathbb{Z})$ with a band structure can be performed by considering its generalized eigenvectors defined by $\psi = \sum_{j\in\mathbb{Z}}\psi(j)e_j$ and

$$(V_\omega(C) - z\mathbb{I})\psi = 0 \quad \text{in } l^2(\mathbb{Z}), \text{ where } z\in\mathbb{U}. \tag{101}$$

Proposition 6.9 *For any $z\in\mathbb{C}^*$, the solutions of (101) are entirely determined by the sequence $\{(\psi(6j-1), \psi(6j))\}_{j\in\mathbb{Z}}$. Moreover, we have the relation*

$$\begin{pmatrix}\psi(6j+5)\\ \psi(6j+6))\end{pmatrix} = T_z(j)\begin{pmatrix}\psi(6j-1)\\ \psi(6j)\end{pmatrix} \quad \textit{for any } j\in\mathbb{Z} \tag{102}$$

for a transfer matrix

$$T_z(j) = T_z(\omega_{6j}+\omega_{6j+3}, \omega_{6j+1}+\omega_{6j+4}, \omega_{6j+2}+\omega_{6j+5}) \in M_2(\mathbb{C}) \tag{103}$$

given by

$$T_z(\alpha,\beta,\gamma) = \frac{(rz^2e^{-i\beta}-1)e^{i\gamma}}{rz^2(z^2e^{-i\beta}-r)}\begin{pmatrix} r^2 & -tr \\ -tr & (\frac{e^{-i(\alpha+\gamma)}z^4(z^2e^{-i\beta}-r)}{(rz^2e^{-i\beta}-1)} + t^2)\end{pmatrix}. \tag{104}$$

Remark 6.10 *If $z = e^{-i\lambda} \in \mathbb{U}$ we have*

$$T_{e^{-i\lambda}}(\alpha, \beta, \gamma) = T_1(\alpha + 2\lambda, \beta + 2\lambda, \gamma + 2\lambda),$$
$$\det(T_1(\alpha, \beta, \gamma)) = \left(\frac{re^{-i\beta} - 1}{e^{-i\beta} - r}\right) e^{i(\gamma-\alpha)} \in \mathbb{U}. \quad (105)$$

Remark 6.11 *The expressions for the remaining coefficients read*

$$\begin{aligned}
\psi(6j+1) &= \frac{te^{i(\omega_{6j+2}-\omega_{6j+4})}(r\psi(6j-1) - t\psi(6j))}{(z^2 e^{-i(\omega_{6j+1}+\omega_{6j+4})} - r)} \quad (106)\\
\psi(6j+2) &= z^{-1}e^{i\omega_{6j+2}}(r\psi(6j-1) - t\psi(6j))\\
\psi(6j+3) &= ze^{-i\omega_{6j}}\psi(6j)\\
\psi(6j+4) &= \frac{te^{i\omega_{6j+2}}(r\psi(6j-1) - t\psi(6j))}{z(z^2 e^{-i(\omega_{6j+1}+\omega_{6j+4})} - r)}.
\end{aligned}$$

Proof: The proof is by explicit computation. ■

Remark 6.12 *In case $C_1^l(r)$ is replaced by $\Phi C_1^l(r)$, on first observes that it amounts to shift the random variables ω_j according to*

$$\begin{aligned}
&\omega_{6j} \mapsto \omega_{6j} + \varphi_2, \quad &\omega_{6j+1} \mapsto \omega_{6j+1} + \varphi_1, \quad &\omega_{6j+2} \mapsto \omega_{6j+2} + \varphi_3,\\
&\omega_{6j+3} \mapsto \omega_{6j+3} + \varphi_1, \quad &\omega_{6j+4} \mapsto \omega_{6j+4} + \varphi_2, \quad &\omega_{6j+5} \mapsto \omega_{6j+5} + \varphi_3,
\end{aligned} \quad (107)$$

for all $j \in \mathbb{Z}$. Consequently, this amounts to replace to the transfer matrix $T_z(\alpha, \beta, \gamma)$ by $\tilde{T}_z(\alpha, \beta, \gamma) = T_z(\alpha + \varphi_1 + \varphi_2, \beta + \varphi_1 + \varphi_2, \gamma + 2\varphi_3)$.

At this point, one observes that the random transfer matrices $\{T_z(j)\}_{j\in\mathbb{Z}}$ are i.i.d. so that we can follow the same route as that described in [BHJ], [HJS1] to prove spectral localization, via Shnol's and Fürstenberg's Theorems. Assuming that $d\nu$ has an absolutely continuous component with support of non empty interior, one needs to show that the group $\mathcal{G}$ generated by products of transfer matrices and their inverses is non compact and that they form an irreducible set of matrices in an appropriate sense, in order to get a positive Lyapunov exponent.

