Few results on the asymptotic of the eigenvalues for the discrete Laplacian

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Let A be a self-adjoint operator acting on a complex and separable Hilbert space. We set:

- $\sigma_{\mathrm{d}}(A) := \{\lambda \in \mathbb{R}, \lambda$ is a isolated eigenvalue of finite multiplicity$\}$.
- $\sigma_{\mathrm{ess}}(A) := \sigma(A) \setminus \sigma_{\mathrm{d}}(A)$.

The spectrum of A is purely discrete if and only if $\sigma_{\mathrm{ess}}(A) = \emptyset$ and if and only if $(A + \mathrm{i})^{-1}$ is a compact operator.

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Question : How to compute the asymptotic of eigenvalues?

Let $\mathscr{V}$ be a countable set and let $\mathscr{E} := \mathscr{V} \times \mathscr{V} \to \{0, 1\}$ be such that

$$\mathscr{E}(x, y) = \mathscr{E}(y, x), \quad \text{for all } x, y \in \mathscr{V}.$$

We say that $\mathscr{G} := (\mathscr{V}, \mathscr{E})$ is an non-oriented graph with edges $\mathscr{E}$ and vertices $\mathscr{V}$.

We say that $x, y \in \mathscr{V}$ are *neighbors* is $\mathscr{E}(x, y) = 1$. We write: $x \sim y$.

The *degree* of $x \in \mathscr{V}$ is its number of neighbors :

$$\deg(x) := |\{y \in \mathscr{E} \mid x \sim y\}|.$$

Hypothesis : $\deg(x) < \infty$, $\mathscr{E}(x, x) = 0$ for all $x \in \mathscr{V}$, and the graph $\mathscr{G}$ is connected.

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We work on $\ell^2(\mathscr{V};\mathbb{C})$, endowed with $\langle f,g\rangle = \sum_{x\in\mathscr{V}} \overline{f(x)}g(x)$.

The Laplacian is given by:

$$\Delta f(x) = \sum_{y\sim x}(f(x) - f(y)),$$

for $f \in \mathcal{C}_c(\mathscr{V}) := \{f : \mathscr{V} \to \mathbb{C}$ such that the support of f is finite$\}$.

The Laplacian is essentially self-adjoint.

(Wojciechowski '07, Jørgensen '08, Colin de Verdière-Torki Hamza-Truc '11, G.'11, G.-Schumacher '11, Milatovic '11, Keller-Lenz'12, Milatovic-Truc'11 ...)

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Proof: Let $f \in \ell^2(\mathscr{V})$ such that $\Delta^* f = -f$. We infer that for all $x \in \mathscr{V}$ that

$$(\deg(x) + 1)f(x) = \sum_{y \sim x} f(y).$$

Therefore, if $f \neq 0$, there is a sequence $(x_n)_n \in \mathscr{V}^{\mathbb{N}}$, such that

$$|f(x_n + 1)| > |f(x_n)|,$$

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a)

$$0 \le \langle f, \Delta f\rangle \le 2\langle f, \deg(\cdot) f\rangle,$$

for all $f \in \mathcal{C}_c(\mathscr{V})$.

Indeed:

$$\langle f, \Delta f\rangle = \frac{1}{2}\sum_{x\in\mathscr{V}}\sum_{y\sim x}|f(x)-f(y)|^2$$
$$\le \sum_{x\in\mathscr{V}}\sum_{y\sim x}(|f(x)|^2+|f(y)|^2) = 2\langle f, \deg(\cdot) f\rangle.$$

b)

$$\Delta \text{ bounded } \iff \deg \text{ bounded }.$$

Indeed:

$$\langle \delta_x, \Delta\delta_x\rangle = \deg(x).$$

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c) The isoperimetric constant associated to $\mathscr{G}$ (and to $\deg$) is given by:

$$\alpha(\mathscr{G}) := \inf_{W \subset \mathscr{V}, \sharp W < \infty} \frac{\langle \mathbf{1}_W, \Delta \mathbf{1}_W \rangle}{\langle \mathbf{1}_W, \deg(\cdot) \mathbf{1}_W \rangle}.$$

We have:

$$\alpha(\mathscr{G}) > 0 \iff \exists c > 0, \quad c\langle f, \deg(\cdot) f \rangle \leq \langle f, \Delta f \rangle,$$

for all $f \in \mathcal{C}_c(\mathscr{V})$.

