

On the scattering theory of asymptotically flat manifolds and Strichartz inequalities

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23 Juin 2016 - Cergy Pontoise

Conférence en l'honneur de Vladimir Georgescu

Purpose of the talk

- ▶ Take the question of **Strichartz inequalities** (for the Schrödinger equation) on **asymptotically flat manifolds** as a case study to review some related **scattering estimates** (resolvent estimates, timedecay, smoothing estimates), either for comparison or because they are crucial inputs in the proofs of Strichartz inequalities
- ▶ Present some recent results (joint with H. Mizutani) on Strichartz inequalities on asymptotically flat manifolds

Strichartz and scattering estimates on the Euclidean space

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provided (p, q) is **admissible** (scaling condition)

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$$\|e^{it\Delta} u_0\|_{L^2(K)} \lesssim_K \|e^{it\Delta} u_0\|_{L^q(\mathbb{R}^n)}, \quad K \in \mathbb{R}^n.$$

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$$\left\| \langle \lambda^{\frac{1}{2}} x \rangle^{-k} \varphi(-\Delta/\lambda) e^{it\Delta} \langle \lambda^{\frac{1}{2}} x \rangle^{-k} \right\|_{L^2 \rightarrow L^2} \lesssim \langle \lambda t \rangle^{-k}$$

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with $\nu > 1/2$. By tracking the dependence on λ , one may obtain the non spectrally localized estimate ($n \geq 3$)

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which is the $\frac{1}{2}$ -**smoothing effect** for the Schrödinger equation.

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Strichartz and scattering estimates on the Euclidean space

Scattering inequalities (end)

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Rem. This correspondence $\lambda \rightarrow t$ also allows to convert resolvent estimates into time decay/propagation estimates (smoothness of $R_0(\lambda \pm i0) \leftrightarrow$ decay of e^{itP})

Strichartz inequalities vs smoothing effect for a wave packet

Strichartz inequalities

Strichartz inequalities vs smoothing effect for a wave packet

Strichartz inequalities

Consider the L^2 normalized semiclassical wave packet

$$G_{z,\zeta,h}(x) = (\pi h)^{-\frac{n}{4}} \exp\left(\frac{i}{h}\zeta \cdot (x - z) - \frac{|x - z|^2}{2h}\right).$$

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for all $h \in (0, 1]$ and $z \in \mathbb{R}^n$.

Strichartz inequalities vs smoothing effect for a wave packet

Smoothing effect (local in time)

$$|\langle D \rangle^s e^{i\frac{t}{2}\Delta} G_{z,\zeta,h}(x)|$$

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$$\|\langle x \rangle^{-\nu} \langle \zeta/h \rangle^s G_{z,\zeta,h}^t\|_{L_x^2}^2 = c_n \langle \zeta/h \rangle^{2s} \langle t/h \rangle^{-n} \int \langle h^{\frac{1}{2}}y+z+t\zeta/h \rangle^{-2\nu} \exp\left(-\frac{y^2}{\langle t/h \rangle^2}\right) dy$$

If we further integrate in time on $[-T, T]_t$,

$$c_n h \langle \zeta/h \rangle^{2s} \int_{-T/h}^{T/h} \langle \tau \rangle^{-n} \int \langle h^{\frac{1}{2}}y+z+\tau\zeta \rangle^{-2\nu} \exp\left(-\frac{y^2}{\langle \tau \rangle^2}\right) dy d\tau$$

which is bounded by

$$c_n h \langle 1/h \rangle^{2s} \int_{-T/h}^{T/h} \int \langle h^{\frac{1}{2}}Y_1\langle \tau \rangle + z_1 + \tau \rangle^{-2\nu} \exp(-Y^2) dY d\tau$$

Remark. Up to the term $Y_1\langle \tau \rangle$, there is no more contribution of the spreading $\langle \tau \rangle$. Here, the main role will be played the translation by $(t/h)\zeta = \tau\zeta$.

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Scattering inequalities turn out to play a crucial role in this problem.

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more precisely, $\partial^\alpha (G_{jk}(x) - \delta_{jk}) = O(\langle x \rangle^{-\mu - |\alpha|})$ for some $\mu > 0$. The geodesic flow is defined analogously with

$$p(x, \xi) = \xi \cdot G(x)^{-1} \xi = \sum_{j,k} G^{jk}(x) \xi_j \xi_k$$

the (principal) symbol of the **Laplace-Beltrami operator**

$$-\Delta_G = - \sum_{j,k} G^{jk}(x) \partial_{x_j} \partial_{x_k} + \sum_{j,k,\ell} G^{jk}(x) \Gamma_{jk}^\ell(x) \partial_{x_\ell}$$

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Question: behavior of $R(\lambda \pm i0)$ and (2) as $\lambda \rightarrow \infty$ (high energy) and $\lambda \rightarrow 0$ (low energy) ?