Concerning the first point we have

Lemma 6.13 *Assume that $0 < r < 1$, and that there exists $\theta_0 \neq \theta_1 \in \mathbb{T}$ in the support of $d\nu$. Then $\mathcal{G}$ is non compact.*

Proof: We first get rid of the dependence in the spectral parameter $z \in \mathbb{C}^*$ by making use of the following identities obtained by explicit computations. For any $z \in \mathbb{C}^*$ and any $0 < r < 1$

$$\begin{aligned}
T_z^{-1}(\alpha, \beta, \gamma)T_z(a, \beta, c) &= \begin{pmatrix} e^{i(c-\gamma)} & \frac{t}{r}(e^{i(\alpha-a)} - e^{i(c-\gamma)}) \\ 0 & e^{i(\alpha-a)} \end{pmatrix} \equiv R(c-\gamma, \alpha - a),\\
T_z(a, \beta, c)T_z^{-1}(\alpha, \beta, \gamma) &= \begin{pmatrix} e^{i(c-\gamma)} & 0 \\ \frac{t}{r}(e^{i(\alpha-a)} - e^{i(c-\gamma)}) & e^{i(\alpha-a)} \end{pmatrix} \equiv L(c-\gamma, \alpha - a). \quad (108)
\end{aligned}$$

Remark 6.14 *The maps R and L are invariant under the replacement of $C_1^l(r)$ by $\Phi C_1^l(r)$.*

Both maps $(\theta,\eta)\mapsto R(\theta,\eta)$ and $(\theta,\eta)\mapsto L(\theta,\eta)$ are group isomorphisms and we have

$$L(\theta,\eta)=R^T(\theta,\eta),\quad R(-\theta,-\eta)=\overline{R}(\theta,\eta)\quad \Rightarrow R(-\theta,-\eta)=L^*(\theta,\eta). \tag{109}$$

We compute

$$L(\theta,\eta)R(\alpha,\beta)=\begin{pmatrix} e^{i(\theta+\alpha)} & \frac{t}{r}e^{i\theta}(e^{i\beta}-e^{i\alpha}) \\ \frac{t}{r}e^{i\alpha}(e^{i\eta}-e^{i\theta}) & e^{i(\eta+\beta)}+\frac{t^2}{r^2}(e^{i\beta}-e^{i\alpha})(e^{i\eta}-e^{i\theta})\end{pmatrix} \tag{110}$$

s.t. $L(\theta,\eta)R(-\theta,-\eta)>0$, has determinant one and

$$\mathrm{tr}(L(\theta,\eta)R(-\theta,-\eta))=2\left(1+\frac{t^2}{r^2}(1-\cos(\theta-\eta))\right). \tag{111}$$

Consequently, one of the eigenvalues of this matrix has modulus larger than one, provided $\theta\neq\eta$ on $\mathbb{T}$. If the support of $d\nu$ is not reduced to a point, this is always true. ■

Concerning the second point, we introduce the map $\tau: M_2(\mathbb{C})\to M_4(\mathbb{R})$ defined by

$$\begin{pmatrix} a & b\\ c & d\end{pmatrix}\to\begin{pmatrix}\Re(a)I+\Im(a)J & \Re(b)I+\Im(b)J\\ \Re(c)I+\Im(c)J & \Re(d)I+\Im(d)J\end{pmatrix}, \tag{112}$$

where

$$I=\begin{pmatrix}1&0\\0&1\end{pmatrix},\quad J=\begin{pmatrix}0&1\\-1&0\end{pmatrix}. \tag{113}$$

This map is a homeomorphism from $M_2(\mathbb{C})$ to $\tau(M_2(\mathbb{C}))$ and, in particular, a group homeomorphisms from the set of matrices in $M_2(\mathbb{C})$ with determinant of modulus one to the set of matrices in $M_4(\mathbb{R})$ with determinant of modulus one. The notion of irreducibility we need is the content of the next lemma.

Lemma 6.15 *The set $\{\tau(T_{e^{-i\lambda}}(\alpha,\beta,\gamma))\in M_4(\mathbb{R}),(\alpha,\beta,\gamma)\in supp\ d\nu+supp\ d\nu\}$ is irreducible in $\mathbb{R}^4$ if the support of $d\nu$ has non empty interior.*

Proof: We first note that it is enough to consider $T_1(\alpha,\beta,\gamma)$ with $(\alpha,\beta,\gamma)\in I^3$, where $I\subset\mathbb{T}$ is an arbitrary open arc. Then, with

$$\frac{re^{-i\beta}-1}{e^{-i\beta}-r}=e^{-i\chi(\beta)},\quad A=\gamma-\chi(\beta),\quad B=-\alpha, \tag{114}$$

we can write

$$\tau(T_1(\alpha,\beta,\gamma)) = \cos(A)M_1+\sin(A)M_2+\cos(B)N_1+\sin(B)N_2 \tag{115}$$

with

$$M_1=\begin{pmatrix} r&0&-t&0\\ 0&r&0&-t\\ -t&0&t^2/r&0\\ 0&-t&0&t^2/r\end{pmatrix}, M_2=\begin{pmatrix}0&r&0&-t\\ -r&0&t&0\\ 0&-t&0&t^2/r\\ t&0&-t^2/r&0\end{pmatrix}, \tag{116}$$

and

$$N_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1/r & 0 \\ 0 & 0 & 0 & 1/r \end{pmatrix}, N_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/r \\ 0 & 0 & -1/r & 0 \end{pmatrix}. \quad (117)$$

Keeping β fixed, any nontrivial subspace $V \subset \mathbb{R}^4$ which is invariant under $\tau(T_1(\alpha, \beta, \gamma))$ has to be invariant under M_1, M_2, N_1 and N_2, since A and B are independent. Since these last two matrices are real (anti) self-adjoint, they leave $V^\perp$ invariant as well. Hence, V and $V^\perp$ are generated by real eigenvectors of these matrices, if they are diagonalizable over $\mathbb{R}$. Hence, if $\dim V = 2$, it can be generated by following set of vectors only, $\{e_1, e_2\}$ or $\{e_3, e_4\}$, where $\{e_1, e_2, e_3, e_4\}$ is the canonical basis of $\mathbb{R}^4$. This is ruled out by the fact the these subspaces are not invariant under M_1. Also, if V is of dimension 1, the only possibility is $V \subset \overline{\text{span}}\{e_1, e_2\}$. The same argument forbids this and since it applies to $V^\perp$ as well, which takes care of the case where $\dim V = 3$. ■

Remark 6.16 *If $C_1^l(r)$ is replaced by $\Phi C_1^l(r)$, the same argument proves the Lemma since β is fixed and A and B are given by γ and $-\alpha$ plus a constant term in that case.*

The arguments provided in [HJS1] prove that Proposition 6.5 derives from these properties.

