Local Spectral Deformation

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Two-body systems

A two-body system, e.g.,

$$-\frac{\Delta_1}{2m_1} - \frac{\Delta_2}{2m_2} - \frac{1}{|x_1 - x_2|}$$

in its center of mass frame takes the form

$$-\frac{\Delta_{\mathrm{CM}}}{2M} - \frac{\Delta_{\mathrm{Rel}}}{2\mu} - \frac{1}{|x_{\mathrm{Rel}}|},$$

where $M = m_1 + m_2$ (total mass) and $\mu = m_1 m_2 / M$ (reduced mass).

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where $M = m_1 + m_2$ (total mass) and $\mu = m_1 m_2 / M$ (reduced mass). If the relative Hamiltonian is in a bound state, e.g., $\psi_0(x) = \exp(-\mu|x|)$ (3 dimensions), then the dynamics of the bound cluster $\varphi(x_{\mathrm{CM}})\psi_0(x_{\mathrm{Rel}})$ will be described by the free Hamiltonian

$$-\frac{\mu}{2} - \frac{\Delta_{\mathrm{CM}}}{2M}.$$

Dispersive Two-body Systems

If we instead consider two dispersive particles

$$\omega_1(p_1) + \omega_2(p_2) - V(x_1 - x_2),$$

one may still pass to "center of mass" coordinates:

$$\omega_1(p_{\mathrm{CM}}/2 + p_{\mathrm{Rel}}) + \omega_2(p_{\mathrm{CM}}/2 - p_{\mathrm{Rel}}) - V(x_{\mathrm{Rel}})$$

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Writing $\omega_\xi(k) = \omega_1(\xi/2 + k) + \omega_2(\xi/2 - k)$, we find the fibrated Hamiltonian

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If $\{\psi_\xi\}_{\xi\in\mathbb{R}^3}$ is a family of bound states $(\omega_\xi - V)\psi_\xi = \Sigma(\xi)\psi_\xi$, then the dynamics of the cluster $\int^{\oplus} \varphi(\xi)\psi_\xi d\xi$ is governed by the operator $\Sigma(p_{\mathrm{CM}})$.

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Semi-analytic sets are convenient because they admit Whitney Stratification into locally finitely many real analytic manifolds.

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$$(\omega_0(p) - V)u = \Delta^2 u - (-\Delta - 1)f = \Delta^2 u - (-\Delta - 1)(-\Delta + 1)u = u,$$

demonstrating that $(1, 0)$ is in the pure point part Σ_{pp} of the energy-momentum spectrum of (H, P_{CM}).

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Then there exists a strictly increasing function $(1 - \epsilon, 1] \ni \sigma \to \lambda_\sigma$ with $\lambda_1 = 1$, which is C^{n_0} but not C^{n_0+1} and real analytic in $(1 - \epsilon, 1)$, satisfying $\sigma \int_{\mathbb{R}} |\phi(k)|^2 (k^2 - \lambda_\sigma)^{-1} = -1$. Then $H_\sigma \psi_\sigma = \lambda_\sigma P_\sigma$, where $\psi_\sigma = (k^2 - \lambda_\sigma)^{-1} \phi$.

What can go wrong II

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$$H(g) = \begin{pmatrix} 0 & 0 \\ 0 & -\Delta - g^2 1[|x| \leq 1] \end{pmatrix}$$

as an operator on $\mathbb{C} \oplus L^2(\mathbb{R}^2)$.

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Also, $\lambda_g \simeq -\exp(-1/g^2)$, as $g \to 0$ (Simon 76), demonstrating that the singularity is not algebraic.

Dilation

Consider again the fiber operator $H(\xi) = \Delta^2 - \frac{3\xi^2}{2}\Delta + \frac{\xi^4}{16} - V$ with $V = u^{-1}(-\Delta - 1)f$ and $u = (-\Delta + 1)^{-1}f > 0$. Recall that 1 is an embedded eigenvalue for $H(0)$.

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Dilate $x \to exp(\theta)x$ such that

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Ignoring for now that V is not dilation analytic, we observe that pushing θ up into the upper half-plane will push the positive part of the continuous spectrum into the lower half-plane. The embedded eigenvalue for $\xi = 0$ will remain at 1 and may now use Kato's analytic perturbation theory.