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High energy estimates ($\lambda \rightarrow +\infty$)

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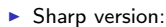
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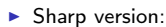
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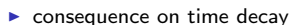
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$$\|e^{i \cdot P} u_0\|_{L^p([-T, T], L^q)} \lesssim_T \|u_0\|_{L^2}$$

[Staffilani-Tataru], [Robbiano-Zuily], [B.-Tzvetkov], [Hassell-Tao-Wunsch]

- ▶ For asymptotically flat manifolds with **small hyperbolic trapped set**

Strichartz on asymptotically flat manifolds

Several results for local in time estimates

- ▶ For general manifolds: estimates with **loss of derivatives**

$$\|e^{i \cdot P} u_0\|_{L^p([-T, T], L^q)} \lesssim_T \|u_0\|_{H^{1/p}(M)} := \|\langle -\Delta_G \rangle^{1/2p} u_0\|_{L^2}$$

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[Burq-Guillarmou-Hassell]

Intuition (non trapping case):

- ▶ Inside a compact set K , combine

$$\|\mathbf{1}_K e^{i \cdot P} u_0\|_{L^2([-T, T], L^{2^*})} \lesssim_T \|u_0\|_{H^{1/2}(M)} \quad \text{and} \quad \|\mathbf{1}_K e^{i \cdot P} v_0\|_{L^2([-T, T], H^{1/2})} \lesssim_T \|v_0\|_{L^2}$$

- ▶ Outside a compact set: use that the geometry is close to a nice model (...)

Strichartz on asymptotically flat manifolds

Few about global in time estimates (partially due to the low energy analysis)

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Results (joint with H. Mizutani)

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Theorem 4 (nonlinear scattering) *Under the assumptions of Theorem 3, the L^2 critical equation*

$$i\partial_t u - Pu = \sigma |u|^{\frac{4}{n}} u, \quad u|_{t=0} = u_0, \quad \sigma = \pm 1,$$

with $\|u_0\|_{L^2} \ll 1$, has a unique solution in (a subspace of) $C(\mathbb{R}, L^2) \cap L^{2+\frac{4}{n}}(\mathbb{R} \times M)$ and

$$\|u(t) - e^{-itP} u_\pm\|_{L^2(M)} \rightarrow 0, \quad t \rightarrow \pm\infty.$$

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Rem. For the localization, $(1 - \chi(\epsilon r)) f(P/\epsilon^2)$,

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$$\int_{\mathbb{R}} \|\chi(\epsilon r) f(P/\epsilon^2) e^{itP} u_0\|_{L^2(\mathbb{R}; L^{2^*})}^2 dt C \|f(P/\epsilon^2) u_0\|_{L^2}^2$$

with C independent of λ (and u_0)

$$\begin{aligned} \|\chi(\epsilon r) f(P/\epsilon^2) e^{itP} u_0\|_{L^{2^*}} &\lesssim \|\nabla_G \chi(\epsilon r) f(P/\epsilon^2) e^{itP} u_0\|_{L^2} \quad (\text{homogeneous Sobolev est.}) \\ &\lesssim \|\epsilon \chi'(\epsilon r) f(P/\epsilon^2) e^{itP} u_0\|_{L^2} + \|\chi(\epsilon r) \nabla_G f(P/\epsilon^2) e^{itP} u_0\|_{L^2} \\ &\lesssim \|\langle r \rangle^{-1} f(P/\epsilon^2) e^{itP} u_0\|_{L^2} + \|\langle \epsilon r \rangle^{-1} P^{\frac{1}{2}} \tilde{f}(P/\epsilon^2) e^{itP} u_0\|_{L^2} \\ &\lesssim \|\langle r \rangle^{-1} f(P/\epsilon^2) e^{itP} u_0\|_{L^2} + \|\langle r \rangle^{-1} \tilde{\tilde{f}}(P/\epsilon^2) e^{itP} u_0\|_{L^2} \end{aligned}$$

où $\tilde{f}, \tilde{\tilde{f}} \in C_0^\infty(0, +\infty)$. One concludes by mean of an optimally weighted resolvent inequality [B-Royer, 2015]

$$\|\langle r \rangle^{-1} f(P/\epsilon^2) e^{i \cdot P} u_0\|_{L^2(\mathbb{R}; L^2)} \lesssim \left(1 + \sup_{|\lambda| \leq 2} \|\langle r \rangle^{-1} (P - \lambda \pm i0)^{-1} \langle r \rangle^{-1}\|_{L^2 \rightarrow L^2}\right) \|u_0\|_{L^2}.$$

Rem. For the localization, $(1 - \chi(\epsilon r)) f(P/\epsilon^2)$, one has “ $|\xi| \sim \epsilon$ ” and “ $|x| \gtrsim \epsilon^{-1}$ ” \Rightarrow no problem of uncertainty principle to use microlocal techniques

Rest of the proof

At infinity: split $f(P/\lambda)e^{itP}$ into sums of

$$T_\lambda(t) = L_\lambda f(P/\lambda)e^{itP}$$

with suitable localization operators L_λ , and show

$$\|T_\lambda(t)\|_{L^2 \rightarrow L^2} \lesssim 1, \quad \|T_\lambda(t)T_\lambda(s)\|_{L^1 \rightarrow L^\infty} \lesssim |t-s|^{-\frac{n}{2}}$$

by writing

$$T_\lambda(t)T_\lambda(s) = \text{approximation} + \text{remainder}$$

- ▶ the “approximation” is explicit enough operator to bound sharply its integral kernel by $|t-s|^{-\frac{n}{2}}$ (dispersion bound)
- ▶ the remainder is a remainder term in a Duhamel formula in which we combine L^2 time decay/propagation estimates (for the time decay) and Sobolev estimates (to replace $L^2 \rightarrow L^2$ by $L^1 \rightarrow L^\infty$) to derive dispersion bounds.

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