On classical and quantum scattering for field equations on the (De Sitter) Kerr metric

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1.1 Black holes

$(\mathcal{M}, g)$ lorentzian manifold, $sign(g) = (+,-,-,-)$. Einstein equations (1915) :

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}.$$

- $R_{\mu\nu}$: Ricci curvature,
- R : scalar curvature,
- g : metric,
- Λ : cosmological constant,
- $T_{\mu\nu}$: energy momentum tensor,
- $\kappa = \frac{8\pi G}{c^4}$: Einstein constant.
- $T_{\mu\nu} = 0$: Einstein vacuum equations.

The Schwarzschild solution

Schwarzschild (1916). $\mathcal{M} = \mathbb{R}_t \times \mathbb{R}_{r>2M} \times S^2_\omega$

$g = Ndt^2 - N^{-1}dr^2 - r^2 d\omega^2$

$N = (1 - \frac{2M}{r})$ (M : mass of the black hole).

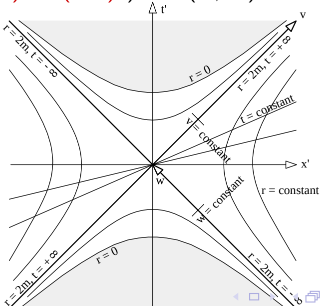
$r = 0$: curvature singularity, $r = 2M$: coordinate singularity.

Regge-Wheeler coordinate : $\frac{dx}{dr} = N^{-1}$, $x \pm t = const$. along spherically symmetric null geodescics.

$v = t + x$, $w = t - x$, $\quad g = Ndvdw - r^2 d\omega^2$.

$v' = \exp(\frac{v}{4M})$, $w' = -\exp(-\frac{w}{4M})$, $t' = \frac{v' + w'}{2}$, $x' = \frac{v' - w'}{2}$

$g = \frac{32M^2}{r}\exp(\frac{-r}{2M})((dt')^2 - (dx')^2) - r^2(t', x')d\sigma^2$.



The (De Sitter) Kerr metric

De Sitter Kerr metric in Boyer-Lindquist coordinates

$\mathcal{M}_{BH} = \mathbb{R}_t \times \mathbb{R}_r \times S^2_\omega$, with spacetime metric

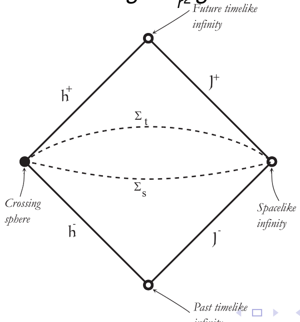
$$\begin{aligned} g &= \frac{\Delta_r - a^2 \sin^2\theta \Delta_\theta}{\lambda^2 \rho^2} dt^2 + \frac{2a \sin^2\theta((r^2+a^2)^2 \Delta_\theta - a^2 \sin^2\theta \Delta_r)}{\lambda^2\rho^2} dt d\varphi \\ &- \frac{\rho^2}{\Delta_r} dr^2 - \frac{\rho^2}{\Delta_\theta} d\theta^2 - \frac{\sin^2\theta \sigma^2}{\lambda^2 \rho^2} d\varphi^2, \\ \rho^2 &= r^2 + a^2\cos^2\theta, \quad \Delta_r = \left(1 - \frac{\Lambda}{3} r^2\right)(r^2+a^2) - 2Mr, \\ \Delta_\theta &= 1 + \frac{1}{3}\Lambda a^2 \cos^2\theta,\ \sigma^2 = (r^2+a^2)^2 \Delta_\theta - a^2 \Delta_r \sin^2\theta,\ \lambda = 1 + \frac{1}{3}\Lambda a^2. \end{aligned}$$

$\Lambda \geq 0$: cosmological constant ($\Lambda = 0$: Kerr), $M > 0$: masse, a : angular momentum per unit masse ($|a| < M$).

- $\rho^2 = 0$ is a curvature singularity, $\Delta_r = 0$ are coordinate singularities. $\Delta_r > 0$ on some open interval $r_- < r < r_+$. $r = r_-$: black hole horizon, $r = r_+$ cosmological horizon.
- ∂_φ and ∂_t are Killing. There exist $r_1(\theta)$, $r_2(\theta)$ s. t. ∂_t is
  - timelike on $\{(t, r, \theta, \varphi) :\ r_1(\theta) < r < r_2(\theta)\}$,
  - spacelike on $\{(t, r, \theta, \varphi) :\ r_- < r < r_1(\theta)\} \cup \{(t, r, \theta, \varphi :\ r_2(\theta) < r < r_+\} =: \mathcal{E}_- \cup \mathcal{E}_+$. The regions $\mathcal{E}_-$, $\mathcal{E}_+$ are called ergospheres.

The Penrose diagram ($\Lambda = 0$)

- Kerr-star coordinates :
$$t^* = t + x, r, \theta, \varphi^* = \varphi + \Lambda(r),\ \frac{dx}{dr} = \frac{r^2 + a^2}{\Delta},\ \frac{d\Lambda(r)}{dr} = \frac{a}{\Delta}.$$
.
Along incoming principal null geodesics : $\dot{t}^* = \dot{\theta} = \dot{\varphi}^* = 0,\ \dot{r} = -1.$
- Form of the metric in Kerr-star coordinates :
$g = g_{tt} dt^{*2} + 2g_{t\varphi} dt^* d\varphi^* + g_{\varphi\varphi} d\varphi^{*2} + g_{\theta\theta} d\theta^2 - 2dt^* dr + 2a\sin^2 d\varphi^* dr.$
- Future event horizon : $\mathfrak{H}^+ := \mathbb{R}_{t^*} \times \{r = r_-\} \times S^2_{\theta,\varphi^*}.$
- The construction of the past event horizon $\mathfrak{H}^-$ is based on outgoing principal null geodesics (star-Kerr coordinates). Similar constructions for future and past null infinities $\mathcal{I}^+$ and $\mathcal{I}^-$ using the conformally rescaled metric $\hat{g} = \frac{1}{r^2} g$.



