Schrödinger operator on tree-shaped networks: stabilization and dispersive effects

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Outline

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- Abstract and notations
- Well-posedness of the system
- Spectral analysis
- Riesz basis
- Energy decreasing
- Energy decreasing using the Riesz basis
- Transfer function analysis
- Stabilization to zero by changing the feedback law

2 Dispersive effects for the Schrödinger operator

- Star-Shaped Network case
- Tadpole graph case

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Abstract and notations

We analyse the spectrum of the dissipative Schrödinger operator on binary tree-shaped networks. As applications, we study the stability of the Schrödinger system using a Riesz basis as well as the transfer function associated to the system.

- A multi-index $\bar{\alpha}$ is a k-tuple $(\alpha_1, \ldots, \alpha_k)$ if k lies in $\mathbb{N} - \{0\}$ and it is empty if $k = 0$. For a fixed integer n, we choose for I the set of multi-indices $\bar{\alpha}$, with length k in $\{0, 1, \ldots, n\}$, such that, if $k \neq 0$, $\alpha_j \in \{1; 2\}$, for all j in $\{1, \ldots, k\}$. Then the set of vertices V of the tree $\mathcal{T}$ is $V := (\cup_{\bar{\alpha} \in I} \mathcal{O}_{\bar{\alpha}}) \cup \{\mathcal{R}\}$ where $\mathcal{R}$ is an additional vertex which will be the root of the tree $\mathcal{T}$.
- The edges are denoted by $e_{\bar{\alpha}}$ with $\bar{\alpha}$ in I. Note that the number of edges is the cardinal of I and it holds: $|I| = N = 2^{n+1} - 1$.

Define, for any non-empty multi-indices $\bar{\alpha} = (\alpha_1, ..., \alpha_k)$ and $\bar{\beta} = (\beta_1, ..., \beta_m)$, the multi-index $\bar{\alpha} \circ \bar{\beta} := (\alpha_1, ..., \alpha_k, \beta_1, ..., \beta_m)$ of length $(k+m)$. Then, for a non-empty multi-index $\bar{\alpha} = (\alpha_1, ..., \alpha_k)$, the edge $e_{\bar{\alpha}}$ is chosen to have the extremities $\mathcal{O}_{\bar{\alpha}}$ and $\mathcal{O}_{\bar{\alpha}'}$ with $\bar{\alpha} = \bar{\alpha}' \circ (\alpha_k)$ and the edge e (corresponding to the case $\bar{\alpha} = \varnothing$) has the extremities $\mathcal{R}$ and $\mathcal{O}$.

- By the multiplicity of a vertex of $\mathcal{T}$ we mean the number of edges that branch out from that vertex. If the multiplicity is equal to one, the vertex is called exterior. Otherwise, it is said to be interior.
- We denote by *Int* the set of the interior vertices of the tree $\mathcal{T}$ and by *Dir* the set of the exterior vertices, except $\mathcal{R}$, which has a particular status in our problem. *Dir* is chosen for Dirichlet. A dissipation law is imposed at the root $\mathcal{R}$ which explains why it is isolated from the other exterior vertices.
- Define $I_{Int} = \{\bar{\alpha};\, \mathcal{O}_{\bar{\alpha}} \in Int\}$, $I_{Dir} = \{\bar{\alpha};\, \mathcal{O}_{\bar{\alpha}} \in Dir\}$ which are the sets of the indices of the interior and exterior vertices, except $\mathcal{R}$, respectively.

Note that the multiplicity of each interior point of the tree $\mathcal{T}$ is equal to 3 and that the integer $(n+1)$ represents the maximum level of the binary tree $\mathcal{T}$. Furthermore, the length of the edge $e_{\bar{\alpha}}$ is equal to 1. Then, $e_{\bar{\alpha}}$ will be parametrized by its arc length by means of the functions $\pi_{\bar{\alpha}}$, defined in $[0,1]$ such that $\pi_{\bar{\alpha}}(0) = \mathcal{O}_{\bar{\alpha}}$ and $\pi_{\bar{\alpha}}(1)$ is the other vertex of this edge.

We study the dissipative Schrödinger operator under the tree-shaped network $\mathcal{T}$. The case $N \geq 3$ is the one we are interested in: it corresponds to $n \geq 1$.

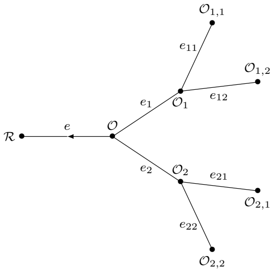

Figure: A Tree-Shaped network

More precisely, we consider the following initial and boundary value problem:

$$\frac{\partial u_{\bar{\alpha}}}{\partial t}(x,t) + i\,\frac{\partial^2 u_{\bar{\alpha}}}{\partial x^2}(x,t) = 0, \quad 0 < x < 1,\ t > 0,\ \bar{\alpha} \in I, \tag{2.1}$$

$$i\,u(1,t) + \frac{\partial u}{\partial x}(1,t) = 0,\ u_{\bar{\alpha}}(0,t) = 0,\ \bar{\alpha} \in I_{Dir},\ t > 0, \tag{2.2}$$

$$u_{\bar{\alpha}\circ(\beta)}(1,t) = u_{\bar{\alpha}}(0,t), \quad t > 0,\ \beta = 1,2,\ \bar{\alpha} \in I_{Int}, \tag{2.3}$$

$$\sum_{\beta=1}^{2} \frac{\partial u_{\bar{\alpha}\circ(\beta)}}{\partial x}(1,t) = \frac{\partial u_{\bar{\alpha}}}{\partial x}(0,t), \quad t > 0,\ \bar{\alpha} \in I_{Int}, \tag{2.4}$$

$$u_{\bar{\alpha}}(x,0) = (u_{\bar{\alpha}})_0(x), \quad 0 < x < 1,\ \bar{\alpha} \in I, \tag{2.5}$$

where $u_{\bar{\alpha}} : [0,1] \times (0,+\infty) \to \mathbb{R}$, $\bar{\alpha} \in I$, is the transverse displacement of the edge $e_{\bar{\alpha}}$. These functions allow us to identify the network with its rest graph. Note that in the problem above, (2.1) is the Schrödinger equation imposed on all the branches of the tree, (2.2) concerns the root and the other exterior nodes (recall that $u = u_{\bar{\alpha}}$ with $\bar{\alpha} = \varnothing$ and that this empty multi-index is chosen for the edge containing the root $\mathcal{R}$).

