Cross-product algebras and the essential spectrum of the N-body hamiltonian

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Abstract

Determination of essential spectrum of N-body Hamiltonians with potentials that have **radial limits** at infinity on X/Y.

Extends the classical HVZ-theorem (zero at infinity on X/Y).

The proof is based on the study of algebras generated by potentials and their cross-products. Technically, the proofs rely on the theory developed by Georgescu and collab. Especially with Damak and Iftimovici on

localizations at infinity via cross-products.

These results are **joint work with Vladimir Georgescu.**

Summary (three parts)

Notation and assumptions

The hamiltonian and (simplified) assumptions*:

- X = f.d. vect sp.; $\overline{X} := X \cup S_X =$ **radial compactification** (S_X is the space of **rays** in X.)
- **Fix** a finite dimensional real vector space X ($\mathbb{R}^{3N}$);
- $H := -\Delta + V$, where $V = \sum_Y V_Y$ (finite sum).
- $V_Y : \overline{X/Y} \to \mathbb{R}$, for a subspace $Y \subset X$, **continuous.** (Also a function on X via $X \to X/Y$.)

* our functions have *radial limits* at infinity; however, we can relax some other assumptions (next slide).

Extensions

Let $T_x : L^2(X) \to L^2(X)$ be the translation by $x \in X$ and

$$h(P)f := \int_{x \in X} \hat{h}(x) T_x(f) dx$$

be the associated **convolution operator** (so P is the momentum).

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We can **relax** our assumptions as follows:

- We can replace Δ with $h(P)$, for a suitable **proper** function $h : X^* \to [0, \infty)$.
- We can include Coulomb type singularities ($\sim r^{-1}$) in each V_Y, so classical N-body interactions are covered.
- In general, $H =$ an operator **affiliated** to our algebra $\mathcal{A}$ (to be constructed) [Damak-Georgescu].

Radial limits

Let $S_X := (X \setminus \{0\})/\mathbb{R}_+^*$ and $\alpha = \mathbb{R}_+^* a \in S_X$.

If $v \in \mathcal{C}(\overline{X})$ (cont on the radial comp), then there exists

$$v(\alpha) = \lim_{r\to\infty} v(ra + x)\,, \quad \forall\, a \in X^*,\ x \in X\,.$$

Let now $v_Y \in \mathcal{C}(\overline{X/Y})$ and $\alpha = \mathbb{R}_+^* a$ (**as always**). Then

$$\lim_{r\to\infty} V_Y(x + ra) = \begin{cases} V_Y(\alpha^*) & \text{if } a \notin Y \\ V_Y(x) & \text{otherwise} \end{cases}$$

$$=: \ \rho_\alpha(V_Y)(x)$$

* the projection $X \to X/Y$ extends to $S_X \setminus S_Y \to S_{X/Y}$.

(Radial) limit operators

We interpret the last relation in terms of **strong (radial) limits.**

Let $f(Q)$ denote the operator of **multiplication** by f on $L^2(X)$.

If $V_Y \in \mathcal{C}(\overline{X/Y})$ and $L = V_Y(Q)$, then we have the following

BASIC RADIAL STRONG LIMIT PROPERTY:

$$\mathrm{s\text{-}}\lim_{r\to\infty} T_{ra} L T_{ra}^* = \begin{cases} L\,, & \text{if } a \in Y \\ V_Y(\alpha)\,, & \text{if } a \notin Y \end{cases} =: \rho_a(L) =: \rho_\alpha(L)$$

$(\alpha = \mathbb{R}_+^* a)$.

$V_L(\alpha) \in \mathbb{C}$ is **simpler** than $L = V_Y(Q)$, an operator.

Statement of main result

If the limit exists, we let

$$\rho_\alpha(L) := \operatorname{s-lim}_{a\to\alpha} T_{ra} L T_{ra}^*\,, \quad \alpha = \mathbb{R}_+^* a.$$

We have seen that the limit exists for $L = V_Y(Q)$.

Also, trivially, $\rho_\alpha(h(P)) = h(P)$, for $L = h(P)$ (conv operator).

Theorem (Georgescu-Nistor)

Let $V_Y \in \mathcal{C}(\overline{X/Y})$, $H = -\Delta + \sum_Y V_Y$, and $\alpha \in S_X$. Then

$$\rho_\alpha(H) = -\Delta + \sum_{Y \supset \alpha} V_Y + \sum_{Y \not\supset \alpha} V_Y(\alpha)\,.$$

and $\sigma_{\mathrm{ess}}(H) = \overline{\cup}_{\alpha \in S_X} \sigma(\rho_\alpha(H))$.

