Wigner Measures and Effective Mass Theorems

Victor Chabu[1], Clotilde Fermanian Kammerer[1], Fabricio Macia [2]

[1]Université Paris Est - Créteil & CNRS

[2]Universidad Politécnica de Madrid, ETSI Navales

Cergy-Pontoise, June 22nd. 2016

Content:

1. Schrödinger Equation in a Lattice
2. Floquet-Bloch theory
3. Quantifying the lack of dispersion
4. Strategy of the proof
5. Back to Effective Mass Theory

The equation:

$$i\hbar\partial_t\Psi + \frac{\hbar^2}{2m}\Delta_x\Psi - Q_{\rm per}(x)\,\Psi - Q_{\rm ext}(x)\Psi = 0, \;\; (t,x)\in\mathbb{R}\times\mathbb{R}^d,$$
$$\Psi_{|t=0} = \Psi_0 \in L^2(\mathbb{R}^d),$$

where $Q_{\rm per}$ is a potential periodic with respect to a lattice $\Gamma = \mathbb{Z}^d$.

Let ε be the ratio between the mean spacing of the lattice and the characteristic length scale of variation of $Q_{\rm ext}$.

$$\varepsilon \ll 1.$$

$\Longrightarrow$ Change of units and rescaling the external potential and the wave function ! (see [Poupaud & Ringhofer 96]).

The equation:

$$i\hbar\partial_t\Psi + \frac{\hbar^2}{2m}\Delta_x\Psi - Q_{\rm per}(x)\,\Psi - Q_{\rm ext}(x)\Psi = 0,\ \ (t,x)\in\mathbb{R}\times\mathbb{R}^d,$$
$$\Psi_{|t=0} = \Psi_0 \in L^2(\mathbb{R}^d),$$

where $Q_{\rm per}$ is a potential periodic with respect to a lattice $\Gamma = \mathbb{Z}^d$.

Let ε be the ratio between the mean spacing of the lattice and the characteristic length scale of variation of $Q_{\rm ext}$.

$$\varepsilon \ll 1.$$

$\Longrightarrow$ Change of units and rescaling the external potential and the wave function ! (see [Poupaud & Ringhofer 96]).

The equation:

$$i\partial_t \psi^\varepsilon + \frac{1}{2}\Delta_x \psi^\varepsilon - \frac{1}{\varepsilon^2} V_{\rm per}\left(\frac{x}{\varepsilon}\right)\psi^\varepsilon - V(x)\psi^\varepsilon = 0, \;\; (t,x)\in\mathbb{R}\times\mathbb{R}^d,$$
$$\psi^\varepsilon_{|t=0} = \psi^\varepsilon_0 \in L^2(\mathbb{R}^d),$$

where $V_{\rm per}$ is a potential periodic with respect to $\mathbb{Z}^d$.

Question: Effective Mass Theory consists in showing situations where $\psi^\varepsilon(t)$ can be approximated by the solution of a Effective Mass Equation:

$$i\partial_t \phi(t,x) + \frac{1}{2}\langle M\, D_x, D_x\rangle \phi(t,x) - V(x)\phi(t,x) = 0.$$

M is a $d\times d$ matrix called the effective mass tensor.

The equation:

$$i\partial_t\psi^\varepsilon + \frac{1}{2}\Delta_x\psi^\varepsilon - \frac{1}{\varepsilon^2}V_{\rm per}\left(\frac{x}{\varepsilon}\right)\psi^\varepsilon - V(x)\psi^\varepsilon = 0, \ \ (t,x)\in\mathbb{R}\times\mathbb{R}^d,$$
$$\psi^\varepsilon_{|t=0} = \psi^\varepsilon_0 \in L^2(\mathbb{R}^d),$$

where $V_{\rm per}$ is a potential periodic with respect to $\mathbb{Z}^d$.

Question: Effective Mass Theory consists in showing situations where $\psi^\varepsilon(t)$ can be approximated by the solution of a Effective Mass Equation:

$$i\partial_t\phi(t,x) + \frac{1}{2}\langle M\,D_x, D_x\rangle\phi(t,x) - V(x)\phi(t,x) = 0.$$

M is a $d\times d$ matrix called the effective mass tensor.

Schrödinger Equation in a Lattice : Our purpose

The two main questions of the literature :

- Finding initial conditions for which the previous analysis holds,
- Finding the corresponding M.

[Bensoussan, Lions & Papanicolaou 78], [Poupaud & Ringhofer 96], [Allaire & Piatniski 05], [Hoefer &Weinstein 11], [Barletti & Ben Abdallah 11].

$\Longrightarrow$ Our purpose :

- Getting rid of assumptions on the initial conditions,
- Clarifying the dependence of M on the parameter of the equation.

Schrödinger Equation in a Lattice : Our purpose

The two main questions of the literature :

- Finding initial conditions for which the previous analysis holds,
- Finding the corresponding M.

[Bensoussan, Lions & Papanicolaou 78], [Poupaud & Ringhofer 96], [Allaire & Piatniski 05], [Hoefer &Weinstein 11], [Barletti & Ben Abdallah 11].

$\Longrightarrow$ Our purpose :

- Getting rid of assumptions on the initial conditions,
- Clarifying the dependence of M on the parameter of the equation.

Schrödinger Equation in a Lattice : our strategy

Remark

If v^ε solves the semiclassical Schrödinger equation

$$i\varepsilon\partial_t v^\varepsilon + \frac{\varepsilon^2}{2}\Delta_x v^\varepsilon - V_{\rm per}\left(\frac{x}{\varepsilon}\right) v^\varepsilon - \varepsilon^2 V(x) v^\varepsilon = 0, \quad v^\varepsilon|_{t=0} = \psi_0^\varepsilon.$$

Then, $\psi^\varepsilon(t,x) = v^\varepsilon\left(\frac{t}{\varepsilon}, x\right)$.

[Gérard], [GMMP] [Poupaud & Ringhofer], [Bechouche, Mauser & Poupaud], [Spohn & Teufel] [Panati, Spohn &Teufel], [Dimassi, Guillot & Ralston] [Allaire &Palombaro] [Carles &Sparber]

$\Longrightarrow$ Perform simultaneously the s.c. limit $\varepsilon \to 0$ with the limit $t/\varepsilon \to +\infty$.

[Macia and his collaborators Anantharaman, Léautaud, Rivière & C.F.K.] (without periodic potential).

Our goal : apply this viewpoint to effective mass theory.
$\Longrightarrow$ a generalized effective mass equation of Heisenberg type.

Remark

If v^ε solves the semiclassical Schrödinger equation

$$i\varepsilon\partial_t v^\varepsilon + \frac{\varepsilon^2}{2}\Delta_x v^\varepsilon - V_{\rm per}\left(\frac{x}{\varepsilon}\right) v^\varepsilon - \varepsilon^2 V(x) v^\varepsilon = 0, \quad v^\varepsilon|_{t=0} = \psi_0^\varepsilon.$$

Then, $\psi^\varepsilon(t,x) = v^\varepsilon\left(\frac{t}{\varepsilon}, x\right)$.

[Gérard], [GMMP] [Poupaud & Ringhofer], [Bechouche, Mauser & Poupaud], [Spohn & Teufel] [Panati, Spohn &Teufel], [Dimassi, Guillot & Ralston] [Allaire &Palombaro] [Carles &Sparber]

$\Longrightarrow$ Perform simultaneously the s.c. limit $\varepsilon \to 0$ with the limit $t/\varepsilon \to +\infty$.

[Macia and his collaborators Anantharaman, Léautaud, Rivière & C.F.K.] (without periodic potential).

Our goal : apply this viewpoint to effective mass theory.

$\Longrightarrow$ a generalized effective mass equation of Heisenberg type.