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In order to realize H_θ as an operator with domain $D(H)$, we demand that $e^{i\theta A} D(H) \subset D(H)$ for $\theta \in \mathbb{R}$ (and that $\sup_{-1 \le \theta \le 1} \|He^{i\theta A}\psi\| < \infty$ for $\psi \in D(H)$).

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If one formally expands H_θ into a power series in θ one finds

$$H_\theta = H - i[H, A]\theta + (-i)^2 [[H, A], A]\theta^2 + \cdots + (-i)^n \mathrm{ad}_A^n(H)\theta^n + \cdots$$

where $\mathrm{ad}_A^n(H)$ denotes n-fold commutator of H with A.

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Engelmann-M-Rasmussen 15

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- The iterated commutators $\mathrm{ad}_A^n(H)$ exists for all n as H-bounded operators, and there exists $C > 0$ such that

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The expansion for H_θ is convergent strongly on $D(H)$ for $|\theta| < R'$ (some $R' > 0$). The graph norms of H and H_θ are equivalent.

The Mourre Estimate

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Being less ambitious, we may also only want to push the spectrum down locally near an embedded eigenvalue $\lambda \in \mathbb{R}$ of H, by imposing the weaker condition:

$$i[H,A] \geq e - C\big(1[|H-\lambda| \geq \kappa](1+|H|) + P\big),$$

where P projects onto to the associated eigenspace and $e, C, \kappa > 0$. This is a so-called *Mourre estimate*.

Clearing out the Essential Spectrum

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$$\sigma_{\mathrm{ess}}(H_\theta) \cap \left\{ z \in \mathbb{C} \,\middle|\, |\mathrm{Re} z - \lambda| < \kappa',\ \mathrm{Im} z > -\tfrac{1}{2} e \mathrm{Im}\theta/2 \right\} = \emptyset.$$

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The proof revolves around: (1) a Feshbach reduction, making use of the undilated eigenprojection P. (2) A proposition that $\bar{P} H_\theta \bar{P}$ has no spectrum in the region in question. The ingredient (2) is the key.

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That eigenfunctions are analytic vectors for A were previously established by M-Westrich 2011 by brute force.

Parameter Dependent Family

We now assume that $\xi \to H(\xi)$ is a family of self-adjoint operators with identical domains, parametrized by $\xi \in U \subset \mathbb{R}^n$ open, and $0 \in U$. Again A denotes a self-adjoint operator.

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For each $\xi \in U$, we assume that $H_\theta(\xi)$ extends analytically to the same strip S_R and satisfies the same bound $\|H_\theta(\xi)(H(\xi)+i)^{-1}\| \leq M$ for all $\theta \in \mathbb{C}$ with $|\theta| < R$.

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Finally we assume that there exists a θ_0 with $\mathrm{Im}\theta_0 > 0$ and $|\theta| < R''$, such that $\xi \to H_{\theta_0}(\xi)$ extends to an analytic family of Type (A) in a complex neighborhood of $\xi = 0$.

Suppose that $U \subset \mathbb{R}$. Using that any real eigenvalue for $H_{\theta_0}(\xi)$ is also an eigenvalue for $H(\xi)$ (and vice versa), we may invoke Kato:

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The pure point spectrum near $(\lambda, 0)$ are graphs of real analytic functions for $\xi \neq 0$ with at most algebraic singularities as $\xi \to 0$.

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For an open set W, we write $\mathcal{O}(W)$ for the ring of sets generated by sets of the form $f = 0$ and $f > 0$, where $f\colon W \to \mathbb{R}$ ranges over real analytic functions. Semi-analytic sets are locally of this form.

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Engelmann-M-Rasmussen 2015

There exists an open neighbourhood W of $(\lambda, 0)$, such that $\Sigma_{\mathrm{pp}} \cap W \in \mathcal{O}(W)$, and for ξ fixed the number of eigenvalues μ with $(\mu, \xi) \in W$ is at most n.