Well-posedness of the system

In order to study system (2.1)-(2.5) we need a proper functional setting. We define the following space

$$H = \prod_{\bar{\alpha} \in I} L^2(0,1)$$

equipped with the inner product

$$< \underline{u}, \underline{\tilde{u}} >_H = \sum_{\bar{\alpha} \in I} \int_0^1 u_{\bar{\alpha}}(x)\, \bar{\tilde{u}}_{\bar{\alpha}}(x)\, dx. \qquad (2.6)$$

It is well-known that system (2.1)-(2.5) may be rewritten as the first order evolution equation

$$\begin{cases} \underline{u}' = \mathcal{A}_d \underline{u}, \\ \underline{u}(0) = \underline{u}_0, \end{cases} \tag{2.7}$$

where the operator $\mathcal{A}_d : \mathcal{D}(\mathcal{A}_d) \subset H \to H$ is defined by

$$\mathcal{A}_d \underline{u} := (-i\, \partial_x^2 u_{\bar{\alpha}})_{\bar{\alpha} \in I},$$

with

$$\mathcal{D}(\mathcal{A}_d) := \left\{ \underline{u} \in \prod_{\bar{\alpha} \in I} H^2(0,1) \; : \text{satisfies (2.8) to (2.10) hereafter} \right\},$$

$$i\, u(1) + \frac{du}{dx}(1) = 0,\ u_{\bar{\alpha}}(0) = 0,\ \bar{\alpha} \in I_{Dir}, \tag{2.8}$$

$$u_{\bar{\alpha}\circ(\beta)}(1) = u_{\bar{\alpha}}(0),\ \beta = 1, 2, \tag{2.9}$$

$$\sum_{\beta=1}^{2} \frac{du_{\bar{\alpha}\circ(\beta)}}{dx}(1) = \frac{du_{\bar{\alpha}}}{dx}(0), \qquad \bar{\alpha} \in I_{Int}. \tag{2.10}$$

The natural energy $E(t)$ of a solution $\underline{u} = (u_{\bar{\alpha}})_{\bar{\alpha}\in I}$ of (2.1)-(2.5) is defined by:

$$E(t) = \frac{1}{2} \sum_{\bar{\alpha}\in I} \int_0^1 |u_{\bar{\alpha}}(x,t)|^2 \, dx. \tag{2.11}$$

Proposition

(i) For an initial datum $\underline{u}_0 \in H$*, there exists a unique solution* $\underline{u} \in C([0, +\infty), H)$ *to problem (2.7). Moreover, if* $\underline{u}_0 \in \mathcal{D}(\mathcal{A}_d)$*, then*

$$\underline{u} \in C([0, +\infty), \mathcal{D}(\mathcal{A}_d)) \cap C^1([0, +\infty), H).$$

(ii) The solution $\underline{u}$ *of* (2.1)-(2.5) *with initial datum in* $\mathcal{D}(\mathcal{A}_d)$ *satisfies the dissipation law:*

$$E'(t) = -\left|u(1,t)\right|^2 \leq 0. \tag{2.12}$$

Therefore the energy is a non-increasing function of the time variable t*.*

Spectral analysis

The goal is to look for the eigenvalues and eigenvectors of the dissipative operator $\mathcal{A}_d$ as well as those of the associated conservative operator $\mathcal{A}_0$. To that end, the operator $\mathcal{A}^\epsilon$ is defined like $\mathcal{A}_d$ except for equation (2.8) which is replaced by:

$$i\epsilon\, u(1) + \frac{du}{dx}(1) = 0,\; u_{\bar{\alpha}}(0) = 0,\; \bar{\alpha} \in I_{Dir}. \tag{2.13}$$

Thus $\mathcal{A}_0$ and $\mathcal{A}_d$ are $\mathcal{A}^\epsilon$ with $\epsilon = 0$ and $\epsilon = 1$ respectively.

Proposition (Spectra of $\mathcal{A}_0$)

Let $\sigma^{(0)}$ be the spectrum of the conservative operator $\mathcal{A}_0$, then

$$\sigma^{(0)} = \sigma_1^{(0)} \cup \sigma_2^{(0)}, \tag{2.14}$$

where

$$\sigma_1^{(0)} = \{i(k\pi)^2 : k \in \mathbb{Z}^*\} \cup \left\{ i\left(k\pi + \frac{\pi}{2}\right)^2 : k \in \mathbb{Z} \right\},$$

and, if n is even,

$$\sigma_2^{(0)} = \left\{ i\left(k\pi + \frac{1}{2}\arg(z_{A,j}^{(n)})\right)^2 : j = 2, \ldots, n+1, k \in \mathbb{Z} \right\}$$

if n is odd,

$$\sigma_2^{(0)} = \left\{ i\left(k\pi + \frac{1}{2}\arg(z_{A,j}^{(n)})\right)^2 : j = 1, \ldots, n+1, k \in \mathbb{Z} \right\}$$

where $z_{A,j}^{(n)}$, $j = 1, \ldots, n+1$ is the family of the complex roots of the polynomial $P_{A,n}$ defined in ici.

Theorem (Spectra of $\mathcal{A}_d$)

Let σ be the spectrum of the dissipative operator $\mathcal{A}_d$, then

$$\sigma = \sigma_1 \cup \sigma_2 \cup \tilde{\sigma_2}, \tag{2.15}$$

where $\sigma_1 = \sigma_1^{(0)}, \ \tilde{\sigma_2} = \{(\lambda_k)_{k\in S} : S \text{ is finite}, \ \Re(\lambda_k) < 0\}$ *and*

$$\sigma_2 = \{i(\omega_{j,k})^2 : j = 1, \dots, n+1, k \in \mathbb{Z}, |k| \geq k_0\},$$

k_0 being an integer.
Moreover

$$\Re(i(\omega_{j,k})^2) < 0, \ \forall i(\omega_{j,k})^2 \in \sigma_2,$$

and the following asymptotic behaviour holds:

$$i(\omega_{j,k})^2 = i\left(k^2\pi^2 + k\pi \arg(z_{A,j}^{(n)}) + \frac{(\arg(z_{A,j}^{(n)}))^2}{4}\right) + 2\pi\gamma_j + o(1) \tag{2.16}$$

where γ_j is a real negative number ($\gamma_j = -\frac{P_{B,n}(z_{A,j}^{(n)})}{2\pi z_{A,j}^{(n)} (P_{A,n})'(z_{A,j}^{(n)})}$). The polynomials $P_{A,n}$ and $P_{B,n}$ are defined by

$$P_{A,m+1}(z) = 2(z+1)\,P_{A,m}(z) + (z-1)\,P_{B,m}(z), \forall m \in \mathbb{N}, \tag{2.17}$$

$$P_{B,m+1}(z) = 2(z-1)\,P_{A,m}(z) + (z+1)\,P_{B,m}(z), \forall m \in \mathbb{N}, \tag{2.18}$$

$$P_{A,0}(z) = z+1, \;\; P_{B,0}(z) = z-1. \tag{2.19}$$

admits $n+1$ distinct complex roots $z_{A,j}^{(n)} \neq 1, \; j = 1, ..., n+1$ with modulus equal to 1.

Note that when n is even, (-1) is a root of $P_{A,n}$ which is denoted by $z_{A,1}^{(n)}$. Since $k\pi + \frac{1}{2}\arg(z_{A,1}^{(n)}) = k\pi + \frac{\pi}{2}$, when n is even, the index j starts from 2 in the definition of $\sigma_2^{(0)}$. This ensures $\sigma_1^{(0)} \cap \sigma_2^{(0)} = \emptyset$ for any value of $n \geq 1$.

The families of eigenvalues

Theorem (Families of eigenvalues of $\mathcal{A}_0$ and $\mathcal{A}_d$)

1 - *The complex $\lambda = i\omega^2$ ($\omega \in \mathbb{C}$) is an eigenvalue of the operator $\mathcal{A}_0$ if and only if*

- *either $\omega = k\pi, k \in \mathbb{Z}^*$ and the dimension of the corresponding eigenspace is equal to: $2^n - 1$,*
- *or $\omega = (\pi/2) + k\pi, k \in \mathbb{Z}$ and the dimension of the corresponding eigenspace is equal to: $\frac{1}{3}2^n + \frac{2}{3}$ if n is even and $\frac{1}{3}(2^n - 2)$ if n is odd,*

- or $\omega = ((\arg(z_{A,j}^{(n)}))/2) + k\pi, k \in \mathbb{Z}$ where $z_{A,j}^{(n)}$ ($j = 1, \ldots, n$ if n is odd, $j = 2, \ldots, n$ if n is even) is the family of the roots of the polynomial $P_{A,n}$ (if n is even, the first root $z_{A,1}^{(n)} = -1$ is excluded). The dimension of the eigenspace associated to each eigenvalue is one.