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(Dodziuk '84, Dodziuk-Kendal '86, Keller-Lenz '09, Keller '10...)

Aim: Find $c_1, C_1, c_2, C_2 > 0$, such that

$$c_1\langle f, \deg(\cdot)f\rangle - C_1\|f\|^2 \leq \langle f, \Delta f\rangle \leq c_2\langle f, \deg(\cdot)f\rangle + C_2\|f\|^2.$$

Or equivalently show that

$$\mathcal{D}(\deg^{1/2}(\cdot)) = \mathcal{D}(\Delta^{1/2}).$$

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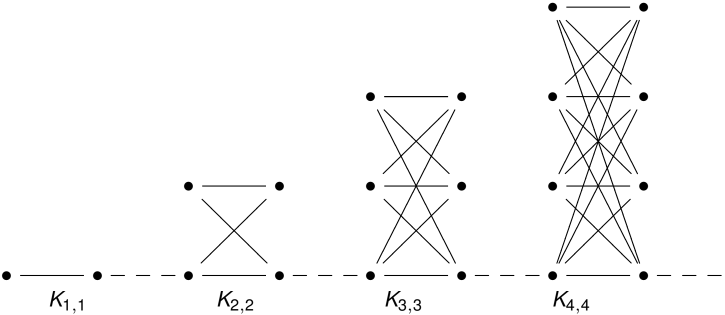
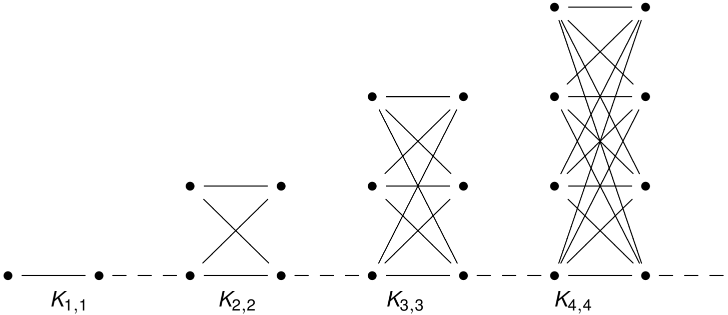


With $f_n := \mathbf{1}_{K_{n,n}}$, we have:

$$\langle f_n, \Delta f_n \rangle = 2, \quad \|f_n\|^2 = 2n, \quad \text{et} \quad \langle f_n, \deg(\cdot) f_n \rangle = 2n^2 + 2.$$

Therefore, $\mathcal{D}(\deg^{1/2}(\cdot)) \neq \mathcal{D}(\Delta^{1/2})$.



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Proposition

Let A, B be two non-negative self-adjoint operators. Suppose that

$$\mathcal{D}(A^{1/2}) \supset \mathcal{D}(B^{1/2}) \text{ and } 0 \leq \langle \psi, A\psi \rangle \leq \langle \psi, B\psi \rangle,$$

for all $\psi \in \mathcal{D}(B^{1/2})$. Then we have $\inf \sigma_{\mathrm{ess}}(A) \leq \inf \sigma_{\mathrm{ess}}(B)$, and

$$N_\lambda(A) \geq N_\lambda(B), \text{ for } \lambda \in [0, \infty) \setminus \{\inf \sigma_{\mathrm{ess}}(B)\},$$

where $N_\lambda(A) := \dim \mathrm{Ran}\, \mathbf{1}_{[0,\lambda]}(A)$.

In particular, if A and B have the same form domain, then $\sigma_{\mathrm{ess}}(A) = \emptyset$ if and only if $\sigma_{\mathrm{ess}}(B) = \emptyset$.

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We have:

$$\inf \sigma_{\text{ess}}(\Delta) \leq 2 \inf \sigma_{\text{ess}}(\deg(\cdot)) \text{ and } N_\lambda(\Delta) \geq N_\lambda(2\deg(\cdot)),$$

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In particular, if $0 \in \sigma_{\text{ess}}(\deg(\cdot))$, *then* $0 \in \sigma_{\text{ess}}(\Delta)$ *and if* Δ *with compact resolvant, then* $\deg(\cdot)$ *is also with compact resolvant.*

In other words: $\sigma_{\text{ess}}(\deg(\cdot)) \neq \emptyset$ *implies* $\sigma_{\text{ess}}(\Delta) \neq \emptyset$.