6.1.5 Spectral Transition for $q = 3$

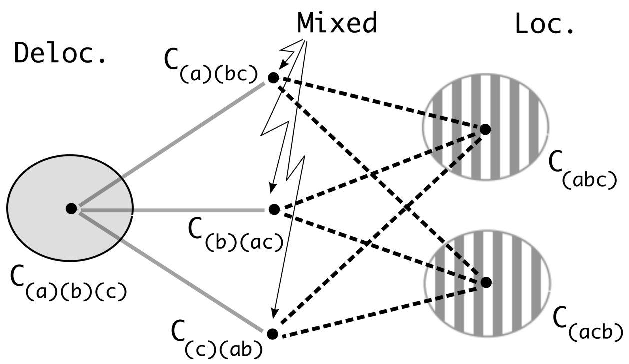

Figure 5: Partial spectral diagram for $q = 3$.

The foregoing shows the existence of six continuous paths in $U(3)$ from a small neighborhood of the set Λ of localizing coin matrices to a small neighborhood of the set $\mathcal{S}$ of delocalizing coin matrices, through elements of the set $\mathcal{M}$ of coin matrices inducing mixed spectra. Each element of Λ is linked to an element of $\mathcal{M}$ by means of the family $C_j^l(r)$, with suitable decorating phase $\Phi(r)$, on which almost sure localization takes place. And each element of $\mathcal{M}$ is linked to the only element of $\mathcal{S}$ by a path of the form $C_j^d(r)$, with suitable decorating phase Φ, which induces absolutely continuous spectrum for all ω for the corresponding walk. The spectral diagram in Figure 5 doesn't show it explicitly, but as mentioned above, it holds for matrices decorated by phases Φ as well.

6.2 $q = 4$

Without attempting to provide a detailed analysis of the spectral diagram for $q = 4$ here, we describe a spectral transition from $\mathcal{L}$ to $\mathcal{D}$ which is different from the case $q = 3$ in the sense that it avoids elements from $\mathcal{M}$.

The sites of the tree $\mathcal{T}_4$ are labeled according to Figure 1 and the coin dependent shift S on $\mathcal{K}_4 = \mathcal{T}_4 \otimes \mathbb{C}^4$ reads

$$S = \sum_{\tau \in A_4} S_\tau \otimes |\tau\rangle\langle\tau|. \tag{118}$$

We also switch to more convenient notations in this case: The alphabet is denoted by $A_4 = \{a, b, a^{-1}, b^{-1}\}$ and the orthonormal basis of basis of the coin Hilbert space is denoted by $\{|a\rangle, |b\rangle, |a^{-1}\rangle, |b^{-1}\rangle\}$. The 4×4 coin matrices C are written in the basis ordered as above. In particular, explicitly for $U_\omega(C) = \mathbb{D}_\omega U(C)$

$$\begin{aligned} &U_\omega(C) x \otimes \tau = \\ &e^{i\omega^a_{xa}} C_{a\tau}\ xa \otimes a + e^{i\omega^b_{xb}} C_{b\tau}\ xb \otimes b + e^{i\omega^{a^{-1}}_{xa^{-1}}} C_{a^{-1}\tau}\ xa^{-1} \otimes a^{-1} + e^{i\omega^{b^{-1}}_{xb^{-1}}} C_{b^{-1}\tau}\ xb^{-1} \otimes b^{-1}. \end{aligned} \tag{119}$$

6.2.1 Propagating, Reducing and Localizing Families

Propagating, respectively reducing, coin matrices take the form

$$\begin{pmatrix} * & * & 0 & * \\ * & * & * & 0 \\ 0 & * & * & * \\ * & 0 & * & * \end{pmatrix} \in \mathcal{P}, \quad \text{respectively} \begin{pmatrix} * & 0 & * & 0 \\ 0 & * & 0 & * \\ * & 0 & * & 0 \\ 0 & * & 0 & * \end{pmatrix} \in \mathcal{R}. \tag{120}$$

In particular, for the two parameter family of reducing matrices

$$[0, 2\pi]^2 \ni (\psi, \xi) \mapsto C^R(\psi, \xi) = \begin{pmatrix} \cos(\psi) & 0 & \sin(\psi) & 0 \\ 0 & \cos(\xi) & 0 & \sin(\xi) \\ -\sin(\psi) & 0 & \cos(\psi) & 0 \\ 0 & -\sin(\xi) & 0 & \cos(\xi) \end{pmatrix} \tag{121}$$

it holds

$$\sigma(U_\omega(C^R(\psi, \xi))) = \sigma_{pp}(U_\omega(C^R(\psi, \xi))) \text{ a.s.} \Leftrightarrow \sin(\psi)\sin(\xi) \neq 0. \tag{122}$$

