# Growth of Sobolev norms for the cubic NLS

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# Introduction

We consider the cubic nonlinear Schrödinger equation

$$(i\partial_t + \Delta) u = |u|^2 u$$

This is a model for dispersive evolution with nonlinear perturbation. We want to understand the following questions:

- What is the influence of the domain?
- What kind of asymptotic behavior is possible?
- Creation of energy at small scales/ Growth of Sobolev norms

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# NLS as a Hamiltonian system

Hamiltonian equation

$$\mathcal{H}(u) = \int_X \left\{ \frac{1}{2}|\nabla_g u|^2 + \frac{1}{4}|u|^4 \right\} d\nu_g,$$

$$\Omega(u, v) = \Im \int_X u\bar{v}\, d\nu_g,$$

In general, only one more conservation law

$$M(u) = \int_X |u|^2 d\nu_g.$$

Natural to study the equation in $H^1(X)$.

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# NLS on $\mathbb{R}^d$

On $\mathbb{R}^d$, the equation is reasonably well understood:

- GWP when $d \leq 4$ and ill-posed when $d \geq 5$ [**Ginibre-Vélo**, **Bourgain**, **Grillakis**, **CKSTT**, **Killip-Visan**, **Kenig-Merle**]
- $2 \leq d \leq 4$: Solutions scatters ,
- $d = 1$, small Solutions modified-scattering (cubic NLS completely integrable) , solutions scatter for quintic nonlinearity.

In particular, smooth solutions satisfy

$$\|u(t)\|_{H^s} \leq C(\|u(0)\|_{H^1})\|u(0)\|_{H^s}$$

uniformly in time.
These results can be extended to some cases of domain with "large volume" (e.g.$\mathbb{H}^3$: **Banica**, **Ionescu-P.-Staffilani**).

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# (modified) Scattering

A solution scatters if it eventually follows the linear flow:

$$u(t) = e^{it\Delta}\left\{f + o(1)\right\}, \qquad t \to \infty.$$

On $\mathbb{R}$, solutions sometimes have a "modified scattering"

$$\widehat{u}(\xi, t) = e^{it\partial_{xx}}\mathcal{F}^{-1}\left\{e^{i|\widehat{f}(\xi)|^2 \log t}\widehat{f}(\xi) + o(1)\right\}, \qquad t \to \infty$$

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# NLS on small domains

For domains with "smaller volume": weaker dispersion, one expects the linear flow to play a less important role. This is what we want to explore.

# Global existence

For GWP: only need control locally in time. Expect same theory as in $\mathbb{R}^d$.

- Verified in lower dimensions ($\mathbb{T}^d$: **Bourgain**, $d = 2$ or $\mathbb{S}^d$: **Burq-Gérard-Tzvetkov**).
- Even true in critical cases, e.g. $\mathbb{T}^4$ (**Herr-Tataru-Tzvetkov**, **Ionescu-P.**).

Idea: Appropriate functional spaces to obtain small data theory. Large data, only obstruction is infinite concentration of energy at a point in space-time $\rightarrow$ blow-up analysis + concentration compactness $\rightarrow$ back to situation on $\mathbb{R}^d$.
However, these results are still consistent with the following picture:

$$\|u(k+1)\|_{H^s} \leq 2\|u(k)\|_{H^s}.$$

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# Asymptotic behavior

Asymptotic behavior much more difficult on a "small" domain. On $\mathbb{T}^d$, various heuristic arguments related to "weak turbulence":

- "generic solutions will explore all of phase space",
- "solutions will cascade to large frequency"
- "creation of small scales"

A related mathematical question was asked by **Bourgain** (00):
*Does there exist a solution such that*

$$\|u(0)\|_{H^2} \lesssim 1, \qquad \limsup_{t\to+\infty} \|u(t)\|_{H^2} = \infty \quad ?$$

# Few evidence for norm growth

Not possible if scattering or on $\mathbb{R}^d$ or on $\mathbb{T}^1$.

Difficult to control solutions globally in time on a compact domain!

