PROPAGATORS ON CURVED SPACETIMES

JAN DEREZIŃSKI

in collaboration with

DANIEL SIEMSSEN

Dep. of Math. Meth in Phys.
Faculty of Physics
University of Warsaw

Consider a Klein–Gordon operator on a globally hyperbolic spacetime M

$$P := |g|^{-\frac{1}{2}}(x)\big(\mathrm{i}\partial_\mu - A_\mu(x)\big)g^{\mu\nu}|g|^{\frac{1}{2}}(x)\big(\mathrm{i}\partial_\nu - A_\nu(x)\big) + m^2(x).$$

We say that G is a bisolution of P if

$$GP = PG = 0.$$

We say that G is an inverse (Green's function or a fundamental solution) if

$$GP = PG = \mathbb{1}.$$

We are looking for distinguished bisolutions and inverses. We will call them propagators. (This word is often used in this context in quantum field theory).

The following "classical propagators" are well known and well defined under general conditions

- the forward/retarded inverse/propagator G^{+},
- the backward/advanced inverse/propagator G^{-},
- the Pauli-Jordan bisolution, also called the causal propagator or the commutator function $G^{\mathrm{PJ}} := G^{+} - G^{-}$.

We are however more interested in "non-classical propagators", typical for quantum field theory. They are less known to pure mathematicians and more difficult to define. They are the Feynman and anti-Feynman inverse and the positive and negative frequency bisolutions.

There exists a well-known paper of Duistermat-Hörmander, which defined Feynman parametrices (a parametrix is an approximate inverse in appropriate sense). There exists large literature devoted to the so-called Hadamard states, which can be interpreted as bisolutons with approximately positive frequencies. These are however large classes. We would like to have distinguished choices.

It is helpful to introduce a time variable t, so that the spacetime is $M = \mathbb{R} \times \Sigma$. Assume that there are no time-space cross terms, so that

$$P = \frac{1}{\beta}(\partial_t + \mathrm{i}V)^2 + L,$$

$$L = -|g|^{-\frac{1}{2}}(\partial_i + \mathrm{i}\vec{A}_i)|g|^{\frac{1}{2}} g_{\Sigma}^{ij}(\partial_j + \mathrm{i}\vec{A}_j) + m^2.$$

We rewrite the Klein-Gordon equation as a 1st order equation given by

$$P_1 := \mathrm{i}\partial_t - B(t),$$

where

$$B(t) := \begin{pmatrix} V(t) & \beta(t) \\ L(t) & V(t) \end{pmatrix}$$

Denote by $U(t,t')$ the dynamics defined by $B(t)$, that is

$$\begin{aligned}\partial_t U(t,t') &= -\mathrm{i}B(t)U(t,t'),\\ U(t,t) &= \mathbb{1}.\end{aligned}$$

Note that if E is a bisolution/inverse of P_1, then E_{12} is a bisolution/inverse of P.

The classical propagators can be easily expressed in terms of the dynamics defined by P_1:

$$
\begin{aligned}
E^{\mathrm{PJ}}(t,t') &:= U(t,t'), & E_{12}^{\mathrm{PJ}} &= G^{\mathrm{PJ}} \\
E^{+}(t,t') &:= \theta(t-t')\,U(t,t'), & E_{12}^{+} &= G^{+} \\
E^{-}(t,t') &:= -\theta(t'-t)\,U(t,t'), & E_{12}^{-} &= G^{-}.
\end{aligned}
$$

We introduce the charge matrix

$$Q := \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}.$$

and the classical Hamiltonian

$$H(t) := QB(t) = \begin{pmatrix} L(t) & V(t) \\ V(t) & \beta(t) \end{pmatrix}.$$

We will assume that $H(t)$ is strictly positive.

