On the scattering theory of asymptotically flat manifolds and Strichartz inequalities

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Introduction

Purpose of the talk

- Take the question of **Strichartz inequalities** (for the Schrödinger equation) on **asymptotically flat manifolds** as a case study to review some related **scattering estimates** (resolvent estimates, timedecay, smoothing estimates), either for comparison or because they are crucial inputs in the proofs of Strichartz inequalities
- Present some recent results (joint with H. Mizutani) on Strichartz inequalities on asymptotically flat manifolds

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provided (p, q) is **admissible** (scaling condition)

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$$||e^{it\Delta} u_0||_{L^2(K)} \lesssim_K ||e^{it\Delta} u_0||_{L^q(\mathbb{R}^n)}, \qquad K \Subset \mathbb{R}^n.$$

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$$\left|\left|\langle\lambda^{\frac{1}{2}}x\rangle^{-k}\varphi(-\Delta/\lambda)e^{it\Delta}\langle\lambda^{\frac{1}{2}}x\rangle^{-k}\right|\right|_{L^2\to L^2} \lesssim \langle\lambda t\rangle^{-k}$$

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with $\nu > 1/2$. By tracking the dependence on λ, one may obtain the non spectrally localized estimate ($n \geq 3$)

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which is the $\frac{1}{2}$-**smoothing effect** for the Schrödinger equation. Note that even locally in time (i.e. with $\mathbb{R}$ replaced by $[-T, T]$) this is non trivial.

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which is the $\frac{1}{2}$-**smoothing effect** for the Schrödinger equation. Note that even locally in time (i.e. with $\mathbb{R}$ replaced by $[-T, T]$) this is non trivial.

Intuition. More on the next slides.

Strichartz and scattering estimates on the Euclidean space

Scattering inequalities (end)

- Integrated decay/ smoothing estimates: Integrated space-time decay estimates are of the form

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Rem. This correspondence $\lambda \to t$ also allows to convert resolvent estimates into time decay/propagation estimates (smoothness of $R_0(\lambda \pm i0) \leftrightarrow$ decay of e^{itP})

Strichartz inequalities vs smoothing effect for a wave packet

Strichartz inequalities

Strichartz inequalities vs smoothing effect for a wave packet

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Consider the L^2 normalized semiclassical wave packet

$$G_{z,\zeta,h}(x) = (\pi h)^{-\frac{n}{4}} \exp\left(\frac{i}{h}\zeta \cdot (x-z) - \frac{|x-z|^2}{2h}\right).$$

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for all $h \in (0,1]$ and $z \in \mathbb{R}^n$.

Strichartz inequalities vs smoothing effect for a wave packet

Smoothing effect (local in time)

$$\left|\langle D\rangle^s e^{i\frac{t}{2}\Delta} G_{z,\zeta,h}(x)\right|$$

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Strichartz inequalities vs smoothing effect for a wave packet

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If we further integrate in time on $[-T, T]_t$,

Strichartz inequalities vs smoothing effect for a wave packet

Smoothing effect (local in time)

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If we further integrate in time on $[-T, T]_t$,

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$$

which is bounded by

$$
c_n {\color{red}h}\langle 1/h\rangle^{2s}\int_{-T/h}^{T/h}\int \langle h^{\frac{1}{2}}Y_1\langle\tau\rangle + z_1 + \tau\rangle^{-2\nu}\exp\left(-Y^2\right)dY d\tau
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Strichartz inequalities vs smoothing effect for a wave packet

Smoothing effect (local in time)

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Remark. Up to the term $Y_1\langle\tau\rangle$, there is no more contribution of the spreading $\langle\tau\rangle$.

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$$\left|\left|\langle x\rangle^{-\nu}\langle\zeta/h\rangle^s G^t_{z,\zeta,h}\right|\right|^2_{L^2_x} = c_n\langle\zeta/h\rangle^{2s}\langle t/h\rangle^{-n}\int\langle h^{\frac{1}{2}}y+z+t\zeta/h\rangle^{-2\nu}\exp\left(-\frac{y^2}{\langle t/h\rangle^2}\right)dy$$

If we further integrate in time on $[-T,T]_t$,

$$c_n h\langle\zeta/h\rangle^{2s}\int_{-T/h}^{T/h}\langle\tau\rangle^{-n}\int\langle h^{\frac{1}{2}}y+z+\tau\zeta\rangle^{-2\nu}\exp\left(-\frac{y^2}{\langle\tau\rangle^2}\right)dyd\tau$$

which is bounded by

$$c_n h\langle 1/h\rangle^{2s}\int_{-T/h}^{T/h}\int\langle h^{\frac{1}{2}}Y_1\langle\tau\rangle+z_1+\tau\rangle^{-2\nu}\exp\left(-Y^2\right)dYd\tau$$

Remark. Up to the term $Y_1\langle\tau\rangle$, there is no more contribution of the spreading $\langle\tau\rangle$. Here, the main role will be played the translation by $(t/h)\zeta=\tau\zeta$.

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Global Strichartz inequalities on asymptotically flat manifolds

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Scattering inequalities turn out to play a crucial role in this problem.

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the (principal) symbol of the **Laplace-Beltrami operator**

$$-\Delta_G = -\sum_{j,k} G^{jk}(x) \partial_{x_j} \partial_{x_k} + \sum_{j,k,\ell} G^{jk}(x) \Gamma_{jk}^{\ell}(x) \partial_{x_\ell}$$

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Question: behavior of $R(\lambda \pm i0)$ and (2) as $\lambda \to \infty$ (high energy) and $\lambda \to 0$ (low energy) ?