1.2 The Dirac and Klein-Gordon equation on the (De Sitter) Kerr metric

The Klein-Gordon equation We now consider the unitary transform

$$U: \begin{array}{ccc} L^2(\mathcal{M}; \frac{\sigma^2}{\Delta_r\Delta_\theta} dr d\omega) & \to & L^2(\mathcal{M}; dr d\omega) \\ \psi & \mapsto & \frac{\sigma}{\sqrt{\Delta_r \Delta_\theta}}\psi \end{array}$$

If ψ fulfills $(\Box_g + m^2)\psi = 0$, then $u = U\psi$ fulfills

(1) $$(\partial_t^2 - 2ik\partial_t + h)u = 0.$$

with

$$\begin{aligned} k &= \frac{a(\Delta_r - (r^2 + a^2)\Delta_\theta)}{\sigma^2} D_\varphi, \\ h &= -\frac{(\Delta_r - a^2 \sin^2\theta\Delta_\theta)}{\sin^2\theta\sigma^2}\partial_\varphi^2 - \frac{\sqrt{\Delta_r\Delta_\theta}}{\lambda\sigma}\partial_r \Delta_r \partial_r \frac{\sqrt{\Delta_r\Delta_\theta}}{\lambda\sigma} \\ &- \frac{\sqrt{\Delta_r\Delta_\theta}}{\lambda \sin\theta\sigma}\partial_\theta \sin\theta\Delta_\theta\partial_\theta \frac{\sqrt{\Delta_r\Delta_\theta}}{\lambda\sigma} + \frac{\rho^2\Delta_r\Delta_\theta}{\lambda^2\sigma^2}m^2. \end{aligned}$$

h is not positive inside the ergospheres. This entails that the natural conserved quantity

$$\tilde{\mathcal{E}}(u) = \|\partial_t u\|^2 + (hu|u)$$

is not positive → superradiance.

Dirac equation

The situation is easier for the Dirac equation ! Weyl equation :

$$\nabla^A_{A'}\phi_A = 0.$$

Conserved current on general globally hyperbolic spacetimes

$$V^a = \phi^A \overline{\phi}^{A'}, \; C(t) = \frac{1}{\sqrt{2}} \int_{\Sigma_t} V_a T^a d\sigma_{\Sigma_t} = const.$$

T^a : normal to Σ_t, $\mathcal{M} = \bigcup_t \Sigma_t$ foliation of the spacetime.

- Newman-Penrose tetrad $l^a, n^a, m^a, \overline{m}^a$:
  $l_a l^a = n_a n^a = m_a m^a = l_a m^a = n_a m^a = 0.$
  - Normalization $l_a n^a = 1$, $m_a \overline{m}^a = -1$
  - l^a, n^a : Scattering directions.
- Spin frame $o^A \overline{o}^{A'} = l^a$, $\iota^A \overline{\iota}^{A'} = n^a$, $o^A \overline{\iota}^{A'} = m^a$
  $\iota^A \overline{o}^{A'} = \overline{m}^a$, $o_A \iota^A = 1$
- Components in the spin frame : $\phi_0 = \phi_A o^A, \phi_1 = \phi_A \iota^A$
- Weyl equation :

$$\left\{ \begin{array}{l} n^a \partial_a \phi_0 - m^a \partial_a \phi_1 + (\mu - \gamma)\phi_0 + (\tau - \beta)\phi_1 = 0, \\ l^a \partial_a \phi_1 - \overline{m}^a \partial_a \phi_0 + (\alpha - \pi)\phi_0 + (\epsilon - \tilde{\rho})\phi_1 = 0. \end{array} \right.$$

Some aspects of the study of field equations on the (De Sitter) Kerr metric

- Superradiance. Exists for entire spin equations (Klein-Gordon, Maxwell), no superradiance for half integer spin equations (Dirac, Rarita Schwinger).
- Local geometry. Trapping. Toy model for Schwarzschild

$$(\partial_t^2 + P)u = 0, \ P = -\partial_x^2 - V\Delta_{S^2}.$$

  V has a non degenerate maximum at $r = 3M$ (photon sphere). $h^{-2} = l(l+1)$ where $l(l+1)$ are the eigenvalues of $-\Delta_{S^2}$ is a good semiclassical parameter. Similar trapping in (De Sitter) Kerr. Normally hyperbolic trapping.
- Geometry at infinity. Schwarzschild. Reinterpretation of P as a perturbation of the Laplacian on a riemannian manifold with two ends :
  - $\Lambda = 0$: one asymptotically euclidean end (corresponding to infinity) and one asymptotically hyperbolic end (corresponding to the black hole horizon).
  - $\Lambda > 0$: two asymptotically hyperbolic ends.

  Consequence : the study of the low frequency behavior is easier in the De Sitter case (case of positive cosmological constant).

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2 Asymptotic completeness for the Klein-Gordon equation on the De Sitter Kerr metric (with C. Gérard and V. Georgescu)

2.1 : 3+1 decomposition, energies, Killing fields

Let $v = e^{-ikt}u$. Then u is solution of (1) if and only if v is solution of

$$(\partial_t^2 + h(t))v = 0, \quad h(t) = e^{-ikt}h_0 e^{ikt}, \quad h_0 = h + k^2 \geq 0.$$

Natural energy :

$$\|\partial_t v\|^2 + (h(t)v|v).$$

Rewriting for u :

$$\dot{\mathcal{E}}(u) = \|(\partial_t - ik)u\|^2 + (h_0 u|u).$$

This energy is positive, but may grow in time $\rightarrow$ superradiance.