Comments

If all the radial limits of the V_Ys are zero (e.g. the usual N-body potentials) then the terms corresponding to $\alpha \not\subset Y$ drop out from the formula for V.

Consequently, we thus recover the Hunziker, van Winter, Zhislin (HVZ) theorem. Many other related results (Georgescu, Gerard, Helfer, Rabinovich, Roch, Simon, ... ; "limit operators").

The proof is based on the "localization at infinity" technique developed in [Damak-Georgescu, Georgescu-Iftimovici] in the context of crossed-product algebras.

One slide introduction to G-I localization at infinity

The Georgescu-Iftimovici (G-I) localization at infinity involves:

- $\mathcal{A}_0 \simeq \mathcal{C}(\Omega_0) \subset \mathcal{C}_{\mathrm{b}}^{\mathrm{u}}(X)$, on which X acts by translations.
- $H = -\Delta + V$, with $V \in \mathcal{A}_0 =$ algebra of potentials.
- Let $\tau_\omega(a) := \mathrm{s\text{-}lim}_{q \to \omega} T_q a T_q^* := \mathrm{s\text{-}lim}_{q \to \omega} \tau_q(a)$. $(\rho_\alpha!)$

Theorem (Georgescu-Iftimovici)

$H_\omega := \tau_\omega(H) = -\Delta + V_\omega$ (**localization at** $\omega \in \Omega$). *Then*

$$\sigma_{\mathrm{ess}}(H) \;=\; \overline{\cup}_{\omega \in \partial\Omega_0}\, \sigma(H_\omega)\,.$$

Proof uses: $(H + \imath)^{-1} \in$ algebra $\mathcal{A}$ generated by products $h(P)g(Q)$ and their adjoints and τ_ω define morphisms on $\mathcal{A}$ whose common kernel consists of compact op. $(\omega \in \partial\Omega_0 := \Omega_0 \smallsetminus X)$.

Idea of the proof of our result

We use **the G-I localization at infinity** for the algebra $\mathcal{A}_0 \simeq \mathcal{C}(\Omega_0) \subset \mathcal{C}_{\mathrm{b}}^{\mathrm{u}}(X)$ generated by all $V_Y \in \mathcal{C}(\overline{X/Y})$.

We then determine Ω_0 and show that **one can obtain all localizations** $\tau_\omega(a)$ **from the radial limits** $\rho_\alpha(a)$**.** More precisely:

1. $\{\rho_\alpha(a)\} \subset \{\tau_\omega(a)\}$, ("$\supset$")
2. If $\omega \in \partial\Omega_0 := \Omega_0 \smallsetminus X$, then there is $\alpha \in S_X$ s.t. ("$\subset$")

$$\sigma(\tau_\omega(a)) \subset \sigma(\rho_\alpha(a)) \,.$$

Hence, for $a \in \mathcal{A}$,

$$\sigma_{\mathrm{ess}}(a) \overset{GI}{=} \overline{\cup}_{\omega\in\partial\Omega_0}\, \sigma(\tau_\omega(a)) = \overline{\cup}_{\alpha\in S_X}\, \sigma(\rho_\alpha(a)) \,.$$

Summary of 2nd section

2nd SECTION:

1. **“Localization at infinity”** (Georgescu + col.) in **four** steps;
2. Then we discuss details on cross prod.

The localization at infinity: first two steps

First of all, by replacing H with $a := (H + \imath)^{-1}$, we may **assume that our operator is bounded** and normal (but not self-adjoint).

(The problem is thus reduced to **the determination of the essential spectrum of** $a := (H + \imath)^{-1}$.)

Let a be a bounded operator on $L^2(X)$.

Second, we use that $z \notin \sigma_{\text{ess}}(a)$ if, and only if, $a - z$ is Fredholm, if, and only if, $a - z$ is **invertible modulo the ideal of compact operators** (on $L^2(X)$), by Atkinson's theorem.

Third step of the localization at infinity

The **third** step is a little bit more specific.

We fix an algebra $\mathcal{A}_0 \subset \mathcal{C}_{\mathrm{b}}^{\mathrm{u}}(X)$ inv by X. Let $\mathcal{A}$ be the **norm closed algebra** generated by all operators of the form $h(P)V(Q)$ and their adjoints, where $V \in \mathcal{A}_0$ and $h \in \mathcal{C}_0(X^*)$ (cont and zero at infinity). In particular, $h(P) \in \mathcal{A}$.