2 - The complex $\lambda = i\omega^2$ ($\omega \in \mathbb{C}$) is an eigenvalue of the operator $\mathcal{A}_d$ if and only if

- either $\omega = k\pi, k \in \mathbb{Z}^*$ and the dimension of the corresponding eigenspace is equal to: $2^n - 1$,
- or $\omega = (\pi/2) + k\pi, k \in \mathbb{Z}$ and the dimension of the corresponding eigenspace is equal to: $\frac{1}{3}(2^n - 1)$ if n is even and $\frac{1}{3}(2^n - 2)$ if n is odd,
- or ω satisfies $\omega \neq k\pi$, $\omega \neq (\pi/2) + k\pi$, $k \in \mathbb{Z}$ and

$$P_{A,n}(z) + \frac{P_{B,n}(z)}{\omega} = 0. \tag{2.20}$$

where $z = e^{2i\omega}$. The dimension of the eigenspace associated to each eigenvalue is one.

Note that the first two families of eigenvalues of $\mathcal{A}^\epsilon$ lie on the imaginary axis. The third one also lies on the imaginary axis for the conservative operator.
The third family of eigenvalues of the dissipative operator lies on the half-plane of complex numbers with a negative real part.

Riesz basis

It is proved that the generalized eigenfunctions of the dissipative operator $\mathcal{A}_d$ associated to the eigenvalues in $\sigma_2 \cup \tilde{\sigma_2}$ form a Riesz basis of the subspace of H which they span.

Theorem

Let $\mathcal{A}$ be a densely defined operator in a Hilbert space $\mathcal{H}$ with compact resolvent. Let $\{\phi_n\}_{n=1}^{\infty}$ be a Riesz basis of $\mathcal{H}$. If there are two integers $N_1, N_2 \geq 0$ and a sequence of generalized eigenvectors $\{\psi_n\}_{n=N_1+1}^{\infty}$ of $\mathcal{A}$ such that

$$\sum_{n=1}^{\infty} \|\phi_{n+N_2} - \psi_{n+N_1}\|_2^2 < \infty,$$

then the set of generalized eigenvectors (or root vectors) of $\mathcal{A}$, $\{\psi_n\}_{n=1}^{\infty}$ forms a Riesz basis of $\mathcal{H}$.

Theorem (Riesz basis for the operator $\mathcal{A}_d$)

Denote by ω^0 the number $\omega^0_{j,k} := k\pi + \frac{1}{2}\arg(z^{(n)}_{A,j})$ which is such that $i(\omega^0)^2 \in \sigma^{(0)}_2$ (except from the case $n = 2$ and $j = 1$: $i(\omega^0_{1,k})^2$ is an eigenvalue in $\sigma^{(0)}_1$).
Denote by $\omega := \omega_{j,k}$ the value which is such that $i(\omega_{j,k})^2 \in \sigma_2 \cup \tilde{\sigma_2}$. The indices j and k are dropped for simplicity since they are fixed here.
Denote by $\varphi^0(\omega^0, \cdot)$ (resp. $\varphi(\omega, \cdot)$) the eigenfunction of $\mathcal{A}_0$ (resp. $\mathcal{A}_d$) associated to the eigenvalue $i(\omega^0)^2$ (resp. $i\omega^2$).

For an odd n and any $i = 1, \dots, n$, there exists an integer k_0 such that, for any $j = 1, \dots, n$:

$$\sum_{|k|>k_0} \|\varphi_i(\omega_{j,k}, \cdot) - \varphi_i^0(\omega_{j,k}^0, \cdot)\|_2^2 < \infty.$$

The index i comes from an indexation based on the level of the vertices of the tree which is read from the leaves to the root here and not the other way round as before.
For an even n and any $i = 1, \dots, n$, there exists an integer k_0 such that, for any $j = 2, \dots, n$:

$$\sum_{|k|>k_0} \|\varphi_i(\omega_{j,k}, \cdot) - \varphi_i^0(\omega_{j,k}^0, \cdot)\|_2^2 < \infty.$$

Energy decreasing

Using the Riesz basis, the energy is proved to decrease exponentially to a non-vanishing value depending on the initial datum. The decay rate is explicitly given since the ω's satisfying $i\omega^2 \in (\sigma_2 \cup \tilde{\sigma_2})$ are the solutions of (2.20). For any fixed value of n, the constant C is computable numerically.

Let us explicit the case $n = 2$. The polynomials $P_{A,2}$ and $P_{B,2}$ are:

$$P_{A,2}(z) = 9z^3 + 7z^2 + 7z + 9,\ P_{B,2}(z) = 9z^3 + z^2 - z - 9.$$

The polynomial $P_{A,2}$ has 3 roots which are:

$$z^{(2)}_{A,1} = -1 = e^{i\pi},\ z^{(2)}_{A,2} = \frac{1}{9}(1 - 4i\sqrt{5}) = e^{-i\,\arctan(4\sqrt{5})},$$
$$z^{(2)}_{A,3} = \frac{1}{9}(1 + 4i\sqrt{5}) = e^{i\,\arctan(4\sqrt{5})}.$$

Thus the spectrum of the conservative operator $\mathcal{A}_0$ is given by (2.14) with

$$\sigma_2^{(0)} = \{i(k\pi \pm \arctan(4\sqrt{5}))^2 : k \in \mathbb{Z}\}.$$

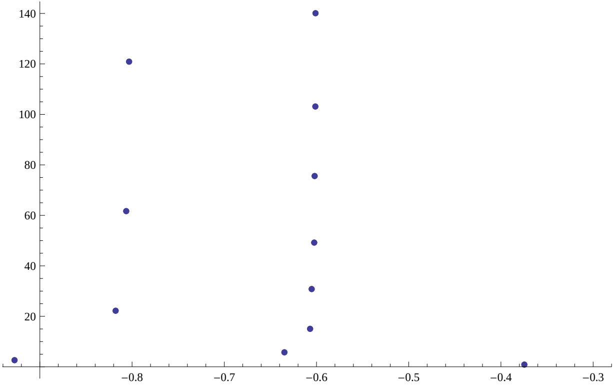

Figure: $n = 2$: the spectrum $\sigma_2 \cup \tilde{\sigma}_2$.

The values of γ_j are: $\gamma_1 = -\frac{2}{5\pi}, \gamma_2 = \gamma_3 = -\frac{3}{10\pi}$.

Then the set σ_2 which is a part of the spectrum of the dissipative operator $\mathcal{A}_d$ has two vertical asymptots:

$$\Re(\lambda) = 2\pi\gamma_1 = -\frac{4}{5},\ \Re(\lambda) = 2\pi\gamma_2 = 2\pi\gamma_3 = -\frac{3}{5},$$

which is consistent with the numerical computation of the spectrum.

At last, numerically the eigenvalue of $\mathcal{A}_d$ with the largest real part is $\lambda \approx -0.37459 + 0.873125i$. Hence the approximate value for the decay rate: $C \approx 0.37459$.