Remark

$k = \Omega D_\varphi$ and Ω has finite limits $\Omega_{-/+}$ when $r \rightarrow r_\mp$. These limits are called angular velocities of the horizons. The Killing fields $\partial_t - \Omega_{-/+}\partial_\varphi$ on the De Sitter Kerr metric are timelike close to the black hole (-) resp. cosmological (+) horizon. Working with these Killing fields rather than with ∂_t leads to the conserved energies :

$$\tilde{\mathcal{E}}_{-/+}(u) = \|(\partial_t - \Omega_{-/+}\partial_\varphi)u\|^2 + (h_0 - (k - \Omega_{-/+}D_\varphi)^2 u|u).$$

Note that in the limit $k \rightarrow \Omega_{-/+}D_\varphi$ the expressions of $\dot{\mathcal{E}}(u)$ and $\tilde{\mathcal{E}}_{-/+}(u)$ coincide.

2.2 The abstract equation

$\mathcal{H}$ Hilbert space. h, k selfadjoint, $k \in \mathcal{B}(\mathcal{H})$.

(2)
$$\begin{cases} (\partial_t^2 - 2ik\partial_t + h)u &= 0, \\ u|_{t=0} &= u_0, \\ \partial_t u|_{t=0} &= u_1. \end{cases}$$

Hyperbolic equation

(A1)
$$h_0 := h + k^2 \geq 0.$$

Formally $u = e^{izt}v$ solution if and only if

$$p(z)v = 0$$

with $p(z) = h_0 - (k-z)^2 = h + z(2k - z)$, $z \in \mathbb{C}$. $p(z)$ is called the quadratic pencil.

Conserved quantities

$$\langle u|u\rangle_\ell := \|u_1 - \ell u_0\|^2 + (p(\ell)u_0|u_0),$$

where $p(\ell) = h_0 - (k-\ell)^2$. Conserved by the evolution, but in general not positive definite, because none of the operators $p(\ell)$ is in general positive.

Spaces and operators

$\mathcal{H}^i$: scale of Sobolev spaces associated to h_0.

(A2) $$0 \notin \sigma_{pp}(h_0);\ h_0^{1/2} k h_0^{-1/2} \in \mathcal{B}(\mathcal{H}).$$

Homogeneous energy spaces

$$\dot{\mathcal{E}} = \Phi(k) h_0^{-1/2} \mathcal{H} \oplus \mathcal{H}, \quad \Phi(k) = \begin{pmatrix} 1\!\!1 & 0 \\ k & 1\!\!1 \end{pmatrix}.$$

where $\dot{\mathcal{E}}$ is equipped with the norm $\|(u_0, u_1)\|^2_{\dot{\mathcal{E}}} = \|u_1 - k u_0\|^2 + (h_0 u_0 | u_0)$.

Klein Gordon operator

$$\begin{aligned} \psi &= (u, \frac{1}{i}\partial_t u), \quad (\partial_t - iH)\psi = 0, \quad H = \begin{pmatrix} 0 & 1\!\!1 \\ h & 2k \end{pmatrix}, \\ (H - z)^{-1} &= p^{-1}(z) \begin{pmatrix} z - 2k & 1\!\!1 \\ h & z \end{pmatrix}. \end{aligned}$$

We note $\dot{H}$ the Klein-Gordon operator on the homogeneous energy space.

2.3 Results in the De Sitter Kerr case

Uniform boundedness of the evolution

$$(3) \qquad \mathcal{H}^n = \{u \in L^2(\mathbb{R} \times S^2) : \ (D_\varphi - n)u = 0\}, \ n \in \mathbb{Z}.$$

We construct the homogeneous energy space $\dot{\mathcal{E}}^n$ as well as the Klein-Gordon operator $\dot{H}^n$ as in Sect. 3.2.

Theorem

There exists $a_0 > 0$ such that for $|a| < a_0$ the following holds : for all $n \in \mathbb{Z}$, there exists $C_n > 0$ such that

$$(4) \qquad \|e^{-it\dot{H}^n}u\|_{\dot{\mathcal{E}}^n} \leq C_n \|u\|_{\dot{\mathcal{E}}^n}, \ u \in \dot{\mathcal{E}}^n, \ t \in \mathbb{R}.$$

Remark

1. Note that for $n = 0$ the Hamiltonian $\dot{H}^n = \dot{H}^0$ is selfadjoint, therefore the only issue is $n \neq 0$.
2. Different from uniform boundedness on Cauchy surfaces crossing the horizon.

Asymptotic dynamics

$x \pm t = const.$ along principal null geodesics. Asymptotic equations :

$$(5) \qquad (\partial_t^2 - 2\Omega_{-/+}\partial_\varphi\partial_t + h_{-/+})u_{-/+} = 0, \quad h_{-/+} = \Omega_{-/+}^2\partial_\varphi^2 - \partial_x^2.$$

The conserved quantities :

$$\begin{aligned} & \|(\partial_t - i\Omega_{-/+}D_\varphi)u_{-/+}\|^2 + ((h_{-/+} - \Omega_{-/+}^2\partial_\varphi^2)u_{-/+}|u_{-/+}) \\ = & \quad \|(\partial_t - i\Omega_{-/+}D_\varphi)u_{-/+}\|^2 + (-\partial_x^2 u_{-/+}|u_{-/+}) \end{aligned}$$

are positive. Let $\ell_{-/+} = \Omega_{-/+}n$. Also let $i_{-/+} \in C^\infty(\mathbb{R})$, $i_- = 0$ in a neighborhood of ∞, $i_+ = 0$ in a neighborhood of $-\infty$ and $i_-^2 + i_+^2 = 1$. Let

$$h^n_{-/+} = -\partial_x^2 - \ell_{-/+}^2, \; k_{-/+} = \ell_{-/+}, \quad H^n_{-/+} = \begin{pmatrix} 0 & \mathbb{1} \\ h_{-/+} & 2k_{-/+} \end{pmatrix}$$

acting on $\mathcal{H}^n$ defined in (3).