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- Partial converse for trapping manifolds: if there are trapped geodesics

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- Robust estimates for powers
  $$||\langle \lambda^{\frac{1}{2}} r\rangle^{-k} (\lambda^{-1} P - 1 \pm i0)^{-k} \langle \lambda^{\frac{1}{2}} r\rangle^{-k}||_{L^2(M)\to L^2(M)} \lesssim 1$$

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- consequence on time decay
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  [Staffilani-Tataru], [Robbiano-Zuily], [B.-Tzvetkov], [Hassell-Tao-Wunsch]

- For asymptotically flat manifolds with **small hyperbolic trapped set**

$$||e^{i\cdot P}u_0||_{L^p([-T,T],L^q)} \lesssim_T ||u_0||_{L^2}$$

  [Burq-Guillarmou-Hassell]

Intuition (non trapping case):

- Inside a compact set K, combine

$$||\mathbf{1}_K e^{i\cdot P}u_0||_{L^2([-T,T],L^{2^*})} \lesssim_T ||u_0||_{H^{1/2}(M)} \quad \text{and} ||\mathbf{1}_K e^{i\cdot P}v_0||_{L^2([-T,T],H^{1/2})} \lesssim_T ||v_0||_{L^2}$$

- Outside a compact set: use that the geometry is close to a nice model (...)

Strichartz on asymptotically flat manifolds

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Theorem 4 (nonlinear scattering) *Under the assumptions of Theorem 3, the L^2 critical equation*

$$i\partial_t u - Pu = \sigma|u|^{\frac{4}{n}}u, \qquad u_{|t=0} = u_0, \qquad \sigma = \pm 1,$$

with $||u_0||_{L^2} \ll 1$, has a unique solution in (a subspace of) $C(\mathbb{R}, L^2) \cap L^{2+\frac{4}{n}}(\mathbb{R}\times M)$ and

$$||u(t) - e^{-itP}u_\pm||_{L^2(M)} \to 0, \qquad t \to \pm\infty.$$

A quarter of the proof

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Rem. For the localization, $(1-\chi(\epsilon r)) f(P/\epsilon^2)$,

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with C independent of λ (and u_0)

$$\begin{aligned}
||\chi(\epsilon r) f(P/\epsilon^2) e^{itP} u_0||_{L^{2*}} &\lesssim ||\nabla_G \chi(\epsilon r) f(P/\epsilon^2) e^{itP} u_0||_{L^2} \quad \text{(homogeneous Sobolev est.)} \\
&\lesssim ||\epsilon \chi'(\epsilon r) f(P/\epsilon^2) e^{itP} u_0||_{L^2} + ||\chi(\epsilon r) \nabla_G f(P/\epsilon^2) e^{itP} u_0||_{L^2} \\
&\lesssim ||\langle r\rangle^{-1} f(P/\epsilon^2) e^{itP} u_0||_{L^2} + ||\langle \epsilon r\rangle^{-1} P^{\frac{1}{2}} \tilde{f}(P/\epsilon^2) e^{itP} u_0||_{L^2} \\
&\lesssim ||\langle r\rangle^{-1} f(P/\epsilon^2) e^{itP} u_0||_{L^2} + ||\langle r\rangle^{-1} \tilde{\tilde{f}}(P/\epsilon^2) e^{itP} u_0||_{L^2}
\end{aligned}$$

où $\tilde{f}, \tilde{\tilde{f}} \in C_0^\infty(0, +\infty)$. One concludes by mean of an optimally weighted resolvent inequality [B-Royer, 2015]

$$||\langle r\rangle^{-1} f(P/\epsilon^2) e^{i\cdot P} u_0||_{L^2(\mathbb{R};L^2)} \lesssim \Big(1 + \sup_{|\lambda|\leq 2} ||\langle r\rangle^{-1} (P - \lambda \pm i0)^{-1} \langle r\rangle^{-1}||_{L^2 \to L^2}\Big) ||u_0||_{L^2}.$$

Rem. For the localization, $(1-\chi(\epsilon r)) f(P/\epsilon^2)$, one has "$|\xi| \sim \epsilon$" and "$|x| \gtrsim \epsilon^{-1}$" $\Rightarrow$ no problem of uncertainty principle to use microlocal techniques

Rest of the proof

At infinity: split $f(P/\lambda)e^{itP}$ into sums of

$$T_\lambda(t) = L_\lambda f(P/\lambda)e^{itP}$$

with suitable localization operators L_λ, and show

$$||T_\lambda(t)||_{L^2\to L^2} \lesssim 1, \qquad ||T_\lambda(t)T_\lambda(s)||_{L^1\to L^\infty} \lesssim |t-s|^{-\frac{n}{2}}$$

by writing

$$T_\lambda(t)T_\lambda(s) = \text{approximation} + \text{remainder}$$

- the "approximation" is explicit enough operator to bound sharply its integral kernel by $|t-s|^{-\frac{n}{2}}$ (dispersion bound)
- the remainder is a remainder term in a Duhamel formula in which we combine L^2 time decay/propagation estimates (for the time decay) and Sobolev estimates (to replace $L^2 \to L^2$ by $L^1 \to L^\infty$) to derive dispersion bounds.

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