Schrödinger Equation in a Lattice : our strategy

Remark

If v^ε solves the semiclassical Schrödinger equation

$$i\varepsilon\partial_t v^\varepsilon + \frac{\varepsilon^2}{2}\Delta_x v^\varepsilon - V_{\rm per}\left(\frac{x}{\varepsilon}\right) v^\varepsilon - \varepsilon^2 V(x) v^\varepsilon = 0, \quad v^\varepsilon|_{t=0} = \psi_0^\varepsilon.$$

Then, $\psi^\varepsilon(t,x) = v^\varepsilon\left(\frac{t}{\varepsilon}, x\right)$.

[Gérard], [GMMP] [Poupaud & Ringhofer], [Bechouche, Mauser & Poupaud], [Spohn & Teufel] [Panati, Spohn &Teufel], [Dimassi, Guillot & Ralston] [Allaire &Palombaro] [Carles &Sparber]

$\Longrightarrow$ Perform simultaneously the s.c. limit $\varepsilon \to 0$ with the limit $t/\varepsilon \to +\infty$.

[Macia and his collaborators Anantharaman, Léautaud, Rivière & C.F.K.] (without periodic potential).

Our goal : apply this viewpoint to effective mass theory.
$\Longrightarrow$ a generalized effective mass equation of Heisenberg type.

Floquet-Bloch theory : Bloch waves and energies

- We use the Ansatz : $\psi^\varepsilon(t,x) = U^\varepsilon\left(t, x, \frac{x}{\varepsilon}\right)$, where $U^\varepsilon(t,x,y)$ is assumed to be $\mathbb{Z}^d$-periodic in y and solves

$$i\varepsilon^2\partial_t U^\varepsilon(t,x,y) = P(\varepsilon D)U^\varepsilon(t,x,y) + \varepsilon^2 V(x)U^\varepsilon(t,x,y), \quad U^\varepsilon|_{t=0} = \psi_0^\varepsilon$$

$$\text{where } P(\xi) = \frac{1}{2}(\xi + D_y)^2 + V_\Gamma(y), \;\; y \in \mathbb{T}^d := \mathbb{R}^d \backslash \mathbb{Z}^d.$$

- The Bloch energies are the eigenvalues of the self-adjoint operator on the torus $P(\xi)$:

$$\lambda_1(\xi) \leq \lambda_2(\xi) \leq \cdots \leq \lambda_n(\xi) \to +\infty.$$

They are $2\pi\mathbb{Z}^d$ periodic and smooth in domain where they are of constant multiplicity.

- The Bloch waves are the orthonormal eigenfunctions of $P(\xi)$

$$P(\xi)\varphi_n(\xi,y) = \lambda_n(\xi)\varphi_n(\xi,y), \;\; n \in \mathbb{N}, \;\; y \in \mathbb{T}^2, \;\; \forall \xi \in \mathbb{R}^d.$$

Floquet-Bloch theory : Bloch waves and energies

- We use the Ansatz : $\psi^\varepsilon(t,x) = U^\varepsilon\left(t, x, \frac{x}{\varepsilon}\right)$, where $U^\varepsilon(t,x,y)$ is assumed to be $\mathbb{Z}^d$-periodic in y and solves

$$i\varepsilon^2\partial_t U^\varepsilon(t,x,y) = P(\varepsilon D)U^\varepsilon(t,x,y) + \varepsilon^2 V(x) U^\varepsilon(t,x,y), \quad U^\varepsilon|_{t=0} = \psi_0^\varepsilon$$

$$\text{where } \; P(\xi) = \frac{1}{2}(\xi + D_y)^2 + V_\Gamma(y), \;\; y \in \mathbb{T}^d := \mathbb{R}^d \backslash \mathbb{Z}^d.$$

- The Bloch energies are the eigenvalues of the self-adjoint operator on the torus $P(\xi)$:

$$\lambda_1(\xi) \leq \lambda_2(\xi) \leq \cdots \leq \lambda_n(\xi) \to +\infty.$$

They are $2\pi\mathbb{Z}^d$ periodic and smooth in domain where they are of constant multiplicity.

- The Bloch waves are the orthonormal eigenfunctions of $P(\xi)$

$$P(\xi)\varphi_n(\xi,y) = \lambda_n(\xi)\varphi_n(\xi,y), \;\; n \in \mathbb{N}, \;\; y \in \mathbb{T}^2, \;\; \forall \xi \in \mathbb{R}^d.$$

Floquet-Bloch theory : Bloch waves and energies

- We use the Ansatz : $\psi^\varepsilon(t,x) = U^\varepsilon\left(t, x, \frac{x}{\varepsilon}\right)$, where $U^\varepsilon(t,x,y)$ is assumed to be $\mathbb{Z}^d$-periodic in y and solves

$$i\varepsilon^2\partial_t U^\varepsilon(t,x,y) = P(\varepsilon D)U^\varepsilon(t,x,y) + \varepsilon^2 V(x)U^\varepsilon(t,x,y), \quad U^\varepsilon|_{t=0} = \psi_0^\varepsilon$$

$$\text{where } \; P(\xi) = \frac{1}{2}(\xi + D_y)^2 + V_\Gamma(y), \;\; y \in \mathbb{T}^d := \mathbb{R}^d \backslash \mathbb{Z}^d.$$

- The Bloch energies are the eigenvalues of the self-adjoint operator on the torus $P(\xi)$:

$$\lambda_1(\xi) \le \lambda_2(\xi) \le \cdots \le \lambda_n(\xi) \to +\infty.$$

  They are $2\pi\mathbb{Z}^d$ periodic and smooth in domain where they are of constant multiplicity.

- The Bloch waves are the orthonormal eigenfunctions of $P(\xi)$

$$P(\xi)\varphi_n(\xi,y) = \lambda_n(\xi)\varphi_n(\xi,y), \;\; n \in \mathbb{N}, \;\; y \in \mathbb{T}^2, \;\; \forall \xi \in \mathbb{R}^d.$$

Floquet Bloch theory : Bloch decomposition

- Consider $(\Pi_n(\xi))_{n\in\mathbb{N}}$ a family of projectors on separated Bloch bands and

$$U_n^\varepsilon(t,x,y) := \Pi_n(\varepsilon D_x)U^\varepsilon(t,x,y) = \int_{\mathbb{R}^d\times\mathbb{R}^d} \Pi_n(\varepsilon\xi)U^\varepsilon(t,w,y)\mathrm{e}^{i\xi\cdot(x-w)}\frac{dwd\xi}{(2\pi)^d} dy$$

$$\textit{so that} \;\; U^\varepsilon(t,x,y) = \sum_{n\in\mathbb{N}} U_n^\varepsilon(t,x,y).$$

- This construction leads to the following representation formula for the solution of the Schrödinger equation

$$\psi^\varepsilon(t,x) = \sum_{n\in\mathbb{N}} U_n^\varepsilon\left(t,x,\frac{x}{\varepsilon}\right).$$

- If $\mathrm{Rk}\,\Pi_n(\xi) = 1$, $\;\mathrm{Range}\,\Pi_n(\xi) = \mathrm{Vect}\,\varphi_n(\xi,\cdot)$, $P(\xi)\varphi_n(\xi) = \lambda_n(\xi)\varphi_n(\xi)$,

$$U_n^\varepsilon(t,x,y) = \varphi_n(\varepsilon D,y)u_n^\varepsilon(t,x) + \mathcal{O}(\varepsilon|t|),$$

where u_n^ε solves

$$i\varepsilon^2\partial_t u_n^\varepsilon = \lambda_n(\varepsilon D_x)u_n^\varepsilon + \varepsilon^2 V(x)u_n^\varepsilon, \;\; u_n^\varepsilon|_{t=0} = u_{n,0}^\varepsilon.$$