We associate to these operators the natural homogeneous energy spaces $\dot{\mathcal{E}}^n_{-/+}$. Let $\mathcal{E}^{fin,n}_{-/+}$ be the subspace of those functions which have finite momenta with respect to $-\Delta_{S^2}$.

Theorem

There exists $a_0 > 0$ *such that for all* $|a| < a_0$ *and* $n \in \mathbb{Z} \setminus \{0\}$ *the following holds :*

- *i) For all* $u \in \mathcal{E}^{fin,n}_{-/+}$ *the limits*

$$W_{-/+}u = \lim_{t\to\infty} e^{it\dot{H}^n} i^2_{-/+} e^{-it\dot{H}^n_{-/+}} u$$

  exist in $\dot{\mathcal{E}}^n$*. The operators* $W_{-/+}$ *extend to bounded operators* $W_{-/+} \in \mathcal{B}(\dot{\mathcal{E}}^n_{-/+}; \dot{\mathcal{E}}^n)$*.*

- *ii) The inverse wave operators*

$$\Omega_{-/+} = \text{s-}\lim_{t\to\infty} e^{it\dot{H}^n_{-/+}} i^2_{-/+} e^{-it\dot{H}^n}$$

  exist in $\mathcal{B}(\dot{\mathcal{E}}^n; \dot{\mathcal{E}}^n_{-/+})$*.*

i), ii) also hold for $n = 0$ *if* $m > 0$*.*

Remark

Results uniform in n recently obtained by Dafermos, Rodnianski, Shlapentokh-Rothman for the wave equation on Kerr.

2.4 Remarks on the proof

- 1st step : $\|p^{-1}(z)u\| \lesssim |z|^{-1}|\mathrm{Im}z|^{-1}\|u\|$, uniformly in $|z| \geq (1+\epsilon)\|k\|_{\mathcal{B}(\mathcal{H})}$, $|\mathrm{Im}z| > 0$. Interpretation : superradiance does not occur for $|z| \geq (1+\epsilon)\|k\|$.
- 2nd step : gluing of asymptotic resolvents using different Killing fields. The poles of the resolvent in the upper half plane are all contained in a large ball (first step). Low frequency behavior ok because of asymptotically hyperbolic character (uses classical result of Mazzeo-Melrose).
- 3 rd step : suitable integrated resolvent estimates hold outside some discrete closed set of so called singular points, link with real resonances.
- 4th step : the conserved energy becomes positive and comparable to the energy norm for high frequencies$\rightarrow$ boundedness for high frequencies.
- 5th step No real resonances on the real line for suitable small a (perturbation argument from $a=0$, see Bony-H., Dyatlov).

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3.1 The Dirac equation and the Newman-Penrose formalism

Weyl equation :

$$\nabla^A_{A'}\phi_A = 0.$$

Conserved current :

$$V^a = \phi^A \overline{\phi}^{A'},\ C(t) = \frac{1}{\sqrt{2}} \int_{\Sigma_t} V_a T^a d\sigma_{\Sigma_t} = const.$$

T^a : normal to Σ_t.

- Newman-Penrose tetrad $l^a, n^a, m^a, \overline{m}^a$: $l_a l^a = n_a n^a = m_a m^a = l_a m^a = n_a m^a = 0$.
  - Normalization $l_a n^a = 1$, $m_a \overline{m}^a = -1$
  - l^a, n^a : Scattering directions.
- Spin frame $o^A \overline{o}^{A'} = l^a$, $\iota^A \overline{\iota}^{A'} = n^a$, $o^A \overline{\iota}^{A'} = m^a$ $\iota^A \overline{o}^{A'} = \overline{m}^a$, $o_A \iota^A = 1$
- Components in the spin frame : $\phi_0 = \phi_A o^A, \phi_1 = \phi_A \iota^A$
- Weyl equation :

$$\begin{cases} n^a \partial_a \phi_0 - m^a \partial_a \phi_1 + (\mu - \gamma)\phi_0 + (\tau - \beta)\phi_1 = 0, \\ l^a \partial_a \phi_1 - \overline{m}^a \partial_a \phi_0 + (\alpha - \pi)\phi_0 + (\epsilon - \tilde{\rho})\phi_1 = 0. \end{cases}$$

A new Newman Penrose tetrad

Problem : The Kerr metric is at infinity a long range perturbation of the Minkowski metric. In the long range situation asymptotic completeness is generically false without modification of the wave operators.

Dirac equation on Schwarzschild :

$$i\partial_t \Psi = \not{D}_S \Psi, \not{D}_S = \Gamma^1 D_x + \frac{(1-\frac{2M}{r})^{1/2}}{r} \not{D}_{S^2} + V.$$

ok because of spherical symmetry.

Tetrad adapted to the foliation : $l^a + n^a = T^a$. Conserved quantity :

$$\frac{1}{\sqrt{2}} \int_{\Sigma_t} (|\phi_0|^2 + |\phi_1|^2) d\sigma_{\Sigma_t}.$$

$l^a, n^a \in \mathrm{span}\{T^a, \partial_r\}$. Ψ spinor multiplied by a certain weight :

$$i\partial_t \Psi = \not{D}_K \Psi, \quad \not{D}_K = h \not{D}_{sym} h + V_\varphi D_\varphi + V.$$

Well adapted to time dependent scattering : $h^2 - 1,\ V_\varphi,\ V$ short range.