On the other hand, for the families $[0, 2\pi)^2 \ni (\psi, \xi) \mapsto C^P_j(\psi, \xi) \in U(4)$, $j \in \{1, \dots, 4\}$, propagating matrices

$$C^P_1(\psi, \xi) = \begin{pmatrix} \cos(\psi) & 0 & 0 & \sin(\psi) \\ 0 & \cos(\xi) & \sin(\xi) & 0 \\ 0 & -\sin(\xi) & \cos(\xi) & 0 \\ -\sin(\psi) & 0 & 0 & \cos(\psi) \end{pmatrix},$$

$$C_2^P(\psi,\xi)) = \begin{pmatrix} 0 & \cos(\xi) & 0 & \sin(\xi) \\ \cos(\psi) & 0 & \sin(\psi) & 0 \\ 0 & -\sin(\xi) & 0 & \cos(\xi) \\ -\sin(\psi) & 0 & \cos(\psi) & 0 \end{pmatrix},$$

$$C_3^P(\psi,\xi) = \begin{pmatrix} \cos(\psi) & \sin(\psi) & 0 & 0 \\ -\sin(\psi) & \cos(\psi) & 0 & 0 \\ 0 & 0 & \cos(\xi) & \sin(\xi) \\ 0 & 0 & -\sin(\xi) & \cos(\xi) \end{pmatrix}, \tag{123}$$

it holds for any realization ω, any $(\psi,\xi) \in [0,2\pi)^2$ and any $j \in \{1,\dots,4\}$,

$$\sigma(U_\omega(C_j^P(\psi,\xi))) = \sigma_{ac}(U_\omega(C_j^P(\psi,\xi))). \tag{124}$$

Moreover, we have existence of localizing families of coin matrices:

Lemma 6.17 *The four one parameter families $[0,2\pi) \ni \psi \mapsto C_j(\psi)$, $j \in \{1,2,3,4\}$,*

$$C_1(\psi) = \begin{pmatrix} 0 & 0 & \cos(\psi) & \sin(\psi) \\ 0 & 0 & -\sin(\psi) & \cos(\psi) \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, C_2(\psi) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \cos(\psi) & \sin(\psi) & 0 & 0 \\ -\sin(\psi) & \cos(\psi) & 0 & 0 \end{pmatrix},$$

$$C_3(\psi) = \begin{pmatrix} 0 & \cos(\psi) & \sin(\psi) & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -\sin(\psi) & \cos(\psi) & 0 \end{pmatrix}, C_4(\psi) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ \cos(\psi) & 0 & 0 & \sin(\psi) \\ -\sin(\psi) & 0 & 0 & \cos(\psi) \\ 0 & 1 & 0 & 0 \end{pmatrix}, \tag{125}$$

are such that for all realizations ω, $U_\omega(C_j(\psi))$ is pure point.

Proof: It is enough to observe that for each $j = 1,2,3,4$, the following four-dimensional subspaces labeled by $x \in \mathcal{T}_4$ are invariant under $U_\omega(C_j(r))$,

$$\mathcal{H}_x^1 = \{x \otimes a, xa^{-1} \otimes a^{-1}, xa^{-1}b \otimes b, xa^{-1} \otimes b^{-1}\} \tag{126}$$

$$\mathcal{H}_x^2 = \{x \otimes a, xb^{-1} \otimes b^{-1}, xa^{-1} \otimes a^{-1}, x \otimes b\} \tag{127}$$

$$\mathcal{H}_x^3 = \{x \otimes a, xa^{-1} \otimes a^{-1}, xa^{-1}b^{-1} \otimes b^{-1}, xa^{-1} \otimes b\} \tag{128}$$

$$\mathcal{H}_x^4 = \{x \otimes a, xb \otimes b, xa^{-1} \otimes a^{-1}, x \otimes b^{-1}\}, \tag{129}$$

and $\oplus_{x \in \mathcal{T}_4} \mathcal{H}_x^j = \mathcal{K}_4$. ∎

Remark 6.18 *The statements (122), (124) and Lemma 6.17 remain true if these matrices are decorated by phases, possibly depending on (ψ,ξ).*

6.2.2 Permutation Coin Matrices

There exist 24 permutations of the alphabet $A_4 = \{a, b, a^{-1}, b^{-1}\}$ giving rise to coin matrices inducing walks with a variety of different spectral properties. As in the case of $q = 3$, a number of them belong to $\mathcal{L}$ and give rise to fully localized walks

$$\Lambda = \{C_{(abb^{-1}a^{-1})}, C_{(aa^{-1}bb^{-1})}, C_{(aa^{-1}b^{-1}b)}, C_{(ab^{-1}ba^{-1})}, C_{(aa^{-1})(bb^{-1})}\}, \tag{130}$$

with $C_{(aa^{-1})(bb^{-1})} \in \mathcal{R}$. These matrices are special cases of Lemma 6.17 and their respective cyclic subspaces labeled by $x \in \mathcal{T}_4$ are $\mathcal{H}_x^4, \mathcal{H}_x^1, \mathcal{H}_x^3, \mathcal{H}_x^2$ and

$$\mathcal{H}_x^{12} = \overline{\text{span}}\,\{x \otimes a, xa^{-1} \otimes a^{-1}\} \oplus \overline{\text{span}}\,\{x \otimes b, xb^{-1} \otimes b^{-1}\}. \tag{131}$$

There are 9 permutation coin matrices that are propagating matrices from $\mathcal{P}$ and give rise to absolutely continuous spectrum for any deterministic $\mathbb{D}_\omega$:

$$\begin{aligned}\Pi_1 = \{&(aba^{-1}b^{-1}), (ab^{-1}a^{-1}b), (ab)(a^{-1}b^{-1}), (ab^{-1})(ba^{-1}), (a)(b)(a^{-1}b^{-1}), \\ &(a)(b^{-1})(ba^{-1}), (ab^{-1})(b)(a^{-1}), (ab)(a^{-1})(b^{-1}), (a)(b)(a^{-1})(b^{-1})\} \in \mathcal{P}.\end{aligned} \tag{132}$$