Energy decreasing using the Riesz basis

Theorem (Energy decreasing of the solution)

Let $E(t)$ be the energy defined by (2.11) and H the Hilbert space. Let H_1 (respectively H_2) be the subspace of H spanned by the $\underline{\psi}^1(\omega,\cdot)$'s (resp. $\underline{\psi}^2(\omega,\cdot)$'s), which are the normalized (in H) eigenfunctions of $\mathcal{A}_d$ associated to the eigenvalues $i\omega^2$ in σ_1 (resp. $\sigma_2 \cup \tilde{\sigma_2}$).

1. *H_1 is orthogonal to H_2.*
2. *Let $\underline{u}_0$ in H be the initial condition of the boundary value problem and $\underline{u}_0^1$ its orthogonal projection onto H_1.*
   Then $E(t)$ decreases exponentially to $E_1(0) := \|\underline{u}_0^1\|_H^2$ when t tends to $+\infty$. More precisely

$$E(t) \leq E_1(0) + e^{-2Ct} E_2(0) \tag{2.21}$$

where $-C := \sup_{\{i\omega^2 \in (\sigma_2 \cup \tilde{\sigma_2})\}} \Re(i\omega^2) < 0$.

Transfer function analysis

In order to obtain the decay properties of the damped problem (2.1)-(2.5) via observability inequalities for the conservative problem, we can use an assumption which consists in the boundedness of the associated transfer function. More precisely, if $\underline{u}$ is the solution of

$$\begin{cases} \underline{u}' = \mathcal{A}_0 \underline{u}, \\ \underline{u}(0) = \underline{u}_0 \in H_2 \end{cases} \tag{2.22}$$

then the observability inequality consists in proving the existence of a time $T > 0$ and a positive constant $C(T)$ such that

$$\int_0^T |u(1,t)|^2 dt \geq C(T) \|\underline{u}_0\|_H^2. \tag{2.23}$$

$$\begin{cases} \lambda z_{\bar{\alpha}} + i \dfrac{d^2 z_{\bar{\alpha}}}{dx^2} = 0, \ (0,1), \ \bar{\alpha} \in I, \\ \dfrac{dz}{dx}(1) = (-i)g, \ z_{\bar{\alpha}}(0) = 0, \ \bar{\alpha} \in I_{Dir}, \\ z_{\bar{\alpha}\circ(\beta)}(1) = z_{\bar{\alpha}}(0), \ \beta = 1,2, \ \bar{\alpha} \in I_{Int}, \\ \displaystyle\sum_{\beta=1}^{2} \dfrac{dz_{\bar{\alpha}\circ(\beta)}}{dx}(1) = \dfrac{dz_{\bar{\alpha}}}{dx}(0), \quad \bar{\alpha} \in I_{Int}. \end{cases} \tag{2.24}$$

where $\Re\lambda > 0$ and the input g belongs to $\mathbb{C}$. Therefore the ouput $z(1)$ has the form $z(1) = H(\lambda)g$ and our goal is to give an estimate of $|H(\lambda)|$ for λ on the line $\Re\lambda = \gamma, \gamma > 0$.

We recall that the operator $\mathcal{A}_0 : \mathcal{D}(\mathcal{A}_0) \subset H \to H$ is defined by

$$\mathcal{A}_0 \underline{u} := (-i\,\partial_x^2 u_{\bar{\alpha}})_{\bar{\alpha}\in I},$$

with

$$\mathcal{D}(\mathcal{A}_0) := \left\{ \underline{u} \in \prod_{\bar{\alpha}\in I} H^2(0,1) \ : \text{satisfies}\,(2.25)\ \text{to}\ (2.27)\ \text{hereafter} \right\},$$

$$\frac{du}{dx}(1) = 0,\ u_{\bar{\alpha}}(0) = 0,\ \bar{\alpha} \in I_{Dir}, \tag{2.25}$$

$$u_{\bar{\alpha}\circ(\beta)}(1) = u_{\bar{\alpha}}(0),\ \beta = 1, 2,\ \bar{\alpha} \in I_{Int}, \tag{2.26}$$

$$\sum_{\beta=1}^{2} \frac{du_{\bar{\alpha}\circ(\beta)}}{dx}(1) = \frac{du_{\bar{\alpha}}}{dx}(0), \quad \bar{\alpha} \in I_{Int}. \tag{2.27}$$

Then we define $B \in \mathcal{L}(\mathbb{C}, \mathcal{D}(\mathcal{A}_0)')$ by its dual B^* such that $B^*(\underline{\Phi}) := \Phi(1)$, for any $\underline{\Phi}$ in $\mathcal{D}(\mathcal{A}_0)$ (the duality is obtained by means of the inner product in H). Then a straightforward computation shows that the solution $\underline{z}$ of problem (2.24) is in H and is equal to $(\lambda I - \mathcal{A}_0)^{-1} Bg$. Consequently $z(1) = B^*(\underline{z}) = B^*(\lambda I - \mathcal{A}_0)^{-1} Bg$ and the transfer function has the form

$$H(\lambda) = B^*(\lambda I - \mathcal{A}_0)^{-1} B.$$

Theorem (Estimate of the transfer function)

Let $\mathcal{A}_0 : \mathcal{D}(\mathcal{A}_0) \subset H \to H$ and B be the operators defined above. The transfer function is given by:

$$H(\lambda) = B^*(\lambda I - \mathcal{A}_0)^{-1} B \in \mathcal{L}(\mathbb{C}), \, \lambda \in \mathbb{C}_+ = \{\lambda \in \mathbb{C}; \, \Re\lambda > 0\}.$$

It satisfies $\sup\limits_{\Re\lambda = \gamma} |H(\lambda)| < \infty, \, \gamma > 0.$

Stabilization to zero by changing the feedback law

We can obtain an exponential stability result of system (2.28)-(2.33) to zero in the energy space by changing the feedback law:

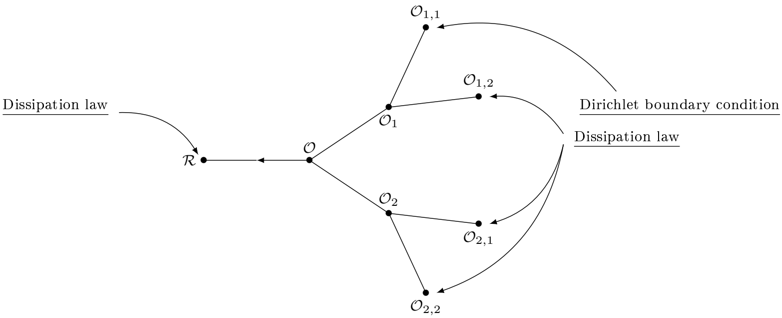

Figure: A Tree-Shaped network with exponential stabilizing feedback

Let $\bar{\alpha}^*$ be an arbitrary element of I_{Dir}. We consider the following initial and boundary value problem :

$$\frac{\partial u_{\bar{\alpha}}}{\partial t}(x,t) + i\,\frac{\partial^2 u_{\bar{\alpha}}}{\partial x^2}(x,t) = 0, \quad 0 < x < 1,\ t > 0,\ \bar{\alpha} \in I, \tag{2.28}$$

$$i\,u(1,t) + \frac{\partial u}{\partial x}(1,t) = 0,\ i\,u_{\bar{\alpha}}(0,t) - \frac{\partial u_{\bar{\alpha}}}{\partial x}(0,t) = 0,\ \bar{\alpha} \in I_{Dir}, \bar{\alpha} \neq \bar{\alpha}^*,\ t > 0, \tag{2.29}$$

$$u_{\bar{\alpha}^*}(0,t) = 0,\ t > 0, \tag{2.30}$$

$$u_{\bar{\alpha}\circ\beta}(1,t) = u_{\bar{\alpha}}(0,t), \quad t > 0,\ \beta = 1,2,\ \bar{\alpha} \in I_{Int}, \tag{2.31}$$

$$\sum_{\beta=1}^{2} \frac{\partial u_{\bar{\alpha}\circ\beta}}{\partial x}(1,t) = \frac{\partial u_{\bar{\alpha}}}{\partial x}(0,t), \quad t > 0,\ \bar{\alpha} \in I_{Int}, \tag{2.32}$$

$$u_{\bar{\alpha}}(x,0) = (u_{\bar{\alpha}})_0(x), \quad 0 < x < 1,\ \bar{\alpha} \in I. \tag{2.33}$$