For $V \in \mathcal{A}_0$ $\quad (\mathcal{A} = \langle h(P)\mathcal{A}_0 \rangle)$, we have

$$H + \imath = -\Delta + \imath + V = (-\Delta + \imath)\big[1 + (-\Delta + \imath)^{-1} V\big] = (-\Delta + \imath) b \,,$$

hence $b := 1 + (-\Delta + \imath)^{-1} V \in \mathcal{A}$ is L^2-inv. So $b^{-1} \in \mathcal{A}$ (!) and

$$a = (H + \imath)^{-1} \;=\; \big[1 + (-\Delta + \imath)^{-1} V\big]^{-1} (-\Delta + \imath)^{-1} \in \mathcal{A} \,.$$

Fourth and last step of the localization at infinity

The **fourth** step of the localization at infinity is the easiest to state, but also the most difficult to solve:

Given $a \in \mathcal{A}$, find **concrete** *conditions on a that will guarantee that it is Fredholm.*

A first result (Georgescu) is that the ideal of compact operators

$$\mathcal{K} = \mathcal{K}(L^2(X)) \subset \mathcal{A},$$

so, in view of **step two**, the fourth step is equivalent to answering

Given $b \in \mathcal{A}/\mathcal{K}$, find **concrete** *conditions on b that will guarantee that it is invertible.*

The general setting for the G.-I. results

TECHNICAL DETAILS OF THE GEORGESCU-IFTIMOVICI LOCALIZATION AT INFINITY RESULTS (general case)

Notation and assumptions:

- $\mathcal{C}_{\mathrm{b}}^{\mathrm{u}}(X) =$ algebra of *bounded uniformly cont* funct on X.
- $\mathcal{C}_0(X) \subset \mathcal{C}_{\mathrm{b}}^{\mathrm{u}}(X)$ the ideal of functions vanishing at infinity.
- $\mathcal{C}_0(X) \subset \mathcal{A}_0 \subset \mathcal{C}_{\mathrm{b}}^{\mathrm{u}}(X)$ a norm-closed, conj. inv. subalgebra, invariant for the action of X by translations on $\mathcal{C}_{\mathrm{b}}^{\mathrm{u}}(X)$.
- Hence $\mathcal{A}_0 = \mathcal{C}(\Omega_0)$, where Ω_0 is a **compact space** containing X as a dense open subset (struct. of commut. C^*-algebras).
- Also, X **acts continuously** on Ω_0, since it acts on $\mathcal{A}$.

The cross-product algebra $\mathcal{A} := \mathcal{A}_0 \rtimes X$

This data ($\mathcal{A}_0 = \mathcal{C}(\Omega_0)$, ...) defines the **cross-product algebra** $\mathcal{A} := \mathcal{A}_0 \rtimes X$ as the norm-closed algebra on $L^2(X)$ generated by

$$h(P)g(Q)\,, \quad h \in \mathcal{C}_0(X^*),\ g \in \mathcal{A}_0\,,$$

and their adjoints.

Thus $\mathcal{A} := \mathcal{A}_0 \rtimes X$ is **the algebra generated by multiplications** with funct in $\mathcal{A}_0$ and **by convolutions.** ($\mathcal{A}_0 =$ **potentials.**)

Reason for $\rtimes$: $H := -\Delta + V$, where $V = V^* \in \mathcal{A}_0 \simeq \mathcal{C}(\Omega_0)$. Then H is self-adjoint, $(-\Delta + \imath)^{-1} V \in \mathcal{A}$ (same calc.), hence

$$(H + \imath)^{-1} \in \mathcal{A}\,.$$

Definition of the localization at infinity

Recall the **definition of localizations at infinity** (functions)

If $q \in X$ and $\varphi : X \to \mathbb{C}$, then $\tau_q(\varphi)(x) := \phi(x-q) =$ its translation by q.

Extends to operators A by $\tau_q(A) := T_q A T_{-q}$.

Recall that $X \subset \Omega$. For $\omega \in \Omega$, we let

$$\tau_\omega(A) := \mathrm{s}\text{-}\lim_{q \to \omega} \tau_q(A)\,, \quad q \in X\,.$$

The s-lim exists for $A = \phi \in \mathcal{A}_0 \subset \mathcal{C}_{\mathrm{b}}^{\mathrm{u}}(X) =$ operator on $L^2(X)$, hence an explicit **algebra morphism:** $(\alpha = \mathbb{R}_+^* a)$

$$\tau_\omega : \mathcal{A} \to \mathcal{C}_{\mathrm{b}}^{\mathrm{u}}(X)\,, \quad \tau_\omega(f)(x) = (\tau_x f)(\omega)\,.$$

Definition of the localization at infinity (cont.)