These matrices are special cases of $C_j^P(\psi, \xi)$ defined in the previous subsection. From those, the subset

$$\mathcal{S} = \{C_{(aba^{-1}b^{-1})}, C_{(ab^{-1}a^{-1}b)}\} \subset \Pi_1 \tag{133}$$

gives rise to a spiral-like walk on the tree and are fully delocalized, see Figure 6. All other

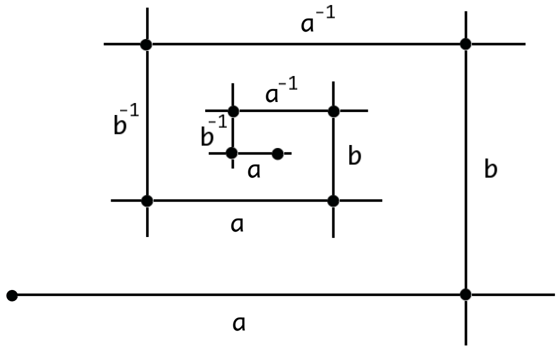

Figure 6: Spiral-like walk for $(aba^{-1}b^{-1})$.

permutations from Π_1 give rise to independent shifts on the tree. For example, the coin matrix corresponding to the permutation $\pi = (ab)(a^{-1}b^{-1})$ has cyclic subspaces given by

$$\overline{\text{span}}\,\{\dots, xa^{-1}b^{-1} \otimes a, xa^{-1} \otimes b, x \otimes a, xb \otimes b, xba \otimes a, \dots\} \tag{134}$$

$$\overline{\text{span}}\,\{\dots, xab \otimes a^{-1}, xa \otimes b^{-1}, x \otimes a^{-1}, xb^{-1} \otimes b^{-1}, xb^{-1}a^{-1} \otimes a^{-1}, \dots\} \tag{135}$$

labeled with $x \in \mathcal{T}_4$. Another set of permutation matrices that give rise to absolutely continuous spectrum but do not belong to $\mathcal{P}$ is given by

$$\begin{aligned}\Pi_2 = \{&(a)(ba^{-1}b^{-1}), (a)(bb^{-1}a^{-1}), (b)(aa^{-1}b^{-1}), (b)(ab^{-1}a^{-1}), \\ &(a^{-1})(abb^{-1}), (a^{-1})(ab^{-1}b), (b^{-1})(aba^{-1}), (b^{-1})(aa^{-1}b)\}.\end{aligned} \tag{136}$$

Let us take a closer look at the operator $U_\omega(C_{(a)(ba^{-1}b^{-1})})$. It leaves the subspace $l^2(\mathcal{T}_4) \otimes |a\rangle$ invariant acting essentially as a shift on the corresponding cyclic subspaces

$$\mathcal{H}_{x_a}^1 = \overline{\text{span}}\,\{\dots, xa^{-1}a^{-1} \otimes a, xa^{-1} \otimes a, x \otimes a, xa \otimes a, xaa \otimes a, \dots\} \tag{137}$$

labeled by $x_a \in \mathcal{T}_4$, see the proof of Lemma 3.3 for the notation, and

$$\mathcal{H}^2_{x_a} = \overline{\text{span}}\{\dots, x \otimes a^{-1}, xb^{-1} \otimes b^{-1}, x \otimes b, xa^{-1} \otimes a^{-1}, xa^{-1}b^{-1} \otimes b^{-1}, xa^{-1} \otimes b, \dots\} \quad (138)$$

labeled by $x_a \in \mathcal{T}_4$ that sum up to $\mathcal{K}_4$. The list of permutation matrices is completed by two coin matrices from $\mathcal{M}$

$$\{C_{(a)(bb^{-1})(a^{-1})}, C_{(aa^{-1})(b)(b^{-1})}\} \subset \mathcal{M}, \quad (139)$$

which are special cases of $C^R(\psi, \xi)$ defined in (121). Taking a closer look at the first of those matrices leaves the subspaces $l^2(\mathcal{T}_4) \otimes |a\rangle$ and $l^2(\mathcal{T}_4) \otimes |a^{-1}\rangle$ invariants, where U_ω is essentially driven by shift S_a and S_a^{-1} acting on the cyclic subspaces $\overline{\text{span}}\{\cdots, xa^{-1} \otimes a, x \otimes a, xa \otimes a, \cdots\}$ and $\overline{\text{span}}\{\cdots, xa \otimes a^{-1}, x \otimes a^{-1}, xa^{-1} \otimes a^{-1}, \cdots\}$ labeled by $x \in \mathcal{T}_4$. On the other hand, for all $x \in \mathcal{T}_4$, the two dimensional subspace $\overline{\text{span}}\{x \otimes b, xb^{-1} \otimes b^{-1}\}$ is invariant under $U_\omega(C_{(a)(bb^{-1})(a^{-1})})$. Therefore the spectrum contains both absolutely continuous and pure point parts. The case of $C_{(aa^{-1})(b)(b^{-1})}$ is similar.