Moreover if $\alpha(\mathscr{G}) > 0$, *we also have*

$$\inf \sigma_{\text{ess}}(\Delta) \geq c \inf \sigma_{\text{ess}}(\deg(\cdot)) \text{ and } N_\lambda(\Delta) \leq N_\lambda(c\deg(\cdot)),$$

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Hardy inequality : For all $m : \mathscr{V} \to (0, \infty)$, we have:

$$\langle f, V_m(\cdot)f\rangle \leq \langle f, \Delta f\rangle, \text{ for all } f \in \mathcal{C}_c(\mathscr{V}),$$

where

$$V_m(x) := \deg(x) - W_m(x)$$

and where

$$W_m(x) := \sum_{y\sim x} \frac{m(y)}{m(x)}.$$

(Haeseler-Keller '11, Colin de Verdière-Torki Hamza-Truc '11)

The inequality is well-known in the continuous setting. It can be seen as an integrated version of Picone's identity (see for example Allegretto). It also appears in the work of Cattiaux-Guillin-Wang-Wu '09.

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Proof: Take $f \in \mathcal{C}_c(\mathscr{V})$,

$$\begin{aligned}
\langle f, V_m(\cdot) f\rangle &= \sum_x \sum_y \mathscr{E}(x,y)\left(|f|^2(x) - \frac{m(y)}{m(x)}|f|^2(x)\right) \\
&= \sum_x \sum_y \mathscr{E}(x,y)\left(|f|^2(x) - \frac{1}{2}\left(\frac{m(y)}{m(x)}|f|^2(x) + \frac{m(x)}{m(y)}|f|^2(y)\right)\right). \\
&\leq \sum_x \sum_y \mathscr{E}(x,y)\left(|f|^2(x) - \Re\left(\overline{f}(x)\sqrt{\frac{m(y)}{m(x)}}\sqrt{\frac{m(x)}{m(y)}}f(y)\right)\right) \\
&= \frac{1}{2}\sum_x \sum_y \mathscr{E}(x,y)\,|f(x) - f(y)|^2 = \langle f, \Delta f\rangle.
\end{aligned}$$

This is the announced result. □

To recover the upper-bound, it is easy if the graph is bi-partite.

We rely on the Upside-Down-Lemma (Bonnefont, G, Keller '14)

Lemma

Let $\mathscr{G} := (\mathscr{V}, \mathscr{E})$ be a graph. Assume there are $a \in (0,1)$, $k \geq 0$ such that for all $f \in \mathcal{C}_c(\mathscr{V})$,

$$(1-a)\langle f, \deg(\cdot) f\rangle - k\|f\|^2 \leq \langle f, \Delta f\rangle,$$

then for all $f \in \mathcal{C}_c(\mathscr{V})$, we also have

$$\langle f, \Delta f\rangle \leq (1+a)\langle f, \deg(\cdot) f\rangle + k\|f\|^2.$$

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Proof: For all $f \in \mathcal{C}_c(\mathscr{V})$,

$$\begin{aligned}\langle f, (2\deg(\cdot) - \Delta) f\rangle &= \frac{1}{2} \sum_{x,y \in \mathscr{V}, x \sim y} (2|f(x)|^2 + 2|f(y)|^2) - |f(x) - f(y)|^2) \\ &= \frac{1}{2} \sum_{x,y,x\sim y} |f(x) + f(y)|^2 \geq \frac{1}{2} \sum_{x,y,x\sim y} ||f(x)| - |f(y)||^2 \\ &= \langle |f|, \Delta |f|\rangle.\end{aligned}$$

Using the assumption gives:

$$\begin{aligned}\langle f, \Delta f\rangle - 2\langle f, \deg(\cdot) f\rangle &\leq -\langle |f|, \Delta |f|\rangle \\ &\leq -(1-a)\langle |f|, \deg(\cdot)|f|\rangle + k\langle |f|, |f|\rangle \\ &= -(1-a)\langle f, \deg(\cdot) f\rangle + k\langle f, f\rangle,\end{aligned}$$

which yields the assertion. □

Theorem (G, 2011)

Let $\mathscr{G}$ be a tree, then
a) for all $\varepsilon > 0$, there is $C_\varepsilon > 0$ such that:

$$(1-\varepsilon)\langle f, \deg(\cdot) f\rangle - C_\varepsilon\|f\|^2 \leq \langle f, \Delta f\rangle \leq (1+\varepsilon)\langle f, \deg(\cdot) f\rangle + C_\varepsilon\|f\|^2,$$

for all $f \in \mathcal{C}_c(\mathscr{V})$.