It is well-known that system (2.28)-(2.33) may be rewritten as the first order evolution equation

$$\begin{cases} \underline{u}' = A_d \underline{u}, \\ \underline{u}(0) = \underline{u}_0, \end{cases} \tag{2.34}$$

where the operator $A_d : \mathcal{D}(A_d) \subset H \to H$ is defined by

$$A_d \underline{u} := (-i\,\partial_x^2 u_{\bar\alpha})_{\bar\alpha \in I},$$

with

$$\mathcal{D}(A_d) := \left\{ \underline{u} \in \prod_{\bar\alpha \in I} H^2(0,1) \ : \text{satisfies (2.35) to (2.38) hereafter} \right\},$$

$$i\,u(1) + \frac{du}{dx}(1) = 0,\ i\,u_{\bar\alpha}(0) - \frac{du_{\bar\alpha}}{dx}(0) = 0,\ \bar\alpha \in I_{Dir}, \bar\alpha \neq \bar\alpha^*, \tag{2.35}$$

$$u_{\bar\alpha^*}(0) = 0, \tag{2.36}$$

$$u_{\bar\alpha\circ\beta}(1) = u_{\bar\alpha}(0),\ \beta = 1,2,\ \bar\alpha \in I_{Int}, \tag{2.37}$$

$$\sum_{\beta=1}^{2} \frac{du_{\bar\alpha\circ\beta}}{dx}(1) = \frac{du_{\bar\alpha}}{dx}(0), \quad \bar\alpha \in I_{Int}. \tag{2.38}$$

- The operator A_d generates a C_0 semigroup of contractions on H.
- We show that the semigroup e^{tA_d} decays to the null steady state with an exponential decay rate. To obtain this, our technique is based on a frequency domain approach method and combines a contradiction argument with the multiplier technique to carry out a special analysis for the resolvent.

Theorem

There exist constants $C, \tau > 0$ such that the semigroup e^{tA_d} satisfies the following estimate

$$\left\| e^{tA_d} \right\|_{\mathcal{L}(H)} \leq C\, e^{-\tau t}, \forall\, t > 0. \qquad (2.39)$$

Lemma (Prüss, Huang)

A C_0 semigroup $e^{t\mathcal{L}}$ on a Hilbert space $\mathcal{H}$ satisfies

$$||e^{t\mathcal{L}}||_{\mathcal{L}(\mathcal{H})} \leq C\, e^{-\tau t},$$

for some constant $C > 0$ and for $\tau > 0$ if and only if

$$\rho(\mathcal{L}) \supset \{ i\beta \mid \beta \in \mathbb{R} \} \equiv i\mathbb{R}, \tag{2.40}$$

and

$$\limsup_{|\beta| \to \infty} \|(i\beta I - \mathcal{L})^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty, \tag{2.41}$$

where $\rho(\mathcal{L})$ denotes the resolvent set of the operator $\mathcal{L}$.

The star-shaped network case

We prove the time decay estimates $L^1(\mathcal{R}) \to L^\infty(\mathcal{R})$, where $\mathcal{R}$ is an infinite star-shaped network, for the Schrödinger group $e^{it(-\frac{d^2}{dx^2}+V)}$ for real-valued potentials V satisfying some regularity and decay assumptions. Further we show that the solution for initial conditions with a lower cutoff frequency tends to the free solution, if the cutoff frequency tends to infinity.

Let $R_i, i = 1, ..., N$, be $N (N \in \mathbb{N}, N \geq 2)$ disjoint sets identified with $(0, +\infty)$ and put $\mathcal{R} := \cup_{k=1}^{N} \overline{R}_k$. We denote by $f = (f_k)_{k=1,...,N} = (f_1, ..., f_N)$ the function on $\mathcal{R}$ taking their values in $\mathbb{R}$ and f_k is the restriction of f to R_k.

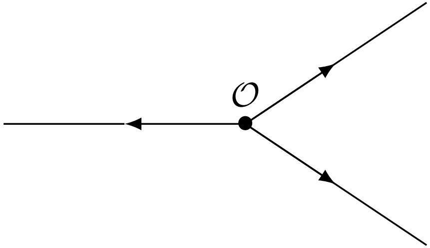

Figure: Star-Shaped Network ($N = 3$)

Define the Hilbert space $\mathcal{H} = \prod_{k=1}^{N} L^2(R_k)$ with scalar product $((u_k),(v_k))_{\mathcal{H}} = \sum_{k=1}^{N}(u_k, v_k)_{L^2(R_k)}$ and introduce the following transmission conditions :

$$(u_k)_{k=1,\dots,N} \in \prod_{k=1}^{N} C(\overline{R_k}) \text{ satisfies } u_i(0) = u_k(0)\, \forall\, i, k = 1, \dots, N, \quad (3.42)$$

$$(u_k)_{k=1,\dots,N} \in \prod_{k=1}^{N} C^1(\overline{R_k}) \text{ satisfies } \sum_{k=1}^{N} \frac{du_k}{dx}(0^+) = 0. \quad (3.43)$$

Let $H_0 : \mathcal{D}(H_0) \to \mathcal{H}$ be the linear operator of $\mathcal{H}$ defined by :

$$\mathcal{D}(H_0) = \left\{(u_k) \in H^2(R_k);\ (u_k) \text{ satisfies } (3.42), (3.43)\right\},$$

$$H_0(u_k) = (H_{0,k} u_k)_{k=1,\ldots,N} = (-\frac{d^2 u_k}{dx^2})_{k=1,\ldots,N} = -\Delta_{\mathcal{R}}(u_k).$$

The operator H_0 defined above is self-adjoint and satisfies that his spectrum $\sigma(H_0)$ is equal to $[0, +\infty)$.

For any $s \in \mathbb{R}$, let us denote by $L^1_s(\mathcal{R})$ the space of all complex-valued measurable functions $\phi = (\phi_1, \ldots, \phi_N)$ defined on $\mathcal{R}$ such that

$$\|\phi\|_{L^1_s(\mathcal{R})} := \int_{\mathcal{R}} |\phi(x)| \langle x \rangle^s \, dx = \sum_{k=1}^{N} \int_{R_k} |\phi_k(x)| \langle x \rangle^s \, dx < \infty,$$

where $\langle x \rangle = (1 + |x|^2)^{1/2}$. This space is a Banach space with the norm $\| \cdot \|_{L^1_s(\mathcal{R})}$.