3.2 Principal results

Comparison dynamics

$\mathcal{H} = L^2((\mathbb{R} \times S^2); dxd\omega); \mathbb{C}^2)$, $\mathbb{D}_H = \gamma D_x - \frac{a}{r_+^2+a^2} D_\varphi, \mathbb{D}_\infty = \gamma D_x$,

$\gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\mathcal{H}^- = \{(\psi_0, 0) \in \mathcal{H}\}$ (resp. $\mathcal{H}^+ = \{(0, \psi_1) \in \mathcal{H}\}$).

Theorem (Asymptotic velocity)

There exist bounded selfadjoint operators s.t. for all $J \in \mathcal{C}_\infty(\mathbb{R})$:

$$\begin{aligned} J(P^\pm) &= s - \lim_{t\to\pm\infty} e^{-it\not{D}_K} J\left(\frac{x}{t}\right) e^{it\not{D}_K} , \\ J(\mp\gamma) &= s - \lim_{t\to\pm\infty} e^{-it\mathbb{D}_H} J\left(\frac{x}{t}\right) e^{it\mathbb{D}_H} \\ &= s - \lim_{t\to\pm\infty} e^{-it\mathbb{D}_\infty} J\left(\frac{x}{t}\right) e^{it\mathbb{D}_\infty} . \end{aligned}$$

In addition we have :

$$\sigma(P^+) = \{-1, 1\} .$$

Theorem (Asymptotic completeness)

The classical wave operators defined by the limits

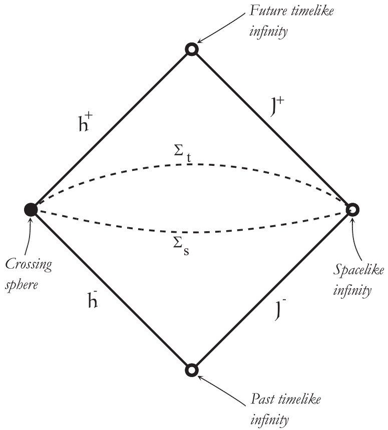
$$
\begin{aligned}
W_H^{\pm} &:= s - \lim_{t \to \pm\infty} e^{-it\not{D}_K} e^{it\mathbb{D}_H} P_{\mathcal{H}^{\mp}} , \\
W_\infty^{\pm} &:= s - \lim_{t \to \pm\infty} e^{-it\not{D}_K} e^{it\mathbb{D}_\infty} P_{\mathcal{H}^{\pm}} , \\
\Omega_H^{\pm} &:= s - \lim_{t \to \pm\infty} e^{-it\mathbb{D}_H} e^{it\not{D}_K} \mathbf{1}_{\mathbb{R}^-}(P^{\pm}) , \\
\Omega_\infty^{\pm} &:= s - \lim_{t \to \pm\infty} e^{-it\mathbb{D}_\infty} e^{it\not{D}_K} \mathbf{1}_{\mathbb{R}^+}(P^{\pm})
\end{aligned}
$$

exist.

Remark

1. Proof based on Mourre theory.
2. The same theorem holds with more geometric comparison dynamics.
3. Generalized by Daudé to the massive charged case.
4. Schwarzschild : Nicolas (95), Melnyk (02), Daudé (04).

3.3 Geometric interpretation



Penrose compactification of block I

- $\mathscr{I}^{\pm}$ are constructed using the conformally rescaled metric $\hat{g} = \frac{1}{r^2} g$.
- The Weyl equation is conformally invariant :
  $\hat{\nabla}^{AA'} \hat{\phi}_A = 0$, where $\hat{\phi}_A = r\phi_A$.

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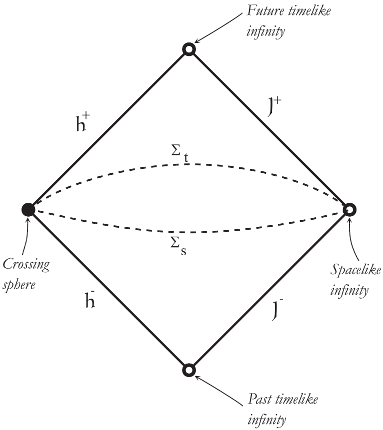

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- $\lim_{r\to r_+} \Psi_0(\gamma^-_{V,\theta,\varphi^\sharp}(r)) =: \Psi_0|_{\mathfrak{H}^+}(0, V, \theta, \varphi^\sharp)$,
  $\lim_{r\to r_+} \Psi_1(\gamma^-_{V,\theta,\varphi^\sharp}(r)) = 0$.
  Ψ is solution of the Dirac equation. $\gamma^-_{V,\theta,\varphi^\sharp}$ is the principal incoming null geodesic meeting $\mathfrak{H}^+$ at $(0, V, \theta, \varphi^\sharp)$.
- Trace operators :

$$\mathcal{T}^+_{\mathfrak{H}} \quad : \quad \begin{array}{ccc} C_0^\infty(\Sigma_0, \mathbb{C}^2) & \to & C^\infty(\mathfrak{H}^+, \mathbb{C}) \\ \Psi_{\Sigma_0} & \mapsto & \Psi_0|_{\mathfrak{H}^+}. \end{array}$$

- $\mathcal{H}$: Hilbert space associated to Σ_0, $\mathcal{H}_{\mathfrak{H}^\pm}$ Hilbert spaces associated to $\mathfrak{H}^\pm$.

Theorem

The trace operators $\mathcal{T}^\pm_{\mathfrak{H}}$ extend in a unique manner to bounded operators from $\mathcal{H}$ to $\mathcal{H}_{\mathfrak{H}^\pm}$.