Remark 6.19 *All results of this section hold true if the permutation matrices $C_\pi \in U(4)$ are replaced by decorated permutation matrices C_π^Φ.*

6.2.3 Spectral Transition for $q = 4$

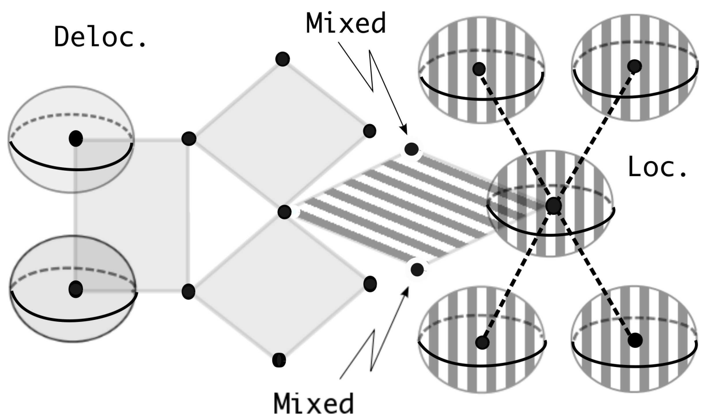

Figure 7: Partial spectral diagram for $q = 4$.

From the foregoing, we get the existence of a continuous path of coin matrices which links localizing matrices from a small neighborhood of Λ to delocalizing matrices from a small neighborhood of $\mathcal{S}$. All elements of Λ are linked by paths described by the one parameter families $C_j(\psi)$, with suitable decorating phases, giving rise to pure point spectrum for all ω. Then $C_{(aa^{-1})(bb^{-1})} \in \Lambda$ is linked to $C_{(a)(b)(a^{-1})(b^{-1})} \in \Pi_1 \subset \mathcal{P}$ by the two parameter family $C^R(\psi, \xi)$ with suitable decorating phases, which gives rise to pure point spectrum, for almost all ω. Eventually, $C_{(a)(b)(a^{-1})(b^{-1})}$ is linked to all other elements of Π_1, in particular to the elements of $\mathcal{S}$, by the the two parameter families $C_j^P(\psi, \xi)$ with suitable decorating phases which yield absolutely continuous spectrum for all ω. This is illustrated in Figure 7, where the elements of Π_2 do not appear since they play no role in this transition.

A Proof of Proposition 4.2:

We argue, following [J3, ABJ, HJS2]. We first control the probability to have spectrum in a neighborhood of any point $z \notin \mathbb{U}$: Lemma 3.5 implies that for any arc $A \subset \mathbb{U}$ of small enough length, we have $\mathbb{P}(\sigma(U_\omega(C^\Phi_{\pi 0})|_{\mathcal{H}_{x_o}}) \cap A = \emptyset) \geq 1 - c|A|$, for some $c > 0$. By independence of the $\{\theta^{x_o}_\omega\}_{x_o \in \mathcal{T}_q}$ and $\dim \mathcal{H}_{\Lambda_L(x_e)} \geq (q-1)^L$,

$$\mathbb{P}(\sigma(U^{\Lambda_L(x_e)}_\omega) \cap A = \emptyset) \geq (1 - c|A|)^{(q-1)^L}. \tag{140}$$

It then follows as in [J3], Lemma 4, that for any $z \notin \mathbb{U}$ and any $\eta > 0$ with $\eta(q-1)^L$ small enough,

$$\mathbb{P}(\mathrm{dist}(z, \sigma(U^{\Lambda_L(x_e)}_\omega)) \leq \eta) \leq c\eta(q-1)^L. \tag{141}$$

We now address the Green functions. We first recall Theorem 3.1 of [HJS2], a result which holds for general unitary operators of the form $\mathbb{D}_\omega S$ with $\mathbb{D}_\omega$ diagonal as in (23) and S deterministic and banded: There exists $C(s) > 0$ such that

$$\mathbb{E}(|G_{a_j,a_k,\omega}(x,y;C,z)|^s) \leq C(s), \tag{142}$$

for any $C \in U(q)$, any $x \otimes a_j, y \otimes a_k \in \mathcal{K}_q$. Moreover, (142) holds independently of and uniformly in the dimension of the underlying Hilbert space. This proves estimate (61) holds for $\alpha = 0$, for any $C \in U(q)$, any $x \otimes a_j, y \otimes a_k \in \mathcal{K}_q$. For $\alpha > 0$, we make use of the properties of $U^{\Lambda_L(x_e)}_\omega(C)$, perturbation theory and (141).

We introduce the relation $x \otimes a_j \sim y \otimes a_k$ which denotes the property $x \otimes a_j$ and $y \otimes a_k$ belong to the same subspace $\mathcal{H}_{x_o}$, for some $x_o \in \mathcal{T}_q$. Note that $x \otimes a_j \sim y \otimes a_k$ implies $d(x,y) \leq 2$, and

$$x \otimes a_j \not\sim y \otimes a_k \;\Rightarrow\; G^{\Lambda_L(x_e)}_{a_j,a_k,\omega}(x,y;C^\Phi_{\pi_o}, z) = 0. \tag{143}$$

We start from the second resolvent identity

$$G^{\Lambda_L}_\omega(C,z) = G^{\Lambda_L}_\omega(C^\Phi_{\pi 0}, z) + G^{\Lambda_L}_\omega(C)\left(U^{\Lambda_L(x_e)}(C^\Phi_{\pi 0}) - U^{\Lambda_L(x_e)}(C)\right) G^{\Lambda_L}_\omega(C^\Phi_{\pi 0}, z), \tag{144}$$

with the notation $G^{\Lambda_L}_\omega(C,z) = (U^{\Lambda_L(x_e)}_\omega(C) - z)^{-1}$ and take into account (56) and the fact that $U_\omega(C)$ couples nearest neighbors on the tree only. We get that for $x \otimes a_j, y \otimes a_k \in \mathcal{H}^{\Lambda_L(x_e)}$ such that $d(x,y) > 2$, and $z \notin \mathbb{U}$,