Definition of localizations at infinity (cross-product)

We have $\tau_\omega(h(P)) := \text{s-lim}_{q\to\omega}\tau_q(h(P)) = \text{s-lim}_{q\to\omega} h(P) = h(P)$.
Hence we obtain

$$\tau_\omega(h(P)g(Q)) := \operatorname*{s-lim}_{q\to\omega}\tau_q(h(P)g(Q)) = h(P)\tau_\omega(g(Q)) .$$

Since $\|\tau_\omega(A)\| \leq \|A\|$ (when the limit exists), we obtain that τ_ω extends to the cross-product algebra $\mathcal{A} := \mathcal{A}_0 \rtimes X$ to yield a **$*$-algebra morphisms**

$$\tau_\omega : \mathcal{A} := \mathcal{A}_0 \rtimes X \to \mathcal{C}_{\mathrm{b}}^{\mathrm{u}}(X) \rtimes X .$$

$a_\omega := \tau_\omega(a)$ is the **the localization of** $a \in \mathcal{A} := \mathcal{A}_0 \rtimes X$ at ω.

Summary and main GI-result

In summary, we have the following results [Georgescu-Iftimovici]:

1. $\omega \in \Omega_0 = \hat{\mathcal{A}}_0$ defines a morphism $\tau_\omega : \mathcal{A} \to \mathcal{C}_{\mathrm{b}}^{\mathrm{u}}(X) \rtimes X$ s.t.

$$\tau_\omega(\varphi(Q)\psi(P)) = \tau_\omega(\varphi)(Q)\psi(P)\,, \quad \varphi \in \mathcal{A}_0(X),\, \psi \in \mathcal{C}_0(X^*)\,.$$

2. If $\omega \in X$, τ_ω is simply the translation by ω; for $\omega \in \partial\Omega_0 := \Omega_0 \smallsetminus X$, τ_ω is a **strong limit of translations.**
3. $\cap_{\omega \in \delta(\mathcal{A})} \ker \tau_\omega = \mathcal{K} =$ (compact ops on $L^2(X)$).
4. Consequently,

$$\sigma_{\mathrm{ess}}(a) \;=\; \overline{\cup}_{\omega \in \partial\Omega_0}\, \sigma(\tau_\omega(a))\,.$$

Summary of 3rd section

3rd SECTION:

1. Some details of the proof of our result;
2. Extensions: Cordes algebras and Lie manifolds.

The algebra of potentials

Our result: the N-body case.

For each linear subspace $Y \subset X$, we let $\mathcal{C}(\overline{X/Y}) \subset \mathcal{C}_{\mathrm{b}}^{\mathrm{u}}(X/Y)$ be the translation invariant subalgebra of functions on X/Y that have uniform radial limits at infinity on X/Y.

Then **our** $\mathcal{A}_0 = \mathcal{A}_0(X) =$ norm closed subalgebra of $\mathcal{C}_{\mathrm{b}}^{\mathrm{u}}(X)$ **generated** by all the subspaces $\mathcal{C}(\overline{X/Y}) \subset \mathcal{C}_{\mathrm{b}}^{\mathrm{u}}(X/Y)$.

Warning: change in notation!!! $\mathcal{A}_0 = \mathcal{A}_0(X) =$ **NOW.**

Comment: for the standard N-body algebra, it may be more natural to consider the subalgebras $\mathcal{C}_0(X/Y)$ of functions that *vanish* at infinity on X/Y. However, the spectrum of this algebra is more cumbersome to understand. (!)

Radial limits and characters of $\mathcal{A}_0(X)$

We want to write $\mathcal{A}_0(X) \simeq \mathcal{C}(\Omega_0)$. We know then that Ω_0 is the set of algebra morphisms $\chi : \mathcal{A}_0(X) \to \mathbb{C}$ (the **characters** of $\mathcal{A}_0(X) \subset \mathcal{C}_{\mathrm{b}}^{\mathrm{u}}(X)$).

- $\chi_x(f) := f(x)$ defines a character $\chi_x : \mathcal{A}_0(X) \to \mathbb{C}$.