Remark

Let $\mathfrak{F}^\pm_{\mathfrak{H}}$ be the $\mathcal{C}^\infty$ diffeomorphisms from $\mathfrak{H}^\pm$ onto Σ_0 defined by identifying points along incoming (resp. outgoing) principal null geodesics and $\Omega^\pm_{H,pn}$ inverse wave operators with comparison dynamics given by the principal null directions. Then $\mathcal{T}^\pm_{\mathfrak{H}} = (\mathfrak{F}^\pm_{\mathfrak{H}})^*\Omega^\pm_{H,pn}$. Comparison dynamics $P_N = \gamma D_{r_*} - \frac{a^2}{r^2+a^2} D_\varphi$.

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Same construction for $\mathcal{T}_{\mathfrak{I}}^{\pm}$ and $\mathcal{H}_{\mathfrak{I}^{\pm}}$. $\mathcal{T}_{\mathfrak{I}}^{\pm}$ can be extended to bounded operators from $\mathcal{H}$ to $\mathcal{H}_{\mathfrak{I}^{\pm}}$.

$$\Pi_F : \begin{array}{ccc} \mathcal{H} & \rightarrow & \mathcal{H}_{\mathfrak{H}^+} \oplus \mathcal{H}_{\mathfrak{I}^+} =: \mathcal{H}_F \\ \Psi_{\Sigma_0} & \mapsto & (\mathcal{T}_{\mathfrak{H}}^{+}\Psi_{\Sigma_0}, \mathcal{T}_{\mathfrak{I}}^{+}\Psi_{\Sigma_0}). \end{array}$$

Theorem (Goursat problem)

Π_F *is an isometry. In particular for all* $\Phi \in \mathcal{H}_F$*, there exists a unique solution of the Dirac equation* $\Psi \in C(\mathbb{R}_t, \mathcal{H})$ *s.t.* $\Phi = \Pi_F \Psi(0)$*.*

Remark

1) First constructions of this type : Friedlander (Minkowski, 80, 01), Bachelot (Schwarzschild, 91).
2) The inverse is possible : Mason, Nicolas (04), Joudioux (10) (asymptotically simple space-times),
Dafermos-Rodnianski-Shlapentokh-Rothman (Kerr).

The Hawking effect as a scattering problem

4.1 The collapse of the star

$$\mathcal{M}_{col} = \bigcup_t \Sigma_t^{col},\ \Sigma_t^{col} = \{(t, \hat{r}, \omega) \in \mathbb{R}_t \times \mathbb{R}_{\hat{r}} \times S^2_\omega;\ \hat{r} \geq \hat{z}(t,\theta)\}.$$

Assumptions :

- For $\hat{r} > \hat{z}(t,\theta)$, the metric is the Kerr Newman metric.
- $\hat{z}(t,\theta)$ behaves asymptotically like certain timelike geodesics in the Kerr-Newman metric. We suppose for the conserved quantities L (angular momentum), $\mathcal{Q}$ (Carter constant) and $\tilde{E}$ (rotational energy) : $L = \mathcal{Q} = \tilde{E} = 0$. We also suppose an asymptotic condition on the surface of the star :

$$\hat{z}(t,\theta) = -t - \hat{A}(\theta)e^{-2\kappa_- t} + \mathcal{O}(e^{-4\kappa_- t}),\ t \to \infty.$$

$\kappa_- > 0$ is the surface gravity of the outer horizon, $\hat{A}(\theta) > 0$.

Remark

1. $\hat{r}$ is a coordinate adapted to simple null geodesics ($t \pm \hat{r} = const.$ along these geodesics).
2. Dirac in $\mathcal{M}_{col}$: we add a boundary condition (MIT)
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4.2 Dirac quantum fields

Dimock '82.

$$\mathcal{M}_{col} = \bigcup_{t\in\mathbb{R}} \Sigma_t^{col}, \quad \Sigma_t^{col} = \{(t, \hat{r}, \theta, \varphi); \hat{r} \geq \hat{z}(t,\theta)\}.$$

Dirac quantum field Ψ_0 and the CAR-algebra $\mathcal{U}(\mathcal{H}_0)$ constructed in the usual way. Fermi-Fock representation.

$$S_{col} : \begin{array}{ccc} (C_0^\infty(\mathcal{M}_{col}))^4 & \to & \mathcal{H}_0 \\ \Phi & \mapsto & S_{col}\Phi := \int_{\mathbb{R}} U(0,t)\Phi(t)dt \end{array}$$

Quantum spin field :

$$\Psi_{col} : \begin{array}{ccc} (C_0^\infty(\mathcal{M}_{col}))^4 & \to & \mathcal{L}(\mathcal{F}(\mathcal{H}_0)) \\ \Phi & \mapsto & \Psi_{col}(\Phi) := \Psi_0(S_{col}\Phi) \end{array}$$

$\mathcal{U}_{col}(\mathcal{O})$ = algebra generated by $\Psi_{col}^*(\Phi^1)\Psi_{col}(\Phi^2)$, $\operatorname{supp}\Phi^j \subset \mathcal{O}$.

$$\mathcal{U}_{col}(\mathcal{M}_{col}) = \overline{\bigcup_{\mathcal{O}\subset\mathcal{M}_{col}} \mathcal{U}_{col}(\mathcal{O})}.$$

Same procedure on $\mathcal{M}_{BH}$:

$$S : \Phi \in (C_0^\infty(\mathcal{M}_{BH}))^4 \mapsto S\Phi := \int_{\mathbb{R}} e^{-itH}\Phi(t)dt.$$

States

- $\mathcal{U}_{col}(\mathcal{M}_{col})$
  Vacuum state :

$$\begin{aligned}\omega_{col}(\Psi^*_{col}(\Phi_1)\Psi_{col}(\Phi_2)) &:= \omega_{vac}(\Psi^*_0(S_{col}\Phi_1)\Psi_0(S_{col}\Phi_2)) \\ &= \langle \mathbf{1}_{[0,\infty)}(H_0)S_{col}\Phi_1, S_{col}\Phi_2\rangle.\end{aligned}$$