Floquet Bloch theory : Bloch decomposition

- Consider $(\Pi_n(\xi))_{n\in\mathbb{N}}$ a family of projectors on separated Bloch bands and

$$U_n^\varepsilon(t,x,y) := \Pi_n(\varepsilon D_x)U^\varepsilon(t,x,y) = \int_{\mathbb{R}^d\times\mathbb{R}^d} \Pi_n(\varepsilon\xi)U^\varepsilon(t,w,y)\mathrm{e}^{i\xi\cdot(x-w)}\frac{dwd\xi}{(2\pi)^d}dy$$

$$\text{so that } \; U^\varepsilon(t,x,y) = \sum_{n\in\mathbb{N}} U_n^\varepsilon(t,x,y).$$

- This construction leads to the following representation formula for the solution of the Schrödinger equation

$$\psi^\varepsilon(t,x) = \sum_{n\in\mathbb{N}} U_n^\varepsilon\left(t,x,\frac{x}{\varepsilon}\right).$$

- If $\mathrm{Rk}\,\Pi_n(\xi) = 1$, $\;\mathrm{Range}\,\Pi_n(\xi) = \mathrm{Vect}\,\varphi_n(\xi,\cdot)$, $P(\xi)\varphi_n(\xi) = \lambda_n(\xi)\varphi_n(\xi)$,

$$U_n^\varepsilon(t,x,y) = \varphi_n(\varepsilon D,y)u_n^\varepsilon(t,x) + \mathcal{O}(\varepsilon|t|),$$

where u_n^ε solves

$$i\varepsilon^2\partial_t u_n^\varepsilon = \lambda_n(\varepsilon D_x)u_n^\varepsilon + \varepsilon^2 V(x)u_n^\varepsilon,\quad u_n^\varepsilon|_{t=0} = u_{n,0}^\varepsilon.$$

Floquet Bloch theory : Bloch decomposition

- Consider $(\Pi_n(\xi))_{n\in\mathbb{N}}$ a family of projectors on separated Bloch bands and

$$U_n^\varepsilon(t,x,y) := \Pi_n(\varepsilon D_x)U^\varepsilon(t,x,y) = \int_{\mathbb{R}^d\times\mathbb{R}^d} \Pi_n(\varepsilon\xi)U^\varepsilon(t,w,y)\mathrm{e}^{i\xi\cdot(x-w)}\frac{dwd\xi}{(2\pi)^d}dy$$

$$\textit{so that}\ \ U^\varepsilon(t,x,y) = \sum_{n\in\mathbb{N}} U_n^\varepsilon(t,x,y).$$

- This construction leads to the following representation formula for the solution of the Schrödinger equation

$$\psi^\varepsilon(t,x) = \sum_{n\in\mathbb{N}} U_n^\varepsilon\left(t,x,\frac{x}{\varepsilon}\right).$$

- If $\mathrm{Rk}\,\Pi_n(\xi)=1$, $\mathrm{Range}\,\Pi_n(\xi) = \mathrm{Vect}\,\varphi_n(\xi,\cdot)$, $P(\xi)\varphi_n(\xi) = \lambda_n(\xi)\varphi_n(\xi)$,

$$U_n^\varepsilon(t,x,y) = \varphi_n(\varepsilon D,y)u_n^\varepsilon(t,x) + \mathcal{O}(\varepsilon|t|),$$

  where u_n^ε solves

$$i\varepsilon^2\partial_t u_n^\varepsilon = \lambda_n(\varepsilon D_x)u_n^\varepsilon + \varepsilon^2 V(x)u_n^\varepsilon,\ \ u_n^\varepsilon|_{t=0} = u_{n,0}^\varepsilon.$$

Quantifying the lack of dispersion : a more general question

- Consider equations of the form

$$\begin{cases} i\varepsilon^2\partial_t u^\varepsilon(t,x) = \lambda(\varepsilon D_x)u^\varepsilon(t,x) + \varepsilon^2 V(x)u^\varepsilon(t,x), \quad (t,x)\in\mathbb{R}\times\mathbb{R}^d, \\ u^\varepsilon|_{t=0} = u_0^\varepsilon. \end{cases} \tag{1}$$

  This equation ceases to be dispersive as soon as $\lambda(\xi)$ has critical points $\xi \neq 0$, and this is always the case if λ is a Bloch energy.

- Heuristically, dispersive time-evolution $\Longrightarrow$ smoothing effect
  i.e. regularization of the high-frequency effects developed by the initial data.
  [Kato 83], [Sjölin 87], [Vega 88], [Constantin & Saut 88], [Kenig, Ponce & Vega 91], [Ben Artzi & Devinatz 91].

- We show that, in the presence of critical points of λ, some of the high-frequency effects developed by the sequence of initial data persist after applying the time evolution.

Quantifying the lack of dispersion : a more general question

- Consider equations of the form

$$\begin{cases} i\varepsilon^2\partial_t u^\varepsilon(t,x) = \lambda(\varepsilon D_x)u^\varepsilon(t,x) + \varepsilon^2 V(x)u^\varepsilon(t,x), \quad (t,x)\in\mathbb{R}\times\mathbb{R}^d, \\ u^\varepsilon|_{t=0} = u_0^\varepsilon. \end{cases} \tag{1}$$

  This equation ceases to be dispersive as soon as $\lambda(\xi)$ has critical points $\xi\neq 0$, and this is always the case if λ is a Bloch energy.

- Heuristically, dispersive time-evolution $\Longrightarrow$ smoothing effect i.e. regularization of the high-frequency effects developed by the initial data. [Kato 83], [Sjölin 87], [Vega 88], [Constantin & Saut 88], [Kenig, Ponce & Vega 91], [Ben Artzi & Devinatz 91].

- We show that, in the presence of critical points of λ, some of the high-frequency effects developed by the sequence of initial data persist after applying the time evolution.

Quantifying the lack of dispersion : The assumptions

Assumptions:

H0 The sequence (u_0^ε) is uniformly bounded in $L^2(\mathbb{R}^d)$ and ε-oscillating :

$$\limsup_{\varepsilon\to 0^+} \int_{|\xi|>R/\varepsilon} |\widehat{u_0^\varepsilon}(\xi)|^2 d\xi \underset{R\to+\infty}{\longrightarrow} 0.$$

H1 $V \in \mathcal{C}^\infty(\mathbb{R}^d)$ and $\lambda \in \mathcal{C}^\infty(\mathbb{R}^d)$ grows at most polynomially; *i.e.* there exist $C, N > 0$ such that:

$$|\lambda(\xi)| \leq C(1+|\xi|)^N, \quad \forall \xi \in \mathbb{R}^d.$$

H2 The set $\Lambda := \left\{\xi \in \mathbb{R}^d : \nabla\lambda(\xi) = 0\right\}$ is a submanifold of $\mathbb{R}^d$ of codimension $0 < p \leq d$ and the Hessian $\nabla^2\lambda$ is of maximal rank over Λ. Moreover, each connected component of Λ is compact.

Remark

If all critical points of λ are non-degenerate, then Λ is a discrete set in $\mathbb{R}^d$. If moreover one has that λ is $\mathbb{Z}^d$-periodic, this set is finite modulo $\mathbb{Z}^d$.

Quantifying the lack of dispersion : The assumptions

Assumptions:

H0 The sequence (u_0^ε) is uniformly bounded in $L^2(\mathbb{R}^d)$ and ε-oscillating :

$$\limsup_{\varepsilon\to 0^+} \int_{|\xi|>R/\varepsilon} |\widehat{u_0^\varepsilon}(\xi)|^2 d\xi \underset{R\to+\infty}{\longrightarrow} 0.$$

H1 $V \in \mathcal{C}^\infty(\mathbb{R}^d)$ and $\lambda \in \mathcal{C}^\infty(\mathbb{R}^d)$ grows at most polynomially; *i.e.* there exist $C, N > 0$ such that:

$$|\lambda(\xi)| \leq C(1+|\xi|)^N, \quad \forall \xi \in \mathbb{R}^d.$$

H2 The set $\Lambda := \{\xi \in \mathbb{R}^d : \nabla\lambda(\xi) = 0\}$ is a submanifold of $\mathbb{R}^d$ of codimension $0 < p \leq d$ and the Hessian $\nabla^2\lambda$ is of maximal rank over Λ. Moreover, each connected component of Λ is compact.