$$\begin{aligned} &|G^{\Lambda_L(x_e)}_{a_j,a_k,\omega}(x,y;C,z)| \leq \\ &\sum_{\substack{x'\otimes a_{j'}, y'\otimes a_{k'} \\ y'\otimes a_{k'} \sim y\otimes a_k, \\ d(x',y')\leq 1}} \left| G^{\Lambda_L(x_e)}_{a_j,a_{j'},\omega}(x,x';C,z) \langle x'\otimes a_{j'} | \left(U^{\Lambda_L(x_e)}_\omega(C) - U^{\Lambda_L(x_e)}_\omega(C^\Phi_{\pi 0})\right) y'\otimes a_{k'}\rangle \right| \\ &\qquad \times \left| G^{\Lambda_L(x_e)}_{a_{k'},a_k,\omega}(y',y;C^\Phi_{\pi 0}, z)\right| \\ &\leq c \frac{\|C - C^\Phi_{\pi 0}\|}{\mathrm{dist}\left(z, \sigma\left(U^{\Lambda_L}_\omega(C^\Phi_{\pi 0})\right)\right)} \sup_{\substack{x'\otimes a_{j'}, y'\otimes a_{k'} \\ y'\otimes a_{k'} \sim y\otimes a_k,\ d(x',y')\leq 1}} |G^{\Lambda_L(x_e)}_{a_j,a_{j'},\omega}(x,x';C,z)|, \end{aligned} \tag{145}$$

where c is some numerical constant and the number of sites involved in the sup is independent of L. The probabilistic control of the denominator is done as follows. Let $z \notin \mathbb{U}$ and $\eta > 0$, and set

$$G_\eta(z) := \left\{\omega \in \Omega \mid \text{dist}\left(z, \sigma\left(U_\omega^{\Lambda_L(x_e)}(C_{\pi_0}^\Phi)\right)\right) > \eta\right\} \tag{146}$$

with $G_\eta^C(z)$, its complement. Denote by χ_A the characteristic function of the set A. Now, for $0 < s < 1$, $p' > 1/(1-s)$ and q' such that $1/p' + 1/q' = 1$, we have $q's < 1$ so that using Hölder's inequality, (142) and (141)

$$\begin{aligned}&\mathbb{E}(\chi_{G_\eta^C(z)}|G_{a_j,a_k,\omega}^{\Lambda_L(x_e)}(x,y;C,z)|^s) \leq \\ &\mathbb{P}(G_\eta^C(z))^{1/p'}\left(\mathbb{E}(|G_{a_j,a_k,\omega}^{\Lambda_L(x_e)}(x,y;C,z)|^{q's}\right)^{1/q'} \leq c(\eta(q-1)^L)^{1/p'}.\end{aligned} \tag{147}$$

If $\|C - C_{\pi_0}^\Phi\| \leq \tilde{c}\eta$ for some $\tilde{c} > 0$, perturbation theory implies

$$\text{dist}\left(\sigma\left(U_\omega^{\Lambda_L(x_e)}(C_{\pi_0}^\Phi)\right), z\right) > \eta \Rightarrow \text{dist}\left(\sigma\left(U_\omega^{\Lambda_L(x_e)}(C)\right), z\right) > \frac{\eta}{2}. \tag{148}$$

Thus, using (145) we get the estimate

$$\chi_{G_\eta(z)}(\omega)|G_{a_j,a_k,\omega}^{\Lambda_L(x_e)}(x,y;C,z)|^s \leq c\frac{\|C - C_{\pi_0}^\Phi\|^s}{\eta^{2s}}, \tag{149}$$

hence,

$$\mathbb{E}\left(\chi_{G_\eta(z)}(\omega)|G_{a_j,a_k,\omega}^{\Lambda_L(x_e)}(x,y;C,z)|^s\right) \leq c\frac{\|C - C_{\pi_0}^\Phi\|^s}{\eta^{2s}}. \tag{150}$$

Altogether, this yields for some numerical constant

$$\mathbb{E}\left(|G_{a_j,a_k,\omega}^{\Lambda_L(x_e)}(x,y;C,z)|^s\right) \leq c\left(\frac{\|C - C_{\pi_0}^\Phi\|^s}{\eta^{2s}} + (\eta(q-1)^L)^{1/p'}\right). \tag{151}$$

One checks that for $c_0(s)$ small enough, uniform in α and $L \geq 3$, (but p'-dependent)

$$\eta = \frac{c_0(s)}{\tilde{c}}e^{-L\alpha/p'}(q-1)^{-L} \tag{152}$$

$$\|C - C_{\pi_0}^\Phi\| \leq c_0(s)e^{-L\alpha(1/s+2/p')}(q-1)^{-2L} \tag{153}$$

imply (61) and satisfy both conditions $\eta(q-1)^L$ small, and $\|C - C_{\pi_0}^\Phi\| \leq \tilde{c}\eta$. ■

B Proof of Proposition 4.3:

Again, we follow [J3, HJS2, ABJ]. The operator T, defined by (62), satisfies

$$\begin{aligned} T^L x \otimes a_j = 0 \quad & \text{if } |d(x,x_e) - L| \geq 2 \\ T^{L^*} x \otimes a_j = 0 \quad & \text{if } |d(x,x_e) - L| \geq 3 \\ \langle y \otimes a_k | T^L x \otimes a_j\rangle = 0 \quad & \text{if } d(x,y) \geq 2 \end{aligned} \tag{154}$$