- $\mathcal{U}_{BH}(\mathcal{M}_{BH})$
  - Vacuum state

$$\omega_{vac}(\Psi^*_{BH}(\Phi_1)\Psi_{BH}(\phi_2)) = \langle \mathbf{1}_{[0,\infty)}(H)S\phi_1, S\phi_2\rangle.$$

  - Thermal Hawking state

$$\begin{aligned}\omega^{\eta,\sigma}_{Haw}(\Psi^*_{BH}(\Phi_1)\Psi_{BH}(\Phi_2)) &= \langle \mu e^{\sigma H}(1+\mu e^{\sigma H})^{-1}S\Phi_1, S\Phi_2\rangle_{\mathcal{H}} \\ &=: \omega^{\eta,\sigma}_{KMS}(\Psi^*(S\Phi_1)\Psi(S\Phi_2)), \\ T_{Haw} &= \sigma^{-1}, \ \mu = e^{\sigma\eta}, \ \sigma > 0.\end{aligned}$$

T_{Haw} Hawking temperature, μ chemical potential.

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The Hawking effect

$$\Phi \in (C_0^\infty(\mathcal{M}_{col}))^4,\ \Phi^T(t,\hat{r},\omega) = \Phi(t-T,\hat{r},\omega).$$

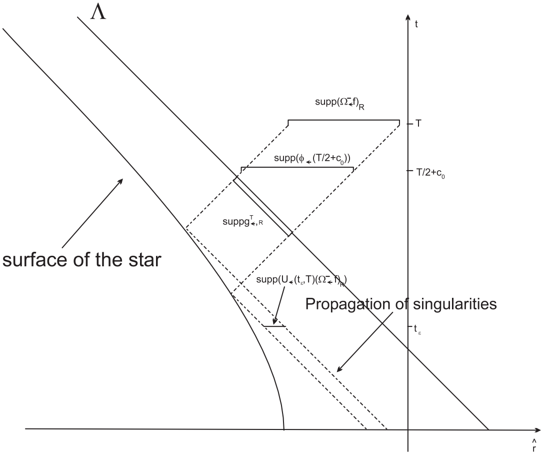
Theorem (Hawking effect)

Let $\Phi_j \in (C_0^\infty(\mathcal{M}_{col}))^4$, $j = 1,2$. *We have*

$$\begin{aligned}
&\lim_{T\to\infty} \omega_{col}(\Psi^*_{col}(\Phi_1^T)\Psi_{col}(\Phi_2^T)) \\
&= \omega_{Haw}^{\eta,\sigma}(\Psi^*_{BH}(\mathbf{1}_{\mathbb{R}^+}(P^-)\Phi_1)\Psi_{BH}(\mathbf{1}_{\mathbb{R}^+}(P^-)\Phi_2)) \\
&+ \omega_{vac}(\Psi^*_{BH}(\mathbf{1}_{\mathbb{R}^-}(P^-)\Phi_1)\Psi_{BH}(\mathbf{1}_{\mathbb{R}^-}(P^-)\Phi_2)),
\end{aligned}$$

$$T_{Haw} = 1/\sigma = \kappa_-/2\pi, \quad \mu = e^{\sigma\eta},\ \eta = \frac{qQr_-}{r_-^2+a^2} + \frac{aD_\varphi}{r_-^2+a^2}.$$

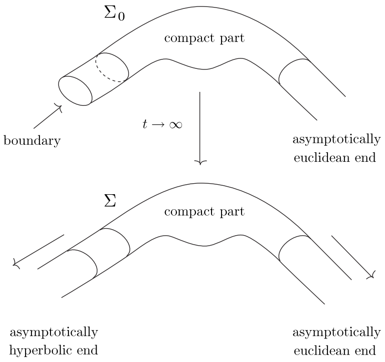
4.3 Explanation



Collapse of the star

Change in frequencies : mixing of positive and negative frequencies.

4.4 The analytic problem



$$
\begin{aligned}
&\lim_{T\to\infty} ||\mathbf{1}_{[0,\infty)}(\not{D}_0)U(0,T)f||_0^2 \\
&= \quad \langle \mathbf{1}_{\mathbb{R}^+}(P^-)f, \mu e^{\sigma\not{D}}(1+\mu e^{\sigma\not{D}})^{-1}\mathbf{1}_{\mathbb{R}^+}(P^-)f\rangle \\
&+ \quad ||\mathbf{1}_{[0,\infty)}(\not{D})\mathbf{1}_{\mathbb{R}^-}(P^-)f||^2. \qquad (6)
\end{aligned}
$$

Remark

1) **Hawking** *1975,*
2) **Bachelot** *(99), Melnyk (04).*

4.5 Toy model : The moving mirror

$$z(t) = -t - Ae^{-2\kappa t};\ A > 0,\ \kappa > 0,$$

$$\begin{cases} \partial_t \psi & = & i\not{D}\psi, \\ \psi_1(t, z(t)) & = & \sqrt{\frac{1-\dot{z}}{1+\dot{z}}}\psi_2(t, z(t)) \\ \psi(t = s, .) & = & \psi_s(.) \end{cases}, \not{D} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} D_x.$$

Solution given by a unitary propagator $U(t, s)$. Conserved L^2 norm :

$$||\psi||^2_{\mathcal{H}_t} = \int_{z(t)}^{\infty} |\psi|^2(t, x)dx.$$

Explicit calculation :

$$\begin{aligned} \lim_{T\to\infty} ||\mathbf{1}_{[0,\infty)}(\not{D}_0)U(0, T)f||_0^2 &= \langle e^{\frac{2\pi}{\kappa}\not{D}} \left(1 + e^{\frac{2\pi}{\kappa}\not{D}}\right)^{-1} P_2 f, P_2 f\rangle \\ &+ ||\mathbf{1}_{[0,\infty)}(\not{D})P_1 f||^2. \end{aligned}$$

Scattering problem : show that the real system behaves the same way.