Remark

If all critical points of λ are non-degenerate, then Λ is a discrete set in $\mathbb{R}^d$. If moreover one has that λ is $\mathbb{Z}^d$-periodic, this set is finite modulo $\mathbb{Z}^d$.

Quantifying the lack of dispersion : non-degenerate case

Theorem (Obstruction to smoothing effects in presence of critical points)

Assume H0 & H1 and that all critical points of λ are non-degenerate.
Then there exists a subsequence $(u_0^{\varepsilon_k})$ such that $\forall a < b$ and $\forall \phi \in \mathcal{C}_c(\mathbb{R}^d)$:

$$\lim_{k\to\infty} \int_a^b \int_{\mathbb{R}^d} \phi(x)|u^{\varepsilon_k}(t,x)|^2 dxdt = \sum_{\xi\in\Lambda} \int_a^b \int_{\mathbb{R}^d} \phi(x)|u_\xi(t,x)|^2 dxdt,$$

where u_ξ solves the Schrödinger equation:

$$i\partial_t u_\xi(t,x) = \nabla^2\lambda(\xi)D_x \cdot D_x u_\xi(t,x) + V(x)u_\xi(t,x),$$

with initial data $u_\xi|_{t=0}$ which is the weak limit in $L^2(\mathbb{R}^d)$ of $\left(\mathrm{e}^{-i\xi/\varepsilon_k\cdot x} u_0^{\varepsilon_k}\right)$.
If $\Lambda = \emptyset$ then the right-hand side above is equal to zero.

Example : If $u_0^\varepsilon(x) = \frac{1}{\varepsilon^{d/4}}\rho\left(\frac{x-x_0}{\sqrt{\varepsilon}}\right)\mathrm{e}^{i\xi_0/\varepsilon\cdot x}$, then $u_\xi = 0$ for all ξ and the Theorem yields that (u^ε) converge to zero in $L^2_{\mathrm{loc}}(\mathbb{R}\times\mathbb{R}^d)$.

Related work : [Ruzhanski & Sugimoto 16]

Quantifying the lack of dispersion : non-degenerate case

Theorem (Obstruction to smoothing effects in presence of critical points)

Assume H0 & H1 *and that all critical points of* λ *are non-degenerate.*
Then there exists a subsequence $(u_0^{\varepsilon_k})$ *such that* $\forall a < b$ *and* $\forall \phi \in \mathcal{C}_c(\mathbb{R}^d)$ *:*

$$\lim_{k\to\infty} \int_a^b \int_{\mathbb{R}^d} \phi(x)|u^{\varepsilon_k}(t,x)|^2 dxdt = \sum_{\xi\in\Lambda} \int_a^b \int_{\mathbb{R}^d} \phi(x)|u_\xi(t,x)|^2 dxdt,$$

where u_ξ *solves the Schrödinger equation:*

$$i\partial_t u_\xi(t,x) = \nabla^2\lambda(\xi) D_x \cdot D_x u_\xi(t,x) + V(x) u_\xi(t,x),$$

with initial data $u_\xi|_{t=0}$ *which is the weak limit in* $L^2(\mathbb{R}^d)$ *of* $\left(\mathrm{e}^{-i\xi/\varepsilon_k\cdot x} u_0^{\varepsilon_k}\right)$*.*
If $\Lambda = \emptyset$ *then the right-hand side above is equal to zero.*

Example : If $u_0^\varepsilon(x) = \frac{1}{\varepsilon^{d/4}} \rho\left(\frac{x-x_0}{\sqrt{\varepsilon}}\right) \mathrm{e}^{i\xi_0/\varepsilon\cdot x}$, then $u_\xi = 0$ for all ξ and the Theorem yields that (u^ε) converge to zero in $L^2_{\mathrm{loc}}(\mathbb{R}\times\mathbb{R}^d)$.

Related work : [Ruzhanski & Sugimoto 16]

Quantifying the lack of dispersion : degenerate case

- When the non-degeneracy of the critical points is replaced by H2, we obtain a similar result which requires some geometric preliminaries.
- Define the tangent bundle of Λ as the union of all tangent spaces to Λ,

$$T\Lambda := \{(x,\xi) \in \mathbb{R}^d \times \Lambda \, : \, x \in T_\xi \Lambda\}.$$

- The normal bundle of Λ is the union of linear subspaces normal to Λ:

$$N\Lambda := \{(y,\xi) \in \mathbb{R}^d \times \Lambda \, : \, y \in N_\xi \Lambda = (T_\xi \Lambda)^\perp\}.$$

  Every point $x \in \mathbb{R}^d$ can be uniquely written as $x = z + y$, where $z \in T_\xi \Lambda$ and $y \in N_\xi \Lambda$.
- Given a function $\phi \in L^\infty(\mathbb{R}^d)$, we write $m_\phi(z,\xi)$, where $z \in T_\xi \Lambda$, to denote the operator acting on $L^2(N_\xi \Lambda)$ by multiplication by $\phi(z + \cdot)$.
- We use the notation Δ_E to denote the Laplacian acting on functions defined on a linear subspace $E \subset \mathbb{R}^d$.

Quantifying the lack of dispersion : degenerate case

- When the non-degeneracy of the critical points is replaced by H2, we obtain a similar result which requires some geometric preliminaries.
- Define the tangent bundle of Λ as the union of all tangent spaces to Λ,

$$T\Lambda := \{(x,\xi) \in \mathbb{R}^d \times \Lambda \,:\, x \in T_\xi \Lambda\}.$$

- The normal bundle of Λ is the union of linear subspaces normal to Λ:

$$N\Lambda := \{(y,\xi) \in \mathbb{R}^d \times \Lambda \,:\, y \in N_\xi \Lambda = (T_\xi \Lambda)^\perp\}.$$

Every point $x \in \mathbb{R}^d$ can be uniquely written as $x = z + y$, where $z \in T_\xi \Lambda$ and $y \in N_\xi \Lambda$.

- Given a function $\phi \in L^\infty(\mathbb{R}^d)$, we write $m_\phi(z,\xi)$, where $z \in T_\xi \Lambda$, to denote the operator acting on $L^2(N_\xi \Lambda)$ by multiplication by $\phi(z + \cdot)$.
- We use the notation Δ_E to denote the Laplacian acting on functions defined on a linear subspace $E \subset \mathbb{R}^d$.

Quantifying the lack of dispersion : degenerate case

- When the non-degeneracy of the critical points is replaced by H2, we obtain a similar result which requires some geometric preliminaries.
- Define the tangent bundle of Λ as the union of all tangent spaces to Λ,

$$T\Lambda := \{(x,\xi) \in \mathbb{R}^d \times \Lambda \, : \, x \in T_\xi \Lambda\}.$$

- The normal bundle of Λ is the union of linear subspaces normal to Λ:

$$N\Lambda := \{(y,\xi) \in \mathbb{R}^d \times \Lambda \, : \, y \in N_\xi \Lambda = (T_\xi \Lambda)^\perp\}.$$

Every point $x \in \mathbb{R}^d$ can be uniquely written as $x = z + y$, where $z \in T_\xi \Lambda$ and $y \in N_\xi \Lambda$.

- Given a function $\phi \in L^\infty(\mathbb{R}^d)$, we write $m_\phi(z,\xi)$, where $z \in T_\xi \Lambda$, to denote the operator acting on $L^2(N_\xi \Lambda)$ by multiplication by $\phi(z + \cdot)$.
- We use the notation Δ_E to denote the Laplacian acting on functions defined on a linear subspace $E \subset \mathbb{R}^d$.