b) We have $\mathcal{D}(\Delta^{1/2}) = \mathcal{D}(\deg(\cdot)^{1/2})$. In particular,

$$\sigma_{\rm ess}(\Delta) = \emptyset \iff \sigma_{\rm ess}(\deg(\cdot)) = \emptyset \iff \lim_{|x|\to\infty} \deg(x) = \infty.$$

c) In the last case, we have:

$$\lambda_n(\Delta) \sim \lambda_n(\deg(\cdot)), \quad \text{for } n \to \infty,$$

where λ_n is the n-th eigenvalue (counted with multiplicity).

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for all $f \in \mathcal{C}_c(\mathscr{V})$.

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Proof: Let $\eta > 0$. Denoting by ω the origin of the tree. We set:

$$m(\omega) := 1 \text{ and } m(x) := \eta\, m(\overleftarrow{x})\deg^{-1/2}(x), \text{ for all } x \in \mathscr{V} \setminus \{\omega\}.$$

We obtain

$$\begin{aligned} \frac{V_m(x)}{\deg(x)} &= 1 - \frac{1}{\deg(x)} \left(\frac{m(\overleftarrow{x})}{m(x)} + \sum_{y \rightsquigarrow x} \frac{m(y)}{m(x)} \right) \\ &= 1 - \frac{1}{\eta} \frac{1}{\deg^{1/2}(x)} - \frac{\eta}{\deg(x)} \sum_{y \rightsquigarrow x} \deg^{-1/2}(y) \\ &\geq 1 - \eta - \frac{1}{\eta} \deg^{-1/2}(x), \end{aligned}$$

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Therefore $\mathcal{D}(\Delta^{1/2}) = \mathcal{D}(\deg^{1/2}(\cdot))$ and

$$\sigma_{\rm ess}(\Delta) = \emptyset \iff \sigma_{\rm ess}(\deg(\cdot)) = \emptyset \iff \lim_{|x|\to\infty} \deg(x) = \infty.$$

For the asymptotic of eigenvalues, we apply twice the min-max theorem to get:

$$1 - \varepsilon \leq \liminf_{N\to\infty} \frac{\lambda_N(\Delta)}{\lambda_N(\deg(\cdot))} \leq \limsup_{N\to\infty} \frac{\lambda_N(\Delta)}{\lambda_N(\deg(\cdot))} \leq 1 + \varepsilon.$$

By letting ε go to 0 we conclude. □

We say that $\mathscr{G} = (\mathscr{V}, \mathscr{E})$ is k-sparse if for all $\mathscr{W} \subset \mathscr{V}$

$$2|\mathscr{E}_{\mathscr{G}_{\mathscr{W}}}| \leq k|\mathscr{W}|,$$

where $\mathscr{G}_{\mathscr{W}}$ denotes the induced graph by $\mathscr{G}$ on $\mathscr{W}$.

Examples:

- Trees are 1-sparse.
- Planar graphs are 3-sparse.
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If $\mathscr{G}$ is k-sparse, then
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Proof: Let $f \in \mathcal{C}_c(\mathscr{V})$ be complex valued.
Assume that

$$\langle f, \deg(\cdot) f\rangle \geq k\|f\|^2.$$

Recalling

$$\deg(\mathscr{W}) = 2|\mathscr{E}_{\mathscr{W}}| + |\partial\mathscr{W}| \quad \text{and} \quad \langle \mathbf{1}_{\mathscr{W}}, \Delta \mathbf{1}_{\mathscr{W}}\rangle = |\partial\mathscr{W}|.$$