No explicit solution whose norm do become unbounded. Besides the growth should be slow: [**Bourgain**]

$$\|u(t)\|_{H^s} \lesssim_A (1+t)^A \qquad \text{(conjecture} \ll (\log t)^A\text{)}.$$

Example of nontrivial globally bounded solutions (KAM results [**Kuksin**, **Bourgain**. . . ]

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# Few evidence for norm growth (II)

Some evidence of finite growth [**Kuksin**]. Importantly

**Arbitrary finite growth on $\mathbb{T}^2$ [CKSTT]**

Given $\varepsilon > 0$, $s > 1$ and $K > 0$, there exists a solution $u$ of cubic NLS on $\mathbb{T}^2$ and a time $T$ such that

$$\|u(0)\|_{H^s} < \varepsilon, \qquad \|u(T)\|_{H^s} > K.$$

Related results [**Hani**, **Kaloshin-Guardia**, **Procesi-Haus**].

**Remark:** Only shows no *a priori* uniform bound; it is possible that the growth saturates.

Recent similar results for the cubic half-wave equation on $\mathbb{R}$ [**Gérard-Lenzman-Pocovnicu-Raphael**].

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# The space $\mathbb{R} \times \mathbb{T}^2$

Nice space to test these questions $\mathbb{R} \times \mathbb{T}^2$:

- Access to nice Fourier analysis.
- Partially compact.

One can ask the following questions:

- what is the threshold for asymptotically linear behavior (i.e. scattering)?
- what happens beyond this?

These questions can be completely answered in the context of noncompact quotients of $\mathbb{R}^d$ ($\mathbb{R} \times \mathbb{T}^2$ is the most interesting example).

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# Scattering

[**Tzvetkov-Visciglia**, **Hani-P.**].
One can have a "nice" scattering theory for

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This only leaves one possibility for an interesting behavior: the cubic equation on $\mathbb{R} \times \mathbb{T}^d$.

**Infinite growth [HPTV, Tzvetkov-P.]**

Let $s > 5/8$, $s \neq 1$. There exists solutions of the cubic NLS on $\mathbb{R} \times \mathbb{T}^2$ such that

$$\limsup_{t \to +\infty} \|U(t)\|_{H^s} = \infty.$$

- Not true (for small data) on $\mathbb{R} \times \mathbb{T}$ (some form of complete integrability).
- True even for some $0 < s < 1$ despite the conservation laws at $s = 0$ and $s = 1$!
- Contrast with recent results of **Killip-Visan, Koch-Tataru** for the completely integrable case.

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# Appearance of the resonant system

Solutions expected to decay over time,

$$(i\partial_t + \Delta)\, u = |u|^2 u \qquad (\sim \varepsilon^2 u)$$

to first order, solutions evolve linearly. Conjugate out the linear flow

$$u(t) = e^{it\Delta_{\mathbb{R}\times\mathbb{T}^d}} F(t), \qquad i\partial_t F = \mathcal{N}[F, F, F]$$

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$$\sum_{q-r+s=p} \int_{\mathbb{R}^2} e^{it\Phi} \widehat{F}_q(\xi - \eta)\overline{\widehat{G}_r}(\xi - \eta - \theta)\widehat{H}_s(\xi - \theta) d\eta d\theta,$$

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Expect $\partial_t F \ll \varepsilon^2$ so main time dependence is in the phase.

If $|\Phi| \geq 1$, can integrate terms through a normal form:

$$i\partial_t \widetilde{F} = \mathcal{N}_{|\Phi|\ll 1}[F, F, F] + O(F^5), \qquad F - \widetilde{F} = O(F^3)$$

Quintic nonlinearities lead to scattering: can be neglected after a long enough time.

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$$\sum_{q-r+s=p} \int_{\mathbb{R}^2} e^{it\Phi} \varphi(\Phi) \widehat{F}_q(\xi - \eta, t) \overline{\widehat{G}_r}(\xi - \eta - \theta, t) \widehat{H}_s(\xi - \theta, t) d\eta d\theta,$$
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Main contribution comes from stationary phase $\eta = \theta = 0$.