Quantifying the lack of dispersion : degenerate case

- When the non-degeneracy of the critical points is replaced by H2, we obtain a similar result which requires some geometric preliminaries.
- Define the tangent bundle of Λ as the union of all tangent spaces to Λ,

$$T\Lambda := \{(x, \xi) \in \mathbb{R}^d \times \Lambda \, : \, x \in T_\xi \Lambda\}.$$

- The normal bundle of Λ is the union of linear subspaces normal to Λ:

$$N\Lambda := \{(y, \xi) \in \mathbb{R}^d \times \Lambda \, : \, y \in N_\xi \Lambda = (T_\xi \Lambda)^\perp\}.$$

  Every point $x \in \mathbb{R}^d$ can be uniquely written as $x = z + y$, where $z \in T_\xi \Lambda$ and $y \in N_\xi \Lambda$.
- Given a function $\phi \in L^\infty(\mathbb{R}^d)$, we write $m_\phi(z, \xi)$, where $z \in T_\xi \Lambda$, to denote the operator acting on $L^2(N_\xi \Lambda)$ by multiplication by $\phi(z + \cdot)$.
- We use the notation Δ_E to denote the Laplacian acting on functions defined on a linear subspace $E \subset \mathbb{R}^d$.

Quantifying the lack of dispersion : degenerate case

- When the non-degeneracy of the critical points is replaced by H2, we obtain a similar result which requires some geometric preliminaries.
- Define the tangent bundle of Λ as the union of all tangent spaces to Λ,

$$T\Lambda := \{(x,\xi) \in \mathbb{R}^d \times \Lambda \,:\, x \in T_\xi \Lambda\}.$$

- The normal bundle of Λ is the union of linear subspaces normal to Λ:

$$N\Lambda := \{(y,\xi) \in \mathbb{R}^d \times \Lambda \,:\, y \in N_\xi \Lambda = (T_\xi \Lambda)^\perp\}.$$

  Every point $x \in \mathbb{R}^d$ can be uniquely written as $x = z + y$, where $z \in T_\xi \Lambda$ and $y \in N_\xi \Lambda$.
- Given a function $\phi \in L^\infty(\mathbb{R}^d)$, we write $m_\phi(z,\xi)$, where $z \in T_\xi \Lambda$, to denote the operator acting on $L^2(N_\xi \Lambda)$ by multiplication by $\phi(z + \cdot)$.
- We use the notation Δ_E to denote the Laplacian acting on functions defined on a linear subspace $E \subset \mathbb{R}^d$.

Quantifying the lack of dispersion : degenerate case

Theorem

Assume H0, H1 & H2. Then there exist a subsequence $(u_0^{\varepsilon_k})$*, a positive measure* $\gamma \in \mathcal{M}_+(T\Lambda)$ *and a measurable fami. of s.-adj., positive, trace-class operators*

$$M_0 : T_\xi\Lambda \ni (z,\xi) \longmapsto M_0(z,\xi) \in \mathcal{L}^1_+(L^2(N_\xi\Lambda)), \quad \mathrm{Tr}_{L^2(N_\xi\Lambda)} M_0(z,\xi) = 1,$$

such that for every $a < b$ *and every* $\phi \in \mathcal{C}_c(\mathbb{R}^d)$ *one has:*

$$\lim_{k\to\infty} \int_a^b \int_{\mathbb{R}^d} \phi(x)|u^{\varepsilon_k}(t,x)|^2 dxdt = \int_a^b \int_{T\Lambda} \mathrm{Tr}_{L^2(N_\xi\Lambda)} \left[m_\phi(z,\xi)M(t,z,\xi)\right] \gamma(dz,d\xi)dt,$$

where $M(\cdot,z,\xi) \in \mathcal{C}(\mathbb{R};\mathcal{L}^1_+(L^2(N_\xi\Lambda))$ *solves the following Heisenberg equation:*

$$i\partial_t M(t,z,\xi) + \left[\frac{1}{2}\Delta_{N_\xi\Lambda} + m_V(z,\xi), M(t,z,\xi)\right] = 0, \quad M|_{t=0} = M_0.$$

Quantifying the lack of dispersions : comments

- The measure γ and the family of operators $M_0(z,\xi)$, for $z \in T_\xi\Lambda$, only depend on the subsequence of initial data $(u_0^{\varepsilon_k})$.
- When Λ is a set of isolated critical points, both Theorems are equivalent : $T\Lambda = \{0\} \times \Lambda$ and

$$\gamma = \sum_{\xi \in \Lambda} \gamma_\xi \delta_\xi, \quad \text{where} \quad \gamma_\xi = ||u_\xi|_{t=0}||^2_{L^2(\mathbb{R}^d)}.$$

In addition, $N_\xi\Lambda = \mathbb{R}^d$ and $M(t,\xi)$ is the orth. proj. onto $u_\xi(t,\cdot)$.

- A consequence of this Theorem is that the weak-$\star$ limit of the densities $|u^{\varepsilon_k}|^2$ is absolutely continuous with respect to the Lebesgue measure $dxdt$ and can be expressed as a superposition of position densities associated to solutions to the family of p-dimensional Schrödinger evolutions:

$$i\partial_t v_{z,\xi}(t,y) + \frac{1}{2}\Delta_y v_{z,\xi}(t,y) + V(z+y)v_{z,\xi}(t,y) = 0, \quad (t,y) \in \mathbb{R} \times N_\xi\Lambda.$$

Quantifying the lack of dispersions : comments

- The measure γ and the family of operators $M_0(z,\xi)$, for $z \in T_\xi\Lambda$, only depend on the subsequence of initial data $(u_0^{\varepsilon_k})$.
- When Λ is a set of isolated critical points, both Theorems are equivalent : $T\Lambda = \{0\} \times \Lambda$ and

$$\gamma = \sum_{\xi\in\Lambda} \gamma_\xi \delta_\xi, \quad \text{where} \quad \gamma_\xi = ||u_\xi|_{t=0}||^2_{L^2(\mathbb{R}^d)}.$$

In addition, $N_\xi\Lambda = \mathbb{R}^d$ and $M(t,\xi)$ is the orth. proj. onto $u_\xi(t,\cdot)$.

- A consequence of this Theorem is that the weak-$\star$ limit of the densities $|u^{\varepsilon_k}|^2$ is absolutely continuous with respect to the Lebesgue measure $dxdt$ and can be expressed as a superposition of position densities associated to solutions to the family of p-dimensional Schrödinger evolutions:

$$i\partial_t v_{z,\xi}(t,y) + \frac{1}{2}\Delta_y v_{z,\xi}(t,y) + V(z+y)v_{z,\xi}(t,y) = 0, \quad (t,y) \in \mathbb{R} \times N_\xi\Lambda.$$

Quantifying the lack of dispersions : comments

- The measure γ and the family of operators $M_0(z,\xi)$, for $z \in T_\xi \Lambda$, only depend on the subsequence of initial data $(u_0^{\varepsilon_k})$.
- When Λ is a set of isolated critical points, both Theorems are equivalent : $T\Lambda = \{0\} \times \Lambda$ and

$$\gamma = \sum_{\xi \in \Lambda} \gamma_\xi \delta_\xi, \quad \text{where} \quad \gamma_\xi = ||u_\xi|_{t=0}||^2_{L^2(\mathbb{R}^d)}.$$

In addition, $N_\xi \Lambda = \mathbb{R}^d$ and $M(t,\xi)$ is the orth. proj. onto $u_\xi(t,\cdot)$.