and set

$$\Omega_t := \{x \in \mathscr{V} \mid |f(x)|^2 > t\}.$$

$$
\begin{aligned}
0 \leq \langle f, \deg(\cdot) f\rangle - k\|f\|^2 &= \int_0^\infty \Big(\deg(\Omega_t) - k|\Omega_t|\Big)dt \\
&= \int_0^\infty \Big(2|\mathscr{E}_{\Omega_t}| + |\partial\Omega_t| - k|\Omega_t|\Big)dt \leq \int_0^\infty |\partial\Omega_t| dt \\
&= \frac{1}{2}\sum_{x,y,x\sim y} \Big||f(x)|^2 - |f(y)|^2\Big| \\
&\leq \frac{1}{2}\sum_{x,y,x\sim y} |(f(x) - f(y))(\overline{f(x)} + \overline{f(y)})| \\
&\leq \frac{1}{2}\left(\sum_{x,y,x\sim y} |f(x) - f(y)|^2\right)^{1/2} \\
&\qquad\qquad \times \left(\sum_{x,y,x\sim y} |f(x) + f(y)|^2\right)^{1/2} \\
&= \langle f, \Delta f\rangle^{\frac{1}{2}} \big(2\langle f, \deg(\cdot) f\rangle - \langle f, \Delta f\rangle\big)^{\frac{1}{2}},
\end{aligned}
$$

Reordering the terms, yields

$$\langle f, \Delta f\rangle^2 - 2\langle f, \deg(\cdot)f\rangle\langle f, \Delta f\rangle + (\langle f, (\deg(\cdot) - k)f\rangle)^2 \leq 0.$$

Resolving the quadratic expression above gives,

$$\langle f, \deg(\cdot)f\rangle - \sqrt{\delta} \leq \langle f, \Delta f\rangle \leq \langle f, \deg(\cdot)f\rangle + \sqrt{\delta},$$

with

$$\begin{aligned}\delta &:= \langle f, \deg(\cdot)f\rangle^2 - (\langle f, (\deg(\cdot) - k)f\rangle)^2. \\ &= k\|f\|^2\langle f, (\deg(\cdot) - k)f\rangle \\ &\leq \left(\varepsilon\langle f, \deg(\cdot)f\rangle + k\left(\frac{1}{\varepsilon} - \varepsilon\right)\|f\|^2\right)^2,\end{aligned}$$

for all $\varepsilon \in (0, 1)$. □

To go further we define :

Let $\mathscr{G} := (\mathscr{V}, \mathscr{E})$ be a graph.

For given $a \geq 0$ and $k \geq 0$, we say that $\mathscr{G}$ is (a, k)-*sparse* if for any finite set $\mathscr{W} \subseteq \mathscr{V}$ the induced subgraph $\mathscr{G}_{\mathscr{W}} := (\mathscr{W}, \mathscr{E}_{\mathscr{W}})$ satisfies

$$2|\mathscr{E}_{\mathscr{W}}| \leq k|\mathscr{W}| + a|\partial\mathscr{W}|.$$

Theorem

Let $\mathscr{G} := (\mathscr{V}, \mathscr{E})$ be a graph. The following assertions are equivalent:

(i) *There are $a, k \geq 0$ such that $(\mathscr{G}, q)$ is (a, k)-sparse.*

(ii) *There are $\tilde{a} \in (0, 1)$ and $\tilde{k} \geq 0$ such that on $\mathcal{C}_c(\mathscr{V})$*

$$(1 - \tilde{a}) \deg(\cdot) - \tilde{k} \leq \Delta \leq (1 + \tilde{a}) \deg(\cdot) + \tilde{k}.$$

(iii) *There are $\tilde{a} \in (0, 1)$ and $\tilde{k} \geq 0$ such that on $\mathcal{C}_c(\mathscr{V})$*

$$(1 - \tilde{a}) \deg(\cdot) - \tilde{k} \leq \Delta.$$

(iv) $\mathcal{D}(\Delta^{1/2}) = \mathcal{D}(\deg(\cdot)^{1/2})$.

Furthermore, Δ has purely discrete spectrum if and only if

$$\liminf_{|x| \to \infty} \deg(x) = \infty.$$

In this case, we obtain

$$1 - \tilde{a} \leq \liminf_{n \to \infty} \frac{\lambda_n(\Delta)}{\lambda_n(\deg(\cdot))} \leq \limsup_{n \to \infty} \frac{\lambda_n(\Delta)}{\lambda_n(\deg(\cdot))} \leq 1 + \tilde{a}.$$

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New aim: Find a magnetic effect which would be specific for

- the asymptotic of eigenvalues
- The form domain

Idea :

- Mimic the works of G.-Mororianu '08 and Morame-Truc '09 where one considers manifolds with cups which are "thin at infinity"

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- Mimic the works of G.-Mororianu '08 and Morame-Truc '09 where one considers manifolds with cups which are "thin at infinity"

Let $\mathscr{V}$ be a countable set and let $\mathscr{E} := \mathscr{V} \times \mathscr{V} \to [0, \infty)$ be such that

$$\mathscr{E}(x, y) = \mathscr{E}(y, x), \quad \text{for all } x, y \in \mathscr{V}.$$

Let $m : \mathscr{V} \to (0, \infty)$. We say that $\mathscr{G} := (\mathscr{V}, \mathscr{E}, m)$ is a weighted non-oriented graph with edges $\mathscr{E}$, vertices $\mathscr{V}$, and weight m.