We now explain how **all the other characters** are obtained. We use **the radial limit morphisms** ($\alpha = \mathbb{R}_+^* a$.)

$$\lim_{r\to\infty} V_Y(x+ra) \ = \ \begin{cases} V_Y(\alpha) & \text{if } a \notin Y \\ V_Y(x) & \text{otherwise} \end{cases} \ =: \ \rho_\alpha(V_Y)(x)\,.$$

Each $\mathcal{C}(\overline{X/Y})$ is invariant with respect to translations by X and is "killed" by ρ_α if $\alpha = \mathbb{R}_+^* a \not\subset Y$. That is $\rho_\alpha(\mathcal{C}(\overline{X/Y})) \subset \mathbb{C}$.

The characters of $\mathcal{A}_0$ (cont.)

Thus ρ_α is already a character on $\mathcal{C}(\overline{X/Y})$, if $\alpha = \mathbb{R}_+^* a \not\subset Y$.

On the other hand, $\rho_\alpha(\mathcal{C}(\overline{X/Y})) = \mathcal{C}(\overline{X/Y})$ if $\alpha \subset Y$. Therefore

$$\rho_\alpha(\mathcal{A}_0(X)) \ = \ \mathcal{A}_0(X/\alpha)\,,$$

where, for $\alpha \in S_X$, we denoted by X/α the quotient of X by the subspace $[\alpha] := \mathbb{R}a$ generated by $\alpha := \mathbb{R}_+^* a$.

$\Rightarrow$ **inductive determination** of the spectrum of $\mathcal{A}_0(X)$ (next).

Inductive construction of the characters of $\mathcal{A}_0$

Iteration of the radial morphisms ρ_α:

1. Fix $\alpha \in S_X$ consider $\rho_\alpha : \mathcal{A}_0(X) \to \mathcal{A}_0(X/\alpha)$.
2. Let $\beta \in S_{X/\alpha}$ and consider $\rho_\beta : \mathcal{A}_0(X/\alpha) \to \mathcal{A}_0(X/[\alpha, \beta])$. ($[\alpha, \beta]$ is the 2-dim subspace of X generated by α and β)
3. We obtain
$$\rho_{\beta,\alpha} := \rho_\beta \circ \rho_\alpha : \mathcal{A}_0(X) \to \mathcal{A}_0(X/[\alpha, \beta]) .$$
4. Similar for families $\overrightarrow{\alpha} = (\alpha_1, \ldots, \alpha_n)$, $n \leq \dim X$, and $\alpha_1 \in S_X$, $\alpha_2 \in S_{X/\alpha_1}$, $\alpha_3 \in S_{X/[\alpha_1,\alpha_2]}$, and so on, thus
$$\rho_{\overrightarrow{\alpha}} := \rho_{\alpha_n} \ldots \rho_{\alpha_1} : \mathcal{A}_0(X) \to \mathcal{A}_0(X/[\alpha_1, \ldots, \alpha_n]) .$$
5. Let $x \in X/[\alpha_1, \ldots, \alpha_n]$ and define
$$\chi_{x,\overrightarrow{\alpha}}(f) := \chi_x(\rho_{\overrightarrow{\alpha}}(f)) = \rho_{\overrightarrow{\alpha}}(f)(x) .$$

Determination of all characters

Theorem (Georgescu-Nistor)

All characters of $\mathcal{A}_0(X)$ are of the form

$$\chi_{x,\overrightarrow{\alpha}}(f) := \chi_x(\rho_{\overrightarrow{\alpha}}(f)) = \rho_{\overrightarrow{\alpha}}(f)(x)\,.$$

with x and $\overrightarrow{\alpha}$ uniquely determined, so $\Omega_0 = \{(x, \overrightarrow{\alpha})\}$.

Warning: The topology is not the obvious one!!!

Moreover, if $\omega = (x, \overrightarrow{\alpha})$, $\omega' = (0, \overrightarrow{\alpha})$, and $a \in \mathcal{A}_0$, then

$$\tau_\omega(a) \sim_u \tau_{\omega'}(a) = \rho_{\overrightarrow{\alpha}}(a) = \rho_{\overrightarrow{\alpha}'}\rho_{\alpha_1}(a)\,.$$

So $\{\rho_\alpha(a)\} \subset \{\tau_\omega(a)\}$ and $\{\tau_\omega(a)\}$ are hom images of $\{\rho_\alpha(a)\}$:

$$\sigma_{\text{ess}}(a) = \overline{\cup}_{\omega\in\partial\Omega_0}\, \sigma(\tau_\omega(a)) = \overline{\cup}_{\alpha\in S_X}\, \sigma(\tau_\alpha(a))\,.$$

Cordes' comparison algebras

Let M be a (non-compact) Riemannian manifold.