4.6 Some remarks on the proof

- We compare to a dynamics for which the radiation can be explicitly computed.
- Can't compare dynamics on Cauchy surfaces $\rightarrow$ characteristic Cauchy problem.
- Three time intervals :
  - $[T/2 + c_0, T]$ no boundary involved $\rightarrow$ use asymptotic completeness+propagation estimates.
  - $[t_\epsilon, T/2 + c_0]$ use Duhamel formula + construction of tetrad and coordinates :
    - There exists a coordinate system $(t, \hat{r}, \omega)$ such that $\hat{r} \rightarrow -t +$ const along incoming simple null geodesics ($L = Q = 0$).
    - There exists a Newman Penrose tetrad such that: $\not{D} = \Gamma \partial_{\hat{r}} + P_\omega + W$, $\Gamma = Diag(1, -1, -1, 1)$, P_ω is a differential operator with derivatives only in the angular directions and W is a potential.
  - $[0, t_\epsilon]$: $\|\mathbf{1}_{[0,\infty]}(\not{D}_0) U(0, t_\epsilon) U_H(t_\epsilon, T) \Omega_H^- f\| \sim \|\mathbf{1}_{[0,\infty)}(\mathbb{D}_{H,0}) U_H(0, T) \Omega_H^- f\|$ if evolution is essentially given by the group (and not the evolution system). For this
    - $U_H(t_\epsilon, T) \Omega_H^- f \rightharpoonup 0$.
    - The hamiltonian flow stays outside the surface of the star for data in the given regime ($|\xi| >> |\Theta|$).
    - Propagation of singularities, compact Sobolev embeddings.

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4.6 Some remarks on the proof

- We compare to a dynamics for which the radiation can be explicitly computed.
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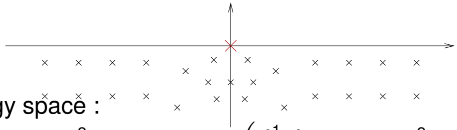
5.1 Local energy decay for the wave equation on the De Sitter Schwarzschild spacetime (a=0)

Distribution of resonances (Sa Barreto-Zworski '97) :



Modified energy space :
$\|(u_0, u_1)\|^2_{\mathcal{E}(\mathrm{mod})} = \|u_1\|^2 + \langle Pu_0, u_0 \rangle + \left(\int_0^1 \int_{\mathbb{S}^2} |u_0(s, \omega)|^2 ds\, d\omega \right)$.

Theorem (Bony-Ha '08)

Let $\chi \in C_0^\infty(\mathcal{M})$. There exists $\varepsilon > 0$ such that $\chi e^{-itH} \chi u = \gamma \begin{pmatrix} r\chi\langle r, \chi u_2 \rangle \\ 0 \end{pmatrix} + R_2(t)u, \quad \|R_2(t)u\|_{\mathcal{E}^{\mathrm{mod}}} \lesssim e^{-\varepsilon t} \left\| - \Delta_\omega u \right\|_{\mathcal{E}^{\mathrm{mod}}}$.

Remark

1. No resonance 0 *for Klein Gordon equation with positive mass of the field $m > 0$.*
2. Similar picture in much more general situations, see Vasy '13.

Consequence for asymptotic completeness

Theorem (Alexis Drouot '15)

Consider u solution in $\mathcal{M}$ of $(m > 0)$

$$(\Box + m^2)u = 0,\ u|_t = 0 = u_0,\ \partial_t u|_t = 0 = u_1$$

with u_0, u_1 in C^1. There exists C^1 functions (called radiation fields of u) $u^*_\pm : \mathcal{M} \to \mathbb{R}$ and $C \in \mathbb{R}$ (depending only on $\mathrm{supp}(u_0; u_1)$) such that

$$u^*_\pm(x, \omega) = 0 \text{ for } x \leq C; \quad u^*_\pm = \mathcal{O}_{C^\infty}(e^{-\nu_0|x|}),$$

and

$$u(t, x, \omega) = u^*_+(-(t + x), \omega) + u^*_-(-t + x, \omega) + \mathcal{O}_{C^\infty(\mathcal{M}_-)}(e^{ct}).$$

Proof uses results of Bony-H. '08 and Melrose-Sa-Barreto-Vasy '14.

Convergence rate for the Hawking effect

Theorem (Alexis Drouot '15)

There exists $\Lambda_0 > 0$ *such that for all* $\Lambda < \Lambda_0$ *the following is true. Let*

$$\mathbb{E}_T(u_0, u_1) = \mathbb{E}^{\mathbb{H}_0, T_0}(u(0), \partial_t u(0)),$$

where u solves for $m > 0$

$$\begin{cases} (\Box_g + m^2) & = & 0, \\ u|_{\mathcal{B}} & = & 0, \\ u(T) & = & u_0, \\ \partial_t u(T) & = & u_1 \end{cases}$$

Then

$$\mathbb{E}_T(u_0, u_1) = \mathbb{E}_+^{D_x^2, T_0}(u_+^*, D_x u_+^*) + \mathbb{E}_-^{D_x^2, T_{Haw}}(u_-^*, D_x u_-^*) + \mathcal{O}(e^{-cT}), \quad T \to \infty.$$

for some $c > 0$.

Comments on the Klein-Gordon case

- Scattering theory
  - The fact that the mixed term has two different limits makes it more complicated than for the Klein-Gordon equation coupled to an electric field. Mourre theory on Krein spaces : Georgescu-Gérard-H. '14.
  - Time dependent scattering should depend only on the behavior of the resolvent on the real axis.
- Hawking effect
  - Proof of a theorem about the Hawking effect for bosons should now work in principle in the same way. Temperature depends on n.
  - Highly idealized model.

Thank you for your attention !