- A consequence of this Theorem is that the weak-$\star$ limit of the densities $|u^{\varepsilon_k}|^2$ is absolutely continuous with respect to the Lebesgue measure $dxdt$ and can be expressed as a superposition of position densities associated to solutions to the family of p-dimensional Schrödinger evolutions:

$$i\partial_t v_{z,\xi}(t,y) + \frac{1}{2}\Delta_y v_{z,\xi}(t,y) + V(z+y) v_{z,\xi}(t,y) = 0, \quad (t,y) \in \mathbb{R} \times N_\xi \Lambda.$$

Strategy of the proof : phase space analysis

- Phase space analysis: Let $W(u^\varepsilon)$ be the Wigner transform of (u^ε),

$$W^\varepsilon(t,x,\xi) = (2\pi)^{-d}\int_{\mathbb{R}^d} \overline{u}^\varepsilon\left(t, x+\varepsilon\frac{v}{2}\right) u^\varepsilon\left(t, x-\varepsilon\frac{v}{2}\right) \mathrm{e}^{iv\cdot\xi} dv.$$

The Wigner transform plays the role of a generalised energy density since

$$|u^\varepsilon(t,x)|^2 = \int_{\mathbb{R}^d} W^\varepsilon(t,x,\xi)d\xi.$$

- Wigner measures of (u^ε) are positive measures $\mu(t)$ satisfying for some subsequence ε_k and for all $a<b$, $c\in\mathcal{C}_0^\infty(\mathbb{R}^{2d})$,

$$\lim_{k\to\infty}\int_a^b\int_{\mathbb{R}^{2d}} c(x,\xi)W^{\varepsilon_k}(t,x,\xi)dxd\xi dt = \int_a^b\int_{\mathbb{R}^{2d}} c(x,\xi)\mu(t,dx,d\xi)dt.$$

- Besides, ε-oscillation $\Longrightarrow$

$$\lim_{k\to\infty}\int_a^b\int_{\mathbb{R}^d}\phi(x)|u^{\varepsilon_k}(t,x)|^2dxdt = \int_a^b\int_{\mathbb{R}^{2d}}\phi(x)\mu(t,dx,d\xi)dt.$$

Strategy of the proof : phase space analysis

- Phase space analysis: Let $W(u^\varepsilon)$ be the Wigner transform of (u^ε),

$$W^\varepsilon(t,x,\xi) = (2\pi)^{-d} \int_{\mathbb{R}^d} \overline{u}^\varepsilon\left(t, x+\varepsilon\frac{v}{2}\right) u^\varepsilon\left(t, x-\varepsilon\frac{v}{2}\right) \mathrm{e}^{iv\cdot\xi} dv.$$

The Wigner transform plays the role of a generalised energy density since

$$|u^\varepsilon(t,x)|^2 = \int_{\mathbb{R}^d} W^\varepsilon(t,x,\xi)d\xi.$$

- Wigner measures of (u^ε) are positive measures $\mu(t)$ satisfying for some subsequence ε_k and for all $a<b$, $c \in \mathcal{C}_0^\infty(\mathbb{R}^{2d})$,

$$\lim_{k\to\infty} \int_a^b \int_{\mathbb{R}^{2d}} c(x,\xi) W^{\varepsilon_k}(t,x,\xi) dx d\xi dt = \int_a^b \int_{\mathbb{R}^{2d}} c(x,\xi)\mu(t,dx,d\xi)dt.$$

- Besides, ε-oscillation $\Longrightarrow$

$$\lim_{k\to\infty} \int_a^b \int_{\mathbb{R}^d} \phi(x)|u^{\varepsilon_k}(t,x)|^2 dx dt = \int_a^b \int_{\mathbb{R}^{2d}} \phi(x)\mu(t,dx,d\xi)dt.$$

Strategy of the proof : phase space analysis

- Phase space analysis: Let $W(u^\varepsilon)$ be the Wigner transform of (u^ε),

$$W^\varepsilon(t,x,\xi) = (2\pi)^{-d} \int_{\mathbb{R}^d} \overline{u}^\varepsilon \left(t, x + \varepsilon \frac{v}{2}\right) u^\varepsilon \left(t, x - \varepsilon \frac{v}{2}\right) \mathrm{e}^{iv\cdot\xi} dv.$$

  The Wigner transform plays the role of a generalised energy density since

$$|u^\varepsilon(t,x)|^2 = \int_{\mathbb{R}^d} W^\varepsilon(t,x,\xi) d\xi.$$

- Wigner measures of (u^ε) are positive measures $\mu(t)$ satisfying for some subsequence ε_k and for all $a < b$, $c \in \mathcal{C}_0^\infty(\mathbb{R}^{2d})$,

$$\lim_{k\to\infty} \int_a^b \int_{\mathbb{R}^{2d}} c(x,\xi) W^{\varepsilon_k}(t,x,\xi) dx d\xi dt = \int_a^b \int_{\mathbb{R}^{2d}} c(x,\xi) \mu(t,dx,d\xi) dt.$$

- Besides, ε-oscillation $\Longrightarrow$

$$\lim_{k\to\infty} \int_a^b \int_{\mathbb{R}^d} \phi(x) |u^{\varepsilon_k}(t,x)|^2 dx dt = \int_a^b \int_{\mathbb{R}^{2d}} \phi(x) \mu(t,dx,d\xi) dt.$$

Strategy of the proof : phase space analysis

- Phase space analysis: Let $W(u^\varepsilon)$ be the Wigner transform of (u^ε),
$$W^\varepsilon(t,x,\xi) = (2\pi)^{-d}\int_{\mathbb{R}^d} \overline{u}^\varepsilon\left(t, x+\varepsilon\frac{v}{2}\right) u^\varepsilon\left(t,x-\varepsilon\frac{v}{2}\right) \mathrm{e}^{iv\cdot\xi}dv.$$
The Wigner transform plays the role of a generalised energy density since
$$|u^\varepsilon(t,x)|^2 = \int_{\mathbb{R}^d} W^\varepsilon(t,x,\xi)d\xi.$$

- Wigner measures of (u^ε) are positive measures $\mu(t)$ satisfying for some subsequence ε_k and for all $a<b$, $c\in\mathcal{C}_0^\infty(\mathbb{R}^{2d})$,
$$\lim_{k\to\infty}\int_a^b\int_{\mathbb{R}^{2d}} c(x,\xi)W^{\varepsilon_k}(t,x,\xi)dxd\xi dt = \int_a^b\int_{\mathbb{R}^{2d}} c(x,\xi)\mu(t,dx,d\xi)dt.$$

- Besides, ε-oscillation $\Longrightarrow$
$$\lim_{k\to\infty}\int_a^b\int_{\mathbb{R}^d}\phi(x)|u^{\varepsilon_k}(t,x)|^2dxdt = \int_a^b\int_{\mathbb{R}^{2d}}\phi(x)\mu(t,dx,d\xi)dt.$$

Set for $\chi \in \mathcal{C}_0(\mathbb{R})$ and $c \in \mathcal{C}_0^\infty(\mathbb{R}^{2d})$,

$$I^\varepsilon(\chi, c) = \int_{\mathbb{R}} \int_{\mathbb{R}^{2d}} \chi(t) c(x,\xi) W^{\varepsilon_k}(t,x,\xi) dx d\xi dt.$$

- Invariance of Wigner measure : Egorov's theorem $\Longrightarrow$

Proposition

Any μ_t is invariant by the flow $\phi^1_s : s \mapsto (x + s\nabla\lambda(\xi), \xi)$.

- Localization of Wigner measures

Corollary

$$\mathrm{Supp}(\mu_t) \subset \{(x,\xi) \in \mathbb{R}^{2d}, \ \nabla\lambda(\xi) = 0\}.$$

Set for $\chi \in \mathcal{C}_0(\mathbb{R})$ and $c \in \mathcal{C}_0^\infty(\mathbb{R}^{2d})$,

$$I^\varepsilon(\chi, c) = \int_{\mathbb{R}} \int_{\mathbb{R}^{2d}} \chi(t) c(x,\xi) W^{\varepsilon_k}(t,x,\xi) dx d\xi dt.$$

- Invariance of Wigner measure : Egorov's theorem $\Longrightarrow$

Proposition

Any μ_t is invariant by the flow $\phi_s^1 : s \mapsto (x + s\nabla\lambda(\xi), \xi)$.