The **Cordes comparison algebra of** M is the (norm-closed) algebra $\mathfrak{A} = \mathfrak{A}(M)$ generated by $P(1-\Delta)^{-m}$, where P is a suitable differential operator of order $2m$ with **bounded** coefficients.

Theorem (Cordes)

$\mathcal{K} \subset \mathfrak{A}$. If M is "very nice", then the quotient $\mathfrak{A}/\mathcal{K}$ is commutative ($\simeq \mathcal{C}(\Omega)$). Hence an operator $a \in \mathfrak{A}$ is Fredholm if, and only if, its "full symbol" $\Sigma(a) \in \mathcal{C}(\Omega)$ is nowhere vanishing.

Ex. if coefficients satisfy $\nabla^k a \in \mathcal{C}_0(M)$, $k \geq 1$.

Comparison algebras and localizations at infinity

Cordes' theorem extends the classical result

"P is **elliptic** $\Leftrightarrow$ P is **Fredholm**"

for differential operators on **compact** manifolds M ($\Omega = S^*M$).

In particular (recall that $\Sigma : \mathcal{A}/\mathcal{K} \simeq \mathcal{C}(\Omega)$),

$$\sigma_{ess}(a) \;=\; \overline{\cup}_{\omega\in\Omega}\, \sigma\big(\Sigma(a)(\omega)\big).$$

$S^*M \subset \Omega$ and gives a part of the essential spectrum of a via its principal symbol ($a =$ order zero pseudo-differential op).

In this sense, $a_\omega := \Sigma(a)(\omega) \in \mathbb{C}$, for $\omega \in \Omega \smallsetminus S^*X$, can be regarded as a "localization at infinity" of a. G-I construction **generalizes this,** but in their case a_ω are operators.

Example: eulidean spaces

The assumptions of this theorem ($\nabla^k a \in \mathcal{C}_0(M)$ for all $a \in \mathcal{A}$ and $k > 0$) are satisfied for asymptotically Euclidean manifolds.

Let $\overline{M}$ be a comp. manifold **with boundary** and h be a metric on $\overline{M}$. Let $r =$ distance to the boundary and $\mathcal{A}_0 := \mathcal{C}^\infty(\overline{M})$. Let

$$g := \frac{h}{r^2} + \kappa\frac{(dr)^2}{r^4}\,, \quad \kappa > 0\,.$$

Then M, the int of $\overline{M}$, with metric g, satisfies the assumptions of Cordes' theorem (so $\mathfrak{A}/\mathcal{K} \simeq \mathcal{C}(\Omega)$). (**Asympt. eucl.** if $\overline{M} = \overline{X}$.)

Cordes' theorem is **not valid** if $\kappa = 0$, **a. hyperbolic case.**

Extension: A. hyperbolic spaces

As before: $\overline{M}$ = comp. man w b, r = dist to b, $\mathcal{A}_0 := \mathcal{C}^\infty(\overline{M})$, h = metric on $\overline{M}$, but $g = r^{-2}h$ ($\kappa = 0$, **asy. hyp.**).

Then we can still define **localizations at infinity** a_ω, $\omega \in \partial\overline{M}$, s.t.

Theorem (Lauter-Monthubert-V.N.)

$a \in \mathfrak{A}$ is Fredholm $\Leftrightarrow$ a is elliptic and a_ω is invertible $\forall\omega \in \partial\overline{M}$.

a_ω is an order zero pseudodiff operator on a solvable Lie group.

The proof is similar, but uses **groupoid C^*-algebras** instead of cross-product C^*-algebras. Also for some **Lie manifolds.**

Extension: The space (spectrum) $\Omega_0 := \widehat{\mathcal{A}_0}$

There are good reasons to study the space $\Omega_0 := \widehat{\mathcal{A}_0}$ for itself. **Fix a finite number** of $Y \subset X$.

1. Even if one is interested in the classical N-body problem, the eigenfunctions live naturally on Ω_0 (nice ends: **Lie man.**).
2. One can study the regularity of eigenfunctions on Ω_0 (work in progress; my initial motivation).
3. It seems that Ω_0 in fact coincides with the space introduced by Vasy (recent progress by Jérémy Mougel).

Thank you!