Let $V \in L^1_1(\mathcal{R})$. Denote by H the self-adjoint realization of the operator $-\frac{d^2}{dx^2} + V(x)$ on $L^2(\mathcal{R})$ and his spectrum $\sigma(H) = [0, +\infty) \cup \{$a finite number of negative eigenvalues$\}$.

We verify that the free Schrödinger group on the star-shaped network $\mathcal{R}$ satisfies the following dispersive estimate

$$\left\| e^{itH_0} \right\|_{L^1(\mathcal{R}) \to L^\infty(\mathcal{R})} \leq C|t|^{-1/2}, \; t \neq 0.$$

Our goal is to assume as little as possible on the potential $V = V(x)$ in terms of decay or regularity. More precisely, we prove the following theorem.

Theorem 1 (Ali Mehmeti-A-Nicaise)

Let $V \in L^1_\gamma(\mathcal{R})$, with $\gamma > 5/2$. Then for all $t \neq 0$,

$$\left\| e^{itH} P_{ac}(H) \right\|_{L^1(\mathcal{R}) \to L^\infty(\mathcal{R})} \leq C \, |t|^{-1/2} \tag{3.44}$$

where C is a positive constant and $P_{ac}(H)$ is the projection onto the absolutely continuous spectral subspace.

As a consequence, we have the following $L^p - L^{p'}$ estimate.

Corollary ($L^p - L^{p'}$ estimate)

Under the assumptions of Theorem 1, for $1 \leq p \leq 2$ *and* $\frac{1}{p} + \frac{1}{p'} = 1$ *we have for all* $t \neq 0$,

$$\left\| e^{itH} P_{ac}(H) \right\|_{L^p(\mathcal{R}) \to L^{p'}(\mathcal{R})} \leq C \, |t|^{-\frac{1}{p} + \frac{1}{2}} \,, \tag{3.45}$$

where $C > 0$ *is a constant.*

Moreover we have the following Strichartz estimates which have been used in the context of the nonlinear Schrödinger equation to obtain well-posedness results.

Corollary (Strichartz estimates)

Let the assumptions of Theorem 1 be satisfied. Then for $2 \leq p, q \leq +\infty$ *and* $\frac{1}{p} + \frac{2}{q} = \frac{1}{2}$ *we have for all* t,

$$\left\| e^{itH} P_{ac}(H) f \right\|_{L^q(\mathbb{R}, L^p(\mathcal{R}))} \leq C \left\| f \right\|_2, \; \forall f \in L^p(\mathcal{R}) \cap L^2(\mathcal{R}), \qquad (3.46)$$

where $C > 0$ *is a constant.*

As a direct consequence, we have the following well-posedness result for a nonlinear Schrödinger equation with potential.

Let $p \in (0,4)$ and suppose that V satisfies the assumptions of Theorem 1. Then, for any $u_0 \in L^2(\mathcal{R})$, there exists a unique solution

$$u \in C(\mathbb{R}; L^2(\mathcal{R})) \cap \bigcap_{(p,q)\,\text{admissible}} L^q_{loc}(\mathbb{R}; L^p(\mathcal{R}))$$

of the equation

$$\begin{cases} iu_t - \Delta_{\mathcal{R}} u + V\,u \pm |u|^p\, u = 0,\ t \neq 0, \\ u(0) = u^0,\ t = 0, \end{cases} \tag{3.47}$$

and where (p,q) is called an admissible pair if (p,q) satisfies that $2 \leq p, q \leq +\infty$ and $\frac{2}{p} + \frac{1}{q} = \frac{1}{2}$.

Remark

Another direct consequence of the dispersive estimate or of the $L^p - L^{p'}$ estimate is that we can construct the scattering operator for the nonlinear Schrödinger equation with potential.

While proving Theorem 1 we obtain as results of independent interest the $L^\infty-$time decay for the high frequency part of the group and a high frequency perturbation estimate:

Theorem 2 (Ali Mehemti-A-Nicaise)

Under the assumptions of Theorem 1 we have

$$\|e^{itH}\chi(H)\|_{1,\infty} \leq (A + B\frac{\|V\|_1}{\sqrt{\lambda_0}})|t|^{-1/2}, t \neq 0, \quad (3.48)$$

$$\|e^{itH}\chi(H) - e^{itH_0}\chi(H_0)\|_{1,\infty} \leq B\frac{\|V\|_1}{\sqrt{\lambda_0}}|t|^{-1/2}, t \neq 0\,. \quad (3.49)$$

Here χ is smoothly cutting off the frequencies below λ_0 and A, B are in terms of the cutoff function independent of λ_0.
In particular we have for any $f \in L^1(\mathcal{R})$ that

$$e^{itH}\chi(H)f \to e^{itH_0}\chi(H_0)f \text{ for } \lambda_0 \to \infty$$

uniformly on $\mathcal{R}$ for every fixed $t > 0$.

The perturbation estimate allows the simultaneous control of the smallness of the difference between perturbed and unperturbed group in terms of the cutoff frequency, the L^1-Norm of the potential and time.

A counterexample

Consider the infinite network $\mathcal{R} = \cup_{n\in\mathbb{N}} e_n$, where each edge $e_n = (n, n+1)$ with the set of vertices $\mathcal{V} = \cup_{n\in\mathbb{N}} v_n$, where $v_n = \{n\}$. For a fixed sequence of positive real numbers $\alpha = (\alpha_n)_{n\in\mathbb{N}}$, we define the Hilbert space $L^2(\mathcal{R}, \alpha)$ as follows

$$L^2(\mathcal{R},\alpha) = \Big\{ u = (u_n)_{n\in\mathbb{N}} : u_n \in L^2(e_n) \forall n \in \mathbb{N}$$

$$\text{such that } \sum_{n\in\mathbb{N}} \alpha_n \int_{e_n} |u_n(x)|^2 \, dx < \infty \Big\},$$

equipped with the inner product

$$(u, v) = \sum_{n\in\mathbb{N}} \alpha_n \int_{e_n} u_n(x) v_n(x) \, dx, \quad \forall u, v \in L^2(\mathcal{R}, \alpha).$$

Similarly for all $k \in \mathbb{N}^*$, we set

$$H^k(\mathcal{R}, \alpha) = \Big\{ u = (u_n)_{n \in \mathbb{N}} \in L^2(\mathcal{R}, \alpha) :$$

$$(u_n^{(\ell)})_{n \in \mathbb{N}} \in L^2(\mathcal{R}, \alpha) \ \forall \ell \in \{1, 2, \dots, k\} \Big\},$$

where $u_n^{(\ell)}$ means the ℓ derivative of u_n with respect to x.

Now we consider the Laplace operator $-\Delta_\alpha$ (depending on α) as follows:

$$\mathcal{D}(-\Delta_\alpha) = \{ u = (u_n)_{n \in \mathbb{N}} \in H^2(\mathcal{R}, \alpha) : \text{ satisfying (3.50), (3.51), (3.52) below } \}$$

$$u_0(0) = 0, \tag{3.50}$$

$$u_n(n+1) = u_{n+1}(n+1), \forall n \in \mathbb{N}, \tag{3.51}$$

$$\alpha_n \frac{du_n}{dx}(n+1) = \alpha_{n+1} \frac{du_{n+1}}{dx}(n+1), \forall n \in \mathbb{N}. \tag{3.52}$$

For all $u \in \mathcal{D}(-\Delta_\alpha)$, we set $-\Delta_\alpha u = (-\frac{d^2 u_n}{dx^2})_{n\in\mathbb{N}}$. This operator is a non negative self-adjoint operator in $L^2(\mathcal{R}, \alpha)$.