Assume now that the problem is static, so that β, L, V, B, H do not depend on time t. Clearly,

$$U(t,t') = \mathrm{e}^{-\mathrm{i}(t-t')B}.$$

The quadratic form H defines the so-called energy scalar product. It is easy to see that B is Hermitian in this product. Under appropriate assumptions B has a gap in its spectrum around 0. Let $\Pi^{(\pm)}$ be the projections onto the positive/negative part of the spectrum of B.

We define the Feynman and anti-Feynman inverse and the positive and negative frequency bisolutions on the level of P_1:

$$E^{(\pm)}(t,t') := \pm \mathrm{e}^{-\mathrm{i}(t-t')B}\Pi^{(\pm)},$$

$$E^{\mathrm{F}}(t,t') := \theta(t-t')\,\mathrm{e}^{-\mathrm{i}(t-t')B}\Pi^{(+)} - \theta(t'-t)\,\mathrm{e}^{-\mathrm{i}(t-t')B}\Pi^{(-)},$$

$$E^{\overline{\mathrm{F}}}(t,t') := \theta(t-t')\,\mathrm{e}^{-\mathrm{i}(t-t')B}\Pi^{(-)} - \theta(t'-t)\,\mathrm{e}^{-\mathrm{i}(t-t')B}\Pi^{(+)}.$$

They lead to corresponding propagators on the level of P:

$$\begin{aligned} G^{(\pm)} &:= E_{12}^{(\pm)}, \\ G^{\mathrm{F}} &:= E_{12}^{\mathrm{F}}, \\ G^{\overline{\mathrm{F}}} &:= E_{12}^{\overline{\mathrm{F}}}. \end{aligned}$$

They satisfy the relations

$$\begin{aligned} G^{\mathrm{PJ}} &= G^{(+)} - G^{(-)}, \\ G^{\mathrm{F}} &= G^{(+)} + G^{-} = G^{(-)} + G^{+}, \\ G^{\overline{\mathrm{F}}} &= -G^{(+)} + G^{+} = -G^{(-)} + G^{-}. \end{aligned}$$

Nonclassical propagators are important in quantum field theory, and they are often called 2-point functions, because they are vacuum expectation values of free fields:

$$
\begin{aligned}
G^{(+)}(x,y) &= \bigl(\Omega|\hat{\phi}(x)\hat{\phi}(y)\Omega\bigr), \\
G^{\mathrm{F}}(x,y) &= \bigl(\Omega|\mathrm{T}\bigl(\hat{\phi}(x)\hat{\phi}(y)\bigr)\Omega\bigr).
\end{aligned}
$$

G^{F} is used to evaluate Feynman diagrams.

It is easy to see that on a general spacetime the Klein-Gordon operator P is Hermitian (symmetric) on $C_{\mathrm{c}}^{\infty}(M)$ in the sense of the Hilbert space $L^2(M)$. In the static case, using Nelson's Commutator Theorem one can show that it is essentially self-adjoint.

Theorem. For $s > \frac{1}{2}$, the operator G^{F} is bounded from the space $\langle t\rangle^{-s}L^2(M)$ to $\langle t\rangle^{s}L^2(M)$. Besides, in the sense of these spaces,

$$\mathrm{s-}\lim_{\epsilon\searrow 0}(P - \mathrm{i}\epsilon)^{-1} = G^{\mathrm{F}}.$$

Let $0 \le \theta \le \pi$. Suppose we replace the metric g by

$$g_\theta := -\mathrm{e}^{-2\mathrm{i}\theta}\beta\,\mathrm{d}t^2 + g_\Sigma$$

and the electric potential V by $V_\theta := \mathrm{e}^{-\mathrm{i}\theta}V$. This replacement is called Wick rotation. The value $\theta = \frac{\pi}{2}$ corresponds to the Riemannian metric

$$g_{\pi/2} = \beta\,\mathrm{d}t^2 + g_\Sigma.$$

We have the Wick rotated Klein-Gordon operator, which is elliptic and even invertible:

$$P_\theta = \frac{\mathrm{e}^{-\mathrm{i}2\theta}}{\beta}(\partial_t + \mathrm{i}V)^2 + L,$$

Theorem. For $s > \frac{1}{2}$, we have

$$\mathrm{s-}\lim_{\theta \searrow 0} P_\theta^{-1} = G^{\mathrm{F}},$$

in the sense of operators from $\langle t\rangle^{-s}L^2(M)$ to $\langle t\rangle^{s}L^2(M)$.