We use the resolvent identity twice, once on Λ_L, once on Λ_{L+3} to get

$$\begin{aligned} G &= G^L - G^L T^L G \\ &= G^L - G^L T^L G^{L+3} + G^L T^L G T^{L+3} G^{L+3}. \end{aligned} \tag{155}$$

The properties (63), (154) yield for vectors $x \otimes a_j$, $y \otimes a_k$ with $d(x, x_e) \leq L$ and $d(x_e, y) \geq L + 5$

$$\langle x \otimes a_j | G\, y \otimes a_k \rangle = \langle x \otimes a_j | G^L T^L G T^{L+3} G^{L+3}\, y \otimes a_k \rangle, \tag{156}$$

which is the geometric resolvent identity to be used below. We define

$$\begin{aligned} \partial \mathcal{H}_{\Lambda_L} &= \left\{ x \otimes a_j, y \otimes a_k \mid \langle y \otimes a_k | T^L x \otimes a_j \rangle \neq 0 \right\} \\ &\subset \left\{ x \otimes a_j, y \otimes a_k \mid |d(x, x_e) - L| \leq 1, |d(y, x_e) - L| \leq 2, d(x, y) \leq 1 \right\} \end{aligned} \tag{157}$$

and we expand (156) over the boundaries of $\mathcal{H}_{\Lambda_L}$ and $\mathcal{H}_{\Lambda_{L+3}}$ to get

$$\begin{aligned} &\langle x \otimes a_j | G\, y \otimes a_k \rangle = \\ &\sum_{\substack{(u,u') \in \partial \mathcal{H}_L \\ (v,v') \in \partial \mathcal{H}_{L+3}}} \langle x \otimes a_j | G^L | u \rangle \langle u |, T^L | u' \rangle \langle u' | G | v \rangle \langle v |, T^{L+3} | v' \rangle \langle v' | G^{L+3} | y \otimes a_k \rangle. \end{aligned} \tag{158}$$

Taking the power $0 < s < 1$ and the expectation, we are lead to

$$\begin{aligned} &\mathbb{E}(|\langle x \otimes a_j | G\, y \otimes a_k \rangle|^s) \leq c\, t^{2s} \\ &\qquad \times \sum_{\substack{(u,u') \in \partial \mathcal{H}_L \\ (v,v') \in \partial \mathcal{H}_{L+3}}} \mathbb{E}\left(|\langle x \otimes a_j | G^L\, u \rangle|^s |\langle u' | G\, v \rangle|^s |\langle v' | G^{L+3}\, y \otimes a_k \rangle|^s \right). \end{aligned} \tag{159}$$

Thanks to (63), the vectors $u \in \mathcal{H}_{\Lambda_L}$ and $v' \in \mathcal{H}_{\Lambda_{L+3}^\perp}$ give stochastically independent contributions. Therefore we resort to a resampling argument to decouple the expectations and to the general estimate (142) to get rid of the full resolvent term. The resampling argument requires $s \in (0, 1/3)$, which we will assume from now on. This result stated as Proposition 13.1 in [HJS2] is based on the structure $U_\omega = \mathbb{D}_\omega S$ and implies, with the notation $u = u_1 \otimes u_2 \in \mathcal{H}^{\Lambda_L}$, $u_1 \in \mathcal{T}_q$, $u_2 \in A_q$:

For every $s \in (0, 1/3)$ there exists a constant $c < \infty$, depending on s, such that

$$\begin{aligned} &\mathbb{E}(|\langle x \otimes a_j | G\, y \otimes a_k \rangle|^s) \leq c\, t^{2s} \\ &\quad \times \sum_{\substack{u \in \mathcal{H}_{\Lambda_L} \\ |d(u_1, x_e) - L| \leq 2}} \mathbb{E}(|\langle x \otimes a_j | G^L\, u \rangle|^s) \sum_{\substack{v' \in \mathcal{H}^\perp_{\Lambda_{L+3}} \\ |d(v'_1, x_e) - (L+3)| \leq 1}} \mathbb{E}(|\langle v' | G^{L+3}\, y \otimes a_k \rangle|^s) \end{aligned} \tag{160}$$

uniformly in $z \notin \mathbb{U}$ with $1/2 < |z| < 2$, $L \in \mathbb{N}$ and $x, y \in \mathcal{T}_q$ with $d(x, x_e) \leq L$ and $d(y, x_e) \geq L + 5$.

Next we relate G^{L+3} to G by means of

$$G^{L+3} = G + G^{L+3} T^{L+3} G, \tag{161}$$

and expand

$$T^{L+3} = \sum_{(w,w')\in\partial\mathcal{H}_{L+3}} |w\rangle\langle w|T^{L+3}|w'\rangle\langle w'|. \tag{162}$$

Altogether we get

$$\begin{aligned}\mathbb{E}(|\langle v'|G^{L+3}\, y\otimes a_k\rangle|^s) &\le \mathbb{E}(|\langle v'|G\, y\otimes a_k\rangle|^s) \\ &+ c\, t^s \sum_{(w,w')\in\partial\mathcal{H}_{L+3}} \mathbb{E}(|\langle v'|G^{L+3}\, w\rangle|^s|\langle w'|G\, y\otimes a_k\rangle|^s).\end{aligned} \tag{163}$$

Then, another application of a resampling argument to factorize the expectations together with estimate (142) and

$$\#\{w\in\mathcal{H}\ |d(w_1,x_e)-(L+k)|\le L+j\}\le c(k,j)(q-1)^L, \tag{164}$$

for k,j fixed, eventually yields (64), in a similar way as what is done to get Proposition 13.2 in [HJS2]. ■

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