Theorem

For all $k \in \mathbb{N}^*$, $k^2\pi^2$ is a simple eigenvalue of $-\Delta_\alpha$ if and only if

$$s = \sum_{n\in\mathbb{N}} \frac{1}{\alpha_n} < \infty. \tag{3.53}$$

In that case the associated orthonormal eigenvector $\varphi^{[k]} = (\varphi^{[k]})_{n\in\mathbb{N}}$ is given by

$$\varphi_n^{[k]} = \sqrt{\frac{2}{s}} \frac{(-1)^{(n-1)k}}{\alpha_n} \sin(k\pi(x-n)), \forall x \in e_n, n \in \mathbb{N}.$$

Now assuming that (3.53) holds, then for any $k \in \mathbb{N}^*$ we consider the solution u of the Schrödinger equation

$$\begin{cases} \partial_t u - i\Delta_\alpha u = 0, \\ u(t=0) = \varphi^{[k]}, \end{cases}$$

or equivalently solution of

$$\begin{cases} \partial_t u_n - i\partial_x^2 u_n = 0, & \text{in} \quad e_n \times \mathbb{R}, \\ u_0(0,t) = 0, & \text{on} \quad \mathbb{R}, \\ u_n(n+1,t) = u_{n+1}(n+1,t) & \text{on} \quad \mathbb{R}, \forall n \in \mathbb{N}, \\ \alpha_n u_n'(n+1,t) = \alpha_{n+1} u_{n+1}'(n+1,t) & \text{on} \quad \mathbb{R}, \forall n \in \mathbb{N}, \\ u(t=0,\cdot) = \varphi^{[k]} & \text{on} \quad \mathcal{R}. \end{cases}$$

This solution is given by $u(t) = e^{-itk^2\pi^2}\varphi^{[k]}$. Moreover simple calculations show that

$$\|u(t)\|_{\infty,\mathcal{R}} = \sqrt{\frac{2}{s}} \sup_{n\in\mathbb{N}} \frac{1}{\alpha_n} \|\sin(k\pi(\cdot - n))\|_{\infty,e_n} = \sqrt{\frac{2}{s}} \sup_{n\in\mathbb{N}} \frac{1}{\alpha_n},$$

which is independent of t and then does not tend to zero as $|t|$ goes to infinity. On the other hand $u(t=0,\cdot)$ belongs to $L^1(\mathcal{R})$, since we have

$$\|u(t)\|_{L^1(\mathcal{R})} = \sqrt{\frac{2}{s}} \sum_{n\in\mathbb{N}} \frac{1}{\alpha_n} \|\sin(k\pi(\cdot - n))\|_{L^1(e_n)} \le \sqrt{2s}.$$

Theorem

If (3.53) holds, then the norm of the Schrödinger operator $e^{it\Delta_\alpha}$ from $L^1(\mathcal{R})$ to $L^\infty(\mathcal{R})$ does not tend to zero as $|t|$ goes to infinity.

This counterexample shows that the decay of the norm of the Schrödinger operator from $L^1(\mathcal{R})$ to $L^\infty(\mathcal{R})$ as $|t|$ goes to infinity is not guaranteed for all infinite networks. Hence the remainder this talk is to give some examples where such a case occurs.

Dispersive estimate for free Schrödinger operator

Theorem (Dispersive estimate)

For all $t \neq 0$,

$$\left\| e^{itH_0} \right\|_{L^1(\mathcal{R}) \to L^\infty(\mathcal{R})} \leq C \, |t|^{-1/2}, \tag{3.54}$$

where $C > 0$ *is a constant.*

We have the following result as a direct consequence for a dispersive estimate for free Schrödinger operator on a star-shaped network.

Corollary ($L^p - L^{p'}$ estimate)

For $1 \leq p \leq 2$ and $\frac{1}{p} + \frac{1}{p'} = 1$ we have for all $t \neq 0$,

$$\left\| e^{itH_0} \right\|_{L^p(\mathcal{R}) \to L^{p'}(\mathcal{R})} \leq C \, |t|^{-\frac{1}{p}+\frac{1}{2}}, \tag{3.55}$$

where $C > 0$ is a constant.

According to (3.54) we have

$$\sup_{t\neq 0} |t|^{\frac{1}{2}} \left\| e^{itH_0} f \right\|_{\infty} \leq C \, \|f\|_1 \, , \forall\, f \in L^1(\mathcal{R}) \cap L^2(\mathcal{R}).$$

Interpolating with the L^2 bound $\left\| e^{itH_0} f \right\|_2 = \|f\|_2$, leads to

$$\sup_{t\neq 0} |t|^{-\frac{1}{2}+\frac{1}{p}} \left\| e^{itH_0} f \right\|_{p'} \leq C \, \|f\|_p \, , \forall\, f \in L^1(\mathcal{R}) \cap L^2(\mathcal{R}), \qquad (3.56)$$

where $1 \leq p \leq 2$.

It is well-known that via T^*T argument (3.56) gives rise to the class of Strichartz estimates

$$\left\| e^{itH_0} f \right\|_{L_t^q(L_x^p)} \leq C \, \|f\|_2 \, , \; \forall \, \frac{2}{q} + \frac{1}{p} = \frac{1}{2}, \; 2 < q \leq +\infty, \; 2 \leq p \leq \infty. \tag{3.57}$$

The endpoint $q = 2$ is not captured by this approach but by the approach devolloped by Kell and Tao. So the estimate (3.57) is valid for all $2 \leq p, q \leq +\infty$ satisfying $\frac{2}{q} + \frac{1}{p} = \frac{1}{2}$ and we have also,

$$\left\| \int_{\mathbb{R}} e^{-itH_0} F(s,.) ds \right\|_{L^2(\mathcal{R})} \leq C \, \|F\|_{L^{q'}(\mathbb{R}, L^{p'}(\mathcal{R}))} \, ,$$

$$\left\| \int_0^t e^{i(t-s)H_0} F(s) ds \right\|_{L^q(\mathbb{R}, L^{r'}(\mathcal{R}))} \leq C \, \|F\|_{L^{r'}(\mathbb{R}, L^{s'}(\mathcal{R}))} \, ,$$

for all admissible pairs (q, p) and (r, s).

According to (3.57), we have for $p \in (0, 4)$, that for any $u_0 \in L^2(\mathcal{R})$ the equation

$$iu_t - \Delta_{\mathcal{R}} u \pm |u|^p u = 0,\ t \neq 0,\ \ u = u_0,\ t = 0,$$

admits a unique solution
$u \in C(\mathbb{R}, L^2(\mathcal{R})) \cap \bigcap_{(q,r)}$ admissible $L^q_{loc}(\mathbb{R}, L^r(\mathcal{R}))$.

The $L^2(\mathcal{R})$-norm is conserved along the time, i.e.,
$\|u(t)\|_{L^2(\mathcal{R})} = \|u_0\|_{L^2(\mathcal{R})}$.