# Appearance of the resonant system

We are left with

$$i\partial_t \widehat{F}(\xi, p, t) = \frac{\pi}{t} \sum_{\substack{q-r+s=p,\\ |q|^2-|r|^2+|s|^2=|p|^2}} \widehat{F}_q(\xi, t)\overline{\widehat{F}_r}(\xi, t)\widehat{F}_s(\xi, t)$$

This is, for each fixed $\xi$ an ODE which is the resonant system of the cubic NLS on $\mathbb{T}^2$.

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# Why $\mathbb{R} \times \mathbb{T}^2$ and not $\mathbb{T}^2$?

Although one would expect growth to appear more easily on $\mathbb{T}^2$, this remains an open question.

All the results about growth so far rely on special solutions for the resonant system that grow.

Key difference between $\mathbb{R} \times \mathbb{T}^2$ and $\mathbb{T}^2$: validity of approximation

$$Equ(u) \quad \simeq \quad RS(u) + O(u^5).$$

on $\mathbb{R} \times \mathbb{T}^2$, quintic terms scatter and thus are perturbative globally in time.

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All the results about growth so far rely on special solutions for the resonant system that grow.

Key difference between $\mathbb{R} \times \mathbb{T}^2$ and $\mathbb{T}^2$: validity of approximation

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Quite challenging

- finding appropriate norms in which to close the nonlinear estimates, especially in the low-regularity case $s < 1$ when solutions are unbounded in $L^\infty$.
- need to control over long time solutions of a nonintegrable ODE whose solutions can grow.

Idea: use 2 norms

- A "Strong norm" which provides good control on the solutions (e.g. $\Delta u$, $xe^{-it\Delta}u \in L^2$) but which grows slowly over time
- A "Weak norm" which remains bounded uniformly in time. Corresponds to a conservation law for the resonant system.

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# Modified scattering

**Modified scattering for the cubic NLS on $\mathbb{R}\times\mathbb{T}^2$ [HPTV, P.-Tzvetkov]**

Let $s>1$, there exists a norm $X$ such that any ID small in $X$ leads to a global solution of the cubic NLS on $\mathbb{R}\times\mathbb{T}^2$. Moreover, this solution satisfies a modified scattering in the sense that there exists a solution of the equation

$$\partial_t \widehat{G}_p(\xi,t) = \sum_{\substack{p+q_2=q_1+q_3\\ |p|^2+|q_2|^2=|q_1|^2+|q_3|^2}} \widehat{G}_{q_1}(\xi,t)\overline{\widehat{G}_{q_2}(\xi,t)}\widehat{G}_{q_3}(\xi,t)$$

such that

$$\|U(t)-e^{it\Delta_{\mathbb{R}\times\mathbb{T}^2}}G(\pi\ln t)\|_{H^s}\to 0.$$

# Exotic solutions

All solutions to the resonant system

$$i\partial_t a_p = \sum_{\substack{q-r+s=p,\\ |q|^2-|r|^2+|s|^2=|p|^2}} a_q \overline{a_r} a_s$$

correspond to an asymptotic behavior of the cubic NLS: many unusual behaviors! Growth, beating effect...

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# A physical space formula for the resonant system

One can also obtain a formula for the resonant system in the physical space: a function

$$f(x,t) = \sum_{p \in \mathbb{Z}^d} a_p(t) e^{i\langle p,x\rangle}$$

is a solution if

$$i\partial_t f = \int_{\alpha=0}^{2\pi} e^{-i\alpha\Delta} \left\{ e^{i\alpha\Delta} f(x,t) \cdot \overline{e^{i\alpha\Delta} f(x,t)} \cdot e^{i\alpha\Delta} f(x,t) \right\} d\alpha$$

which is the Hamiltonian associated to the "averaged perturbation"

$$\mathcal{H}_{av} = \int_{\alpha=0}^{2\pi} \int_X |e^{i\alpha\Delta} f|^4 d\nu d\alpha.$$