We say that $x, y \in \mathscr{V}$ are *neighbors* is $\mathscr{E}(x, y) > 0$.

The *weighted degree* of $x \in \mathscr{V}$ is :

$$\deg(x) := \frac{1}{m(x)} \sum_{y \in \mathscr{V}} \mathscr{E}(x, y).$$

We work on $\ell^2(\mathscr{V}; \mathbb{C})$, endowed with $\langle f, g \rangle = \sum_{x \in \mathscr{V}} m(x) \overline{f(x)} g(x)$.

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Let $\theta := \mathscr{V} \times \mathscr{V} \to \mathbb{R}/2\pi\mathbb{Z}$ such that

$$\theta(x, y) = -\theta(y, x), \quad \text{for all } x, y \in \mathscr{V}.$$

The magnetic Laplacian is given by:

$$\Delta_\theta f(x) := \frac{1}{m(x)} \sum_{y \in \mathscr{V}} \mathscr{E}(x, y) e^{\mathrm{i}\theta(x,y)} (f(x) - f(y)) .$$

It is associated to the *magnetic potential* θ.

We still have:

$$0 \le \langle f, \Delta_\theta f \rangle \le 2 \langle f, \deg(\cdot) f \rangle,$$

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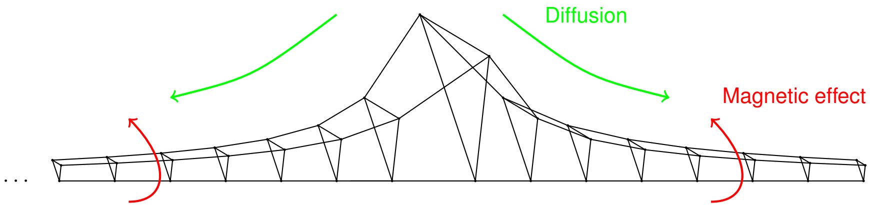

A cusp-like representation of $\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$:

The magnetic field traps the particle by spinning it,

whereas its absence lets the particle diffuse.

Theorem (G., Truc '15)

Let $a \geq 1$. There exist a graph $\mathscr{G} := (\mathscr{E}, \mathscr{V}, m)$, a magnetic potential θ, a constant $\nu > 0$ such that for all $\kappa \in \mathbb{R}$

$$\sigma_{\text{ess}}(\Delta_{\kappa\theta}) = \emptyset \Leftrightarrow \mathcal{D}\left(\Delta_{\kappa\theta}^{1/2}\right) = \mathcal{D}\left(\deg^{1/2}(\cdot)\right) \Leftrightarrow \kappa \notin \mathbb{R}/\nu\mathbb{Z}.$$

Moreover:

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Let $n \geq 3$ *be an integer. There exist a graph* $\mathscr{G} := (\mathscr{E}, \mathscr{V}, m)$, *a magnetic potential* θ, *a constant* $\nu > 0$ *such that for all* $\kappa \in \mathbb{R}$

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2) *When* $\kappa \in \mathbb{R}/\nu\mathbb{Z}$, *the absolutely continuous part of the* $\Delta_{\kappa\theta}$ *is*

$$\sigma_{\mathrm{ac}}(\Delta_{\kappa\theta}) = \left[e^{1/2} + e^{-1/2} - 2, e^{1/2} + e^{-1/2} + 2\right],$$

with multiplicity 1 *and*

$$\lim_{\lambda\to\infty} \frac{N_\lambda\left(\Delta_{\kappa\theta} P_{\mathrm{ac},\kappa}^{\perp}\right)}{N_\lambda(\deg(\cdot))} = \frac{n-1}{n},$$

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- Compared with the first point, the constant $(n-1)/n$ that appears in the second point encodes the fact that a part of the wave packet diffuses.

- Switching on the magnetic field is not a gentle perturbation because the form domain of the operator is changed.

Thank you for your attention

and have a great conference diner