Tadpole graph case

We consider the free Schrödinger group $e^{-it\frac{d^2}{dx^2}}$ on a tadpole graph $\mathcal{R}$. We first show that the time decay estimates $L^1(\mathcal{R}) \to L^\infty(\mathcal{R})$ is in $|t|^{-\frac{1}{2}}$ with a constant independent of the length of the circle. Our proof is based on an appropriate decomposition of the kernel of the resolvent. Further we derive a dispersive perturbation estimate, which proves that the solution on the queue of the tadpole converges uniformly, after compensation of the underlying time decay, to the solution of the Neumann half-line problem, as the circle shrinks to a point. To obtain this result, we suppose that the initial condition fulfills a high frequency cutoff.

Let $R_i, i = 1, 2$, be two disjoint sets identified with a closed path of measure equal to $L > 0$ for R_2 and to $(0, +\infty)$, for R_1, see figure. We set $\mathcal{R} := \cup_{k=1}^{2} \overline{R}_k$. We denote by $f = (f_k)_{k=1,2} = (f_1, f_2)$ the functions on $\mathcal{R}$ taking their values in $\mathbb{C}$ and let f_k be the restriction of f to R_k.

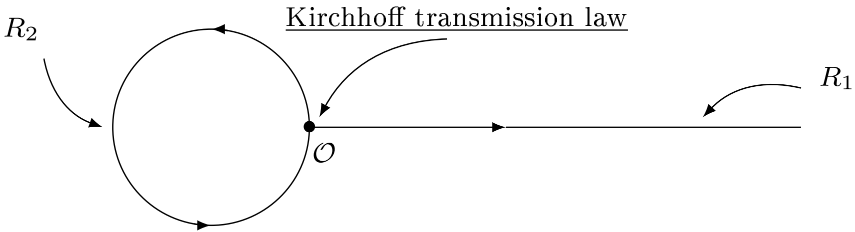

Figure: Tadpole graph

Define the Hilbert space $\mathcal{H} = \oplus_{k=1}^{2} L^2(R_k) = L^2(\mathcal{R})$ with inner product

$$((u_k),(v_k))_{\mathcal{H}} = \sum_{k=1}^{2} (u_k, v_k)_{L^2(R_k)}$$

and introduce the following transmission conditions:

$$(u_k)_{k=1,2} \in \oplus_{k=1}^{2} C(\overline{R_k}) \text{ satisfies } u_1(0) = u_2(0) = u_2(L), \tag{3.58}$$

$$(u_k)_{k=1,2} \in \oplus_{k=1}^{2} C^1(\overline{R_k}) \text{ satisfies } \sum_{k=1}^{2} \frac{du_k}{dx}(0^+) - \frac{du_2}{dx}(L^-) = 0. \tag{3.59}$$

Let $H : \mathcal{D}(H) \subset \mathcal{H} \to \mathcal{H}$ be the linear operator on $\mathcal{H}$ defined by :

$$\mathcal{D}(H) = \left\{ (u_k) \in \oplus_{k=1}^{2} H^2(R_k);\ (u_k)_{k=1,2} \text{ satisfies } (3.42), (3.43) \right\},$$

$$H(u_k) = (H_k u_k)_{k=1,2} = \left(-\frac{d^2 u_k}{dx^2} \right)_{k=1,2} = -\Delta_{\mathcal{R}}(u_k).$$

This operator H is self-adjoint and its spectrum $\sigma(H)$ is equal to $[0, +\infty)$. The self-adjointness and non-negativity of H can be shown by Friedrichs extension.

Here, we prove that the free Schrödinger group on the tadpole graph $\mathcal{R}$ satisfies the standard $L^1 - L^\infty$ dispersive estimate. More precisely, we will prove the following theorem.

Theorem (Ali Mehmeti-A-Nicaise)

For all $t \neq 0$,

$$\left\| e^{itH} P_{ac} \right\|_{L^1(\mathcal{R}) \to L^\infty(\mathcal{R})} \leq C \, |t|^{-1/2}, \tag{3.60}$$

where C is a positive constant independent of L and t, $P_{ac} f$ is the projection onto the absolutely continuous spectral subspace and $L^1(\mathcal{R}) = \oplus_{k=1}^2 L^1(R_k)$, $L^\infty(\mathcal{R}) = \oplus_{k=1}^2 L^\infty(R_k)$.

An important point is that this estimate is independent of the length L of the circle, which also follows from the fact that the problem is scale invariant.

Let H_0 be the negative laplacian on the half line with Neumann boundary conditions. Then holds the following dispersive perturbation estimate:

Theorem (Ali Mehmeti-A-Nicaise)

Let $0 \leq a < b < \infty$. Let $u_0 \in \mathcal{H} \cap L^1(R_1)$ such that

$$\mathrm{supp}\ u_0 \subset R_1 \ . \tag{3.61}$$

Then for all $t \neq 0$, we have

$$\| \ e^{itH} \mathbb{I}_{(a,b)}(H) u_0 - e^{itH_0} \mathbb{I}_{(a,b)}(H_0) u_0 \ \|_{L^\infty(R_1)}$$

$$\leq t^{-1/2} L \ 2\sqrt{2} \ \left(4(2\sqrt{b} - \sqrt{a}) + L(b-a)\right) \ \|u_0\|_{L^1(R_1)} \, .$$

- This last result implies that the solution of the Schrödinger equation on the queue R_1 of the tadpole with an upper frequency cutoff tends uniformly to the solution of the half-line Neumann problem with the same upper frequency cutoff, if the initial condition has its support in the queue, after compensation of the underlying $t^{-1/2}$-decay. In physical terms the frequency cutoff makes that the localization of the signals is limited and thus they have increasing difficulties to enter into the head of the tadpole.
- Without the high frequency cutoff, this result would not be possible, as the problem is scale invariant.

Some references

K. Ammari, D. Mercier and V. Régnier, Spectral analysis of the Schrödinger operator on binary tree-shaped networks and applications, *Journal of Differential Equations*, **259** (2015), 6923-6959.

K. Ammari and S. Nicaise, *Stabilization of elastic systems by collocated feedback*, Lecture Notes in Mathematics, 2124, Springer, Cham, 2015.

K. Ammari and D. Mercier, Boundary feedback stabilization of a chain of serially connected strings, *Evol. Eq. Control. Theory*, **1** (2015), 1–19.

F. Ali Mehemti, K. Ammari and S. Nicaise, Dispersive effects and high frequency behaviour for the Schrödinger equation in star-shaped networks, *Port. Math.*, **72** (2015), 309-355.

F. Ali Mehemti, K. Ammari and S. Nicaise, Dispersive effects for the Schrödinger equation on a tadpole graph, arXiv:1512.05269.

V. Banica and L. Ignat, Dispersion for the Schrödinger Equation on Networks, *J. Math. Phys.* (52), 083703, 2011.

T. Cazenave, Semilinear Schrödinger equations, Courant Lect. Notes Math. 10, American society, Providence, RI, Courant Institute of Mathematical Sciences, New york, 2003.

P. Deift and E. Trubowitz, Inverse scattering on the line, *Comm. Pure Appl. Math.*, **XXXII** (1979), 121-251.

M. Goldberg and W. Schlag, Dispersive estimates for Schrödinger operators in dimensions one and three, *Commun. Math. Phys.*, **251** (2004), 157-178.

R. Weder, $L^p - L^{p'}$ estimates for the Schrödinger equation on the line and inverse scattering for the nonlinear Schrödinger equation with a potential, *Journal of Functional Analysis*, **170** (2000), 37-68.

Thank you for your attention