- Localization of Wigner measures

Corollary

$$\mathrm{Supp}(\mu_t) \subset \{(x,\xi) \in \mathbb{R}^{2d}, \ \nabla\lambda(\xi) = 0\}.$$

Strategy of the proof : localisation of Wigner measures

Set for $\chi \in \mathcal{C}_0(\mathbb{R})$ and $c \in \mathcal{C}_0^\infty(\mathbb{R}^{2d})$,

$$I^\varepsilon(\chi, c) = \int_{\mathbb{R}} \int_{\mathbb{R}^{2d}} \chi(t) c(x,\xi) W^{\varepsilon_k}(t,x,\xi) dx d\xi dt.$$

- Invariance of Wigner measure : Egorov's theorem $\Longrightarrow$

Proposition

Any μ_t is invariant by the flow $\phi^1_s : s \mapsto (x + s\nabla\lambda(\xi), \xi)$.

- Localization of Wigner measures

Corollary

$$\mathrm{Supp}(\mu_t) \subset \{(x,\xi) \in \mathbb{R}^{2d}, \;\; \nabla\lambda(\xi) = 0\}.$$

Strategy of the proof : Two scale observables

We add to the phase space $\mathbb{R}^{2d}$ a new variable $\eta \in \overline{\mathbb{R}^d}$.

[CK], [Nier], [Miller], [FFK &Gérard], [Laser & Teufel], [Harris, Lukkarinen, Teufel& Theil], [Macia], [Anantharaman & Macia]

With $c = c(x, \xi, \eta) \in \mathcal{C}^\infty(\mathbb{R}^{3d})$ satisfying additional properties, which satisfy :

1. there exists a compact K such that for all $\eta \in \mathbb{R}^d$, $(x, \xi) \mapsto c(x, \xi, \eta)$ is a smooth function compactly supported in K;
2. there exists a function $c_\infty(x, \xi, \omega)$ defined on $R^{2d} \times \mathbf{S}^{d-1}$ and $R_0 > 0$ such that if $|\eta| > R_0$, then $c(x, \xi, \eta) = c_\infty(x, \xi, \eta/|\eta|)$.

Assume $\Lambda = \xi_0 + 2\pi\mathbb{Z}^d$. We associate with such c, the two-scale observable

$$c_\varepsilon^\sharp(x, \xi) = c\left(x, \xi, \frac{\xi - \xi_0}{\varepsilon}\right).$$

Remarks : 1) If $c \in \mathcal{C}_0^\infty(\mathbb{R}^{2d})$, c is admissible.
2) Wigner transform acts on two-scale observables.

Strategy of the proof : Two scale observables

We add to the phase space $\mathbb{R}^{2d}$ a new variable $\eta \in \overline{\mathbb{R}^d}$.

[CK], [Nier], [Miller], [FFK &Gérard], [Laser & Teufel], [Harris, Lukkarinen, Teufel& Theil], [Macia], [Anantharaman & Macia]

With $c = c(x, \xi, \eta) \in \mathcal{C}^\infty(\mathbb{R}^{3d})$ satisfying additional properties, which satisfy :

1. there exists a compact K such that for all $\eta \in \mathbb{R}^d$, $(x, \xi) \mapsto c(x, \xi, \eta)$ is a smooth function compactly supported in K;
2. there exists a function $c_\infty(x, \xi, \omega)$ defined on $R^{2d} \times \mathbf{S}^{d-1}$ and $R_0 > 0$ such that if $|\eta| > R_0$, then $c(x, \xi, \eta) = c_\infty(x, \xi, \eta/|\eta|)$.

Assume $\Lambda = \xi_0 + 2\pi\mathbb{Z}^d$. We associate with such c, the two-scale observable

$$c_\varepsilon^\sharp(x, \xi) = c\left(x, \xi, \frac{\xi - \xi_0}{\varepsilon}\right).$$

Remarks : 1) If $c \in \mathcal{C}_0^\infty(\mathbb{R}^{2d})$, c is admissible.

2) Wigner transform acts on two-scale observables.

We add to the phase space $\mathbb{R}^{2d}$ a new variable $\eta \in \overline{\mathbb{R}^d}$.

[CK], [Nier], [Miller], [FFK &Gérard], [Laser & Teufel], [Harris, Lukkarinen, Teufel& Theil], [Macia], [Anantharaman & Macia]

With $c = c(x, \xi, \eta) \in \mathcal{C}^\infty(\mathbb{R}^{3d})$ satisfying additional properties, which satisfy :

1. there exists a compact K such that for all $\eta \in \mathbb{R}^d$, $(x, \xi) \mapsto c(x, \xi, \eta)$ is a smooth function compactly supported in K;
2. there exists a function $c_\infty(x, \xi, \omega)$ defined on $R^{2d} \times \mathbf{S}^{d-1}$ and $R_0 > 0$ such that if $|\eta| > R_0$, then $c(x, \xi, \eta) = c_\infty(x, \xi, \eta/|\eta|)$.

Assume $\Lambda = \xi_0 + 2\pi\mathbb{Z}^d$. We associate with such c, the two-scale observable

$$c_\varepsilon^\sharp(x, \xi) = c\left(x, \xi, \frac{\xi - \xi_0}{\varepsilon}\right).$$

Remarks : 1) If $c \in \mathcal{C}_0^\infty(\mathbb{R}^{2d})$, c is admissible.

2) Wigner transform acts on two-scale observables.

Strategy of the proof : Two scale observables

We add to the phase space $\mathbb{R}^{2d}$ a new variable $\eta \in \overline{\mathbb{R}^d}$.

[CK], [Nier], [Miller], [FFK &Gérard], [Laser & Teufel], [Harris, Lukkarinen, Teufel& Theil], [Macia], [Anantharaman & Macia]

With $c = c(x, \xi, \eta) \in \mathcal{C}^\infty(\mathbb{R}^{3d})$ satisfying additional properties, which satisfy :

1. there exists a compact K such that for all $\eta \in \mathbb{R}^d$, $(x, \xi) \mapsto c(x, \xi, \eta)$ is a smooth function compactly supported in K;
2. there exists a function $c_\infty(x, \xi, \omega)$ defined on $R^{2d} \times \mathbf{S}^{d-1}$ and $R_0 > 0$ such that if $|\eta| > R_0$, then $c(x, \xi, \eta) = c_\infty(x, \xi, \eta/|\eta|)$.

Assume $\Lambda = \xi_0 + 2\pi\mathbb{Z}^d$. We associate with such c, the two-scale observable

$$c^\sharp_\varepsilon(x, \xi) = c\left(x, \xi, \frac{\xi - \xi_0}{\varepsilon}\right).$$

Remarks : 1) If $c \in \mathcal{C}_0^\infty(\mathbb{R}^{2d})$, c is admissible.
2) Wigner transform acts on two-scale observables.

Strategy of the proof : Two scale Wigner measures

Theorem

There exist, $\varepsilon_n \underset{n\to+\infty}{\longrightarrow} 0$, $\nu \in L^\infty(\mathbb{R}, \mathcal{M}^+(\mathbb{R}^d \times \mathbf{S}^{d-1}))$, $\Phi \in \mathcal{C}^0(\mathbb{R}, L^2(\mathbb{R}^d))$ *such that*

$$I^{\varepsilon_n}(\chi, c^\sharp_{\varepsilon_n}) \underset{n\to+\infty}{\longrightarrow} \int_{\mathbb{R}} \chi(t)\left(a(x,\xi_0,D)\Phi(t),\Phi(t)\right) dt + \int_{\mathbb{R}} \chi(t)\langle a_\infty(\cdot,\xi_0,\cdot),\nu_t\rangle dt.$$

1. Φ *solves the effective mass equation*

$$i\partial_t\Phi = \mathrm{Hess}\,\lambda(\xi_0)D\cdot D\,\Phi + V_{ext}(x)\Phi, \quad \Phi(0) = \Phi_0,$$

where Φ_0 *is a weak limit in* $L^2(\mathbb{R}^d)$ *of the sequence* $x \mapsto \mathrm{e}^{\frac{i}{\varepsilon}\xi_0\cdot x}u_0^\varepsilon(x)$.