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There exists $s_1, s_2 > 0$ and $\widetilde{C} > 0$ such that for any multi-index α:

$$|\partial^\alpha \omega_j(k)| \leq \widetilde{C} \langle k \rangle^{s_j}, \quad \omega_j(k) \geq \frac{1}{\widetilde{C}} \langle k \rangle^{s_j} - \widetilde{C}.$$

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$$|\partial^\alpha \omega_j(k)| \leq \widetilde{C} \langle k \rangle^{s_j}, \quad \omega_j(k) \geq \frac{1}{\widetilde{C}} \langle k \rangle^{s_j} - \widetilde{C}.$$

Let $d' = d + 2$. We suppose $V \in C^{d'}(\mathbb{R}^d)$ and that there exists $a > 0$, such that for multi-indicies α wih $|\alpha| \leq d'$, we have

$$\sup_x e^{a|x|} |\partial^\alpha V(x)| < \infty.$$

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- For each ξ, $\sigma_{\rm pp}(H(\xi))\setminus\mathcal{T}(\xi)$ is a locally finite subset of $\mathbb{R}\setminus\mathcal{T}(\xi)$.

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Momentum Representation

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A first choice of conjugate operator A would be $\mathrm{Re}\nabla_k\omega_\xi \cdot i\nabla_k$, but the growth of ω may cause problems. The solution is to keep in mind that momentum is bounded so one may instead use $A_\xi = \mathrm{Re} v_\xi \cdot i\nabla_k$, where $v_\xi(k) = e^{-k^2}\nabla_k\omega_\xi(k)$.

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A conjugate operator of this type was previously employed by Nakamura 1990, also in a momentum representation.

Complex Deformation

Denote by $\gamma_\xi^t(k)$ the solution of $\dot{y} = v_\xi(y)$ with $y(0) = k$. Then

$$e^{itA_\xi}\omega_\xi(k)e^{-itA_\xi} = \omega_\xi(\gamma_\xi^t(k)).$$

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The Mourre estimate essentially follows from the computation

$$i[\omega_\xi(k), A_\xi] = e^{-k^2}|\nabla_k\omega_\xi(k)|^2.$$

Sketch of Proof of Main Theorem I

The key point is to show that for $\mathrm{Im}\theta > 0$ with $|\theta| < R''$ small enough, $\overline{P} H_\theta \overline{P}$ has no spectrum in a region of the form $(\lambda - \rho, \lambda + \rho) + i(-e\mathrm{Im}\theta/2, \infty)$.

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Suppose towards a contradiction that $\mu \in \sigma(\overline{P}H_\theta\overline{P})$ is in this region. We may wlog assume that there exists a normalized sequence $\psi_n \in D(H)$ with $o_n := \|\overline{P}(H_\theta - \mu)\overline{P}\psi_n\| \to 0$.

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The result follows from the computation

$$
\begin{aligned}
\mathrm{Im}\mu &= \mathrm{Im}\langle\overline{P}\psi_n,(\mu-H_\theta)\overline{P}\psi_n\rangle + \mathrm{Im}\langle\overline{P}\psi_n,H_\theta\overline{P}\psi_n\rangle\\
&= \mathrm{Im}\langle\overline{P}\psi_n,(\mu-H_\theta)\overline{P}\psi_n\rangle - \mathrm{Im}\theta\langle\overline{P}\psi_n,i[H,A]\overline{P}\psi_n\rangle\\
&\quad + O(R''\mathrm{Im}\theta)\\
&\le o_n - \mathrm{Im}\theta(e-cR''-C\langle\overline{P}\psi_n,1(|H-\lambda|\ge\kappa)\langle H\rangle\overline{P}\psi_n\rangle)\\
&\le o_n - \mathrm{Im}\theta(e-c'R''-c''\rho).
\end{aligned}
$$

Sketch of Proof of Main Theorem II

It remains to show that H_θ has no essential spectrum near λ. That the spectrum consists of isolated points near λ follows from the preceding result and isospectrality of the Feschbach map, which ensures that $\mu \in \sigma(H_\theta)$ if and only if $\det(F_P(\mu)) = 0$, where

$$F_P(\mu) = P(H_\theta - \mu)P - PH_\theta\overline{P}(\overline{P}H_\theta\overline{P} - \mu)^{-1}H_\theta P.$$

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In order to show that the remaining points in the spectrum are not essential spectrum, one must show that the corresponding Riesz projection have finite rank. Here on can use the Feshbach Reconstruction Formula and Cauchy's Integral Theorem to express the path integral of $(H_\theta - z)^{-1}$ as a sum of finite rank operators.