Can one generalize non-classical propagators to non-static spacetimes? We will assume that the spacetime is close to being static and for large times it approaches a static spacetime sufficiently fast.

It seems that there are 3 natural possibilities:

1. incoming positive/negative frequency bisolution and Feynman propagator;
2. outgoing positive/negative frequency bisolution and Feynman propagator;
3. “canonical” positive/negative frequency bisolution and Feynman propagator.

The incoming positive frequency bisolution is obtained by cutting the phase space with the projections $\Pi_{-\infty}^{(\pm)}$ onto the positive/negative part of the spectrum of $B(-\infty)$. Then the incoming Feynman propagator is obtained by the usual relation. One can argue, that this is the most physical choice, since states are usually prepared in a distant past.

Analogously, the outgoing propagators are defined using the projections $\Pi_{\infty}^{(\pm)}$ onto the positive/negative part of the spectrum of $B(\infty)$.

For spacetimes very close to static, there is a third possibility, which is in some sense the most "canonical". Hence what is obtained we call canonical propagators Note that the projection $\Pi_{-\infty}^{(+)}$ can be transported by the dynamics to any time t, obtaining the projection $\Pi_{-\infty}^{(+)}(t)$. Similarly we obtain the projection $\Pi_{\infty}^{(-)}(t)$. We will say that the Klein-Gordon equation is asymptotically complementary if for some t (and hence for all t) the subspaces

$$\mathrm{Ran}\,\Pi_{-\infty}^{(+)}(t), \mathrm{Ran}\,\Pi_{+\infty}^{(-)}(t)$$

are complementary.

Assume that asymptotic complementarity holds. Define $\Pi_{\rm can}^{(+)}(t)$, $\Pi_{\rm can}^{(-)}(t)$ to be the unique pair of projections corresponding to

$$\mathrm{Ran}\,\Pi_{-\infty}^{(+)}(t), \mathrm{Ran}\,\Pi_{+\infty}^{(-)}(t)$$

The canonical Feynman propagator is defined as

$$\begin{aligned} E_{\rm can}^{\rm F}(t_2,t_1) &:= \theta(t_2-t_1)U(t_2,t_1)\Pi_{\rm can}^{(+)}(t_1) \\ &\quad -\theta(t_1-t_2)U(t_2,t_1)\Pi_{\rm can}^{(-)}(t_1), \\ G_{\rm can}^{\rm F} &:= E_{{\rm can},12}^{\rm F}. \end{aligned}$$

In a somewhat different setting, in the case of massless Klein-Gordon operator $G^{\mathrm{F}}_{\mathrm{can}}$ was considered before by A.Vasy et al. A similar construction can be found in a recent paper of Gerard-Wrochna.

$G^{\mathrm{F}}_{\mathrm{can}}$ has a flaw from the physical point of view: in general it is not associated with a positive state. However, it also seems to have advantages.

Conjecture. For small enough, compactly supported perturbations of the static case, the following holds:

1. Asymptotic complementarity holds, so that we can define $G^{\mathrm{F}}_{\mathrm{can}}$.
2. The Klein-Gordon operator P is essentially self-adjoint on $C_{\mathrm{c}}^{\infty}(M)$.
3. In the sense of operators from $\langle t\rangle^{-s}L^2(M)$ to $\langle t\rangle^{s}L^2(M)$,

$$\mathrm{s-}\lim_{\epsilon\searrow 0}(P-\mathrm{i}\epsilon)^{-1}=G^{\mathrm{F}}.$$