2. ν^t *is invariant by the flow* $\phi^2_s : (x,\omega) \mapsto (x + s\mathrm{Hess}\,\lambda(\xi_0)\omega, \omega)$.

Corollary

If $\mathrm{Hess}\,\lambda(\xi_0)$ *is non degenerated, then* $\nu_t = 0$ *and* $\mu_t(x,\xi)\mathbf{1}_{\xi=\xi_0} = |\Phi(t,x)|^2dx$.

Strategy of the proof : Two scale Wigner measures

Theorem

There exist, $\varepsilon_n \underset{n\to+\infty}{\longrightarrow} 0$, $\nu \in L^\infty(\mathbb{R}, \mathcal{M}^+(\mathbb{R}^d \times \mathbf{S}^{d-1}))$, $\Phi \in \mathcal{C}^0(\mathbb{R}, L^2(\mathbb{R}^d))$ *such that*

$$I^{\varepsilon_n}(\chi, c^\sharp_{\varepsilon_n}) \underset{n\to+\infty}{\longrightarrow} \int_{\mathbb{R}} \chi(t)\left(a(x,\xi_0,D)\Phi(t), \Phi(t)\right) dt + \int_{\mathbb{R}} \chi(t)\langle a_\infty(\cdot,\xi_0,\cdot), \nu_t\rangle dt.$$

1. Φ *solves the effective mass equation*

$$i\partial_t\Phi = \mathrm{Hess}\,\lambda(\xi_0)D\cdot D\,\Phi + V_{ext}(x)\Phi, \quad \Phi(0) = \Phi_0,$$

where Φ_0 *is a weak limit in* $L^2(\mathbb{R}^d)$ *of the sequence* $x \mapsto \mathrm{e}^{\frac{i}{\varepsilon}\xi_0\cdot x} u_0^\varepsilon(x)$.

2. ν^t *is invariant by the flow* $\phi^2_s : (x,\omega) \mapsto (x + s\mathrm{Hess}\,\lambda(\xi_0)\omega, \omega)$.

Corollary

If $\mathrm{Hess}\,\lambda(\xi_0)$ *is non degenerated, then* $\nu_t = 0$ *and* $\mu_t(x,\xi)\mathbf{1}_{\xi=\xi_0} = |\Phi(t,x)|^2 dx$.

Back to effective mass theory : assumptions on the initial data

Let $I \subset \mathbb{N}$, a set of indices n such that the multiplicity of the Bloch energy $\lambda_n(\xi)$ is constant for every $\xi \in \mathbb{R}^d$

- Assume that H2 holds for any λ_n, $n \in I$
- Assume that ψ_0^ε is ε-oscillating and

$$\psi_0^\varepsilon = \sum_{n \in I} \psi_{n,0}^\varepsilon, \quad \psi_{n,0}^\varepsilon = U_n^\varepsilon\left(0, x, \frac{x}{\varepsilon}\right),$$

where $\widehat{U}_n^\varepsilon(0,\xi)$ is in the eigenspace of $\lambda_n(\xi)$.

Back to effective mass theory : application of the Theorem

Then, if (ψ^ε) is the solution to the Schrödinger equation issued for data (ψ_0^ε),

- For every $\phi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$, the family $(\phi\psi^\varepsilon(t))$ is ε-oscillating.
- $\psi^\varepsilon(t,x) = \sum_{n\in I} \psi_n^\varepsilon(t,x)$ with $\psi_n^\varepsilon(t,x) = U_n^\varepsilon\left(t, x, \frac{x}{\varepsilon}\right)$, For each $n \in \mathbb{N}$,

$$\begin{cases} i\varepsilon^2\partial_t\psi_n^\varepsilon(t,x) = \lambda_n(\varepsilon D_x)\psi_n^\varepsilon(t,x) + \varepsilon^2 V(x)\psi_n^\varepsilon(t,x) + \varepsilon^2 f_n^\varepsilon(t,x), \\ ||f_n^\varepsilon(t,\cdot)||_{L^2(\mathbb{R}^d)} \leq C\varepsilon, \quad t \in \mathbb{R},. \end{cases}$$

- There exist a subsequence ε_k such that, for every $a < b$, $\phi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$,

$$\lim_{k\to\infty} \int_a^b \int_{\mathbb{R}^d} \phi(x)|\psi^\varepsilon(t,x)|^2 dxdt = \sum_{n\in I} \int_a^b \int_{\mathbb{R}^d} |\phi(x)|^2 \mu_t^n(dx)dt,$$

  where, for each $n \in \mathbb{N}$, the measures $\mu_t^n \in \mathcal{M}_+(\mathbb{R}^d \times \mathbb{R}^d)$ are Wigner measures of (ψ_n^ε).

Conclusion

- Second microlocalisation along Λ has led to a complete description of the mechanism for any (ε-oscillating) initial data.
- In non standard cases (when Λ is a submanifold with H2), we have introduced a generalized effective mass equation with an operator-valued macroscopic item satisfying a Heisenberg equation (instead of a function satisfying a Schrödinger equation).
- In those non standard cases, the second microlocalisation does not concern "all the variable ξ" and the remaining part is responsible of the quantum feature at macroscopic level in the derived effective mass equation which becomes a Heisenberg equation.
- The next step should consist in treating a Bloch band containing two eigenvalues presenting a conical intersection (work in progress).

Thank you for your attention !

Conclusion

- Second microlocalisation along Λ has led to a complete description of the mechanism for any (ε-oscillating) initial data.
- In non standard cases (when Λ is a submanifold with H2), we have introduced a generalized effective mass equation with an operator-valued macroscopic item satisfying a Heisenberg equation (instead of a function satisfying a Schrödinger equation).
- In those non standard cases, the second microlocalisation does not concern "all the variable ξ" and the remaining part is responsible of the quantum feature at macroscopic level in the derived effective mass equation which becomes a Heisenberg equation.
- The next step should consist in treating a Bloch band containing two eigenvalues presenting a conical intersection (work in progress).

Thank you for your attention !

Conclusion

- Second microlocalisation along Λ has led to a complete description of the mechanism for any (ε-oscillating) initial data.
- In non standard cases (when Λ is a submanifold with H2), we have introduced a generalized effective mass equation with an operator-valued macroscopic item satisfying a Heisenberg equation (instead of a function satisfying a Schrödinger equation).
- In those non standard cases, the second microlocalisation does not concern "all the variable ξ" and the remaining part is responsible of the quantum feature at macroscopic level in the derived effective mass equation which becomes a Heisenberg equation.
- The next step should consist in treating a Bloch band containing two eigenvalues presenting a conical intersection (work in progress).

Thank you for your attention !

Conclusion

- Second microlocalisation along Λ has led to a complete description of the mechanism for any (ε-oscillating) initial data.
- In non standard cases (when Λ is a submanifold with H2), we have introduced a generalized effective mass equation with an operator-valued macroscopic item satisfying a Heisenberg equation (instead of a function satisfying a Schrödinger equation).
- In those non standard cases, the second microlocalisation does not concern "all the variable ξ" and the remaining part is responsible of the quantum feature at macroscopic level in the derived effective mass equation which becomes a Heisenberg equation.
- The next step should consist in treating a Bloch band containing two eigenvalues presenting a conical intersection (work in progress).

Thank you for your attention !

Conclusion

- Second microlocalisation along Λ has led to a complete description of the mechanism for any (ε-oscillating) initial data.
- In non standard cases (when Λ is a submanifold with H2), we have introduced a generalized effective mass equation with an operator-valued macroscopic item satisfying a Heisenberg equation (instead of a function satisfying a Schrödinger equation).
- In those non standard cases, the second microlocalisation does not concern "all the variable ξ" and the remaining part is responsible of the quantum feature at macroscopic level in the derived effective mass equation which becomes a Heisenberg equation.
- The next step should consist in treating a Bloch band containing two eigenvalues presenting a conical intersection (work in progress).

Thank you for your attention !