# Random repeated interaction quantum systems

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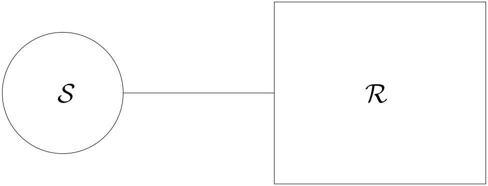
Goal: understand the asymptotic ($t \to +\infty$) behaviour of the system $\mathcal{S}$ (asymptotic state, thermodynamical properties).



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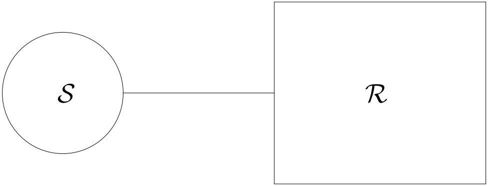


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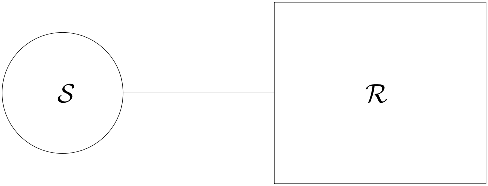


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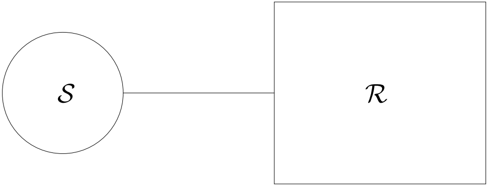


2 approaches: Hamiltonian / Markovian

- Hamiltonian: full description, spectral analysis, scattering theory.
- Markovian: effective description of $\mathcal{S}$, obtained by weak-coupling type limits or if $\mathcal{S}$ undergoes stochastic forces (Langevin equation).

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Interactions:

- Interaction operators $V_k$ acting on $\mathcal{H}_{\mathcal{S}} \otimes \mathcal{H}_{\mathcal{E}_k}$.

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i.e. the full system is governed by

$$\widetilde{H}_n = H_{\mathcal{S}} + H_{\mathcal{E}_n} + V_n + \sum_{k \neq n} H_{\mathcal{E}_k} = H_n + \sum_{k \neq n} H_{\mathcal{E}_k}.$$

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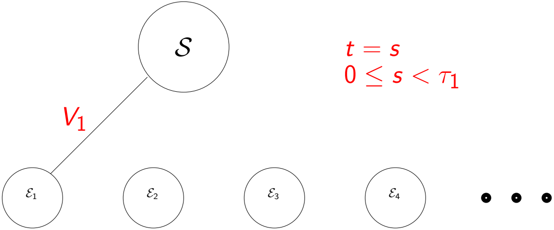


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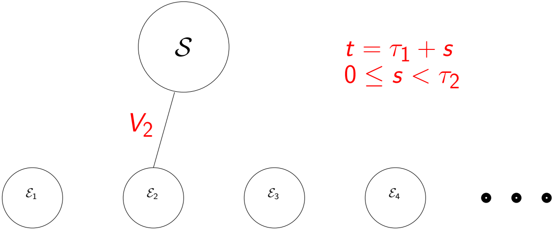


$t = \tau_1 + s$
$0 \leq s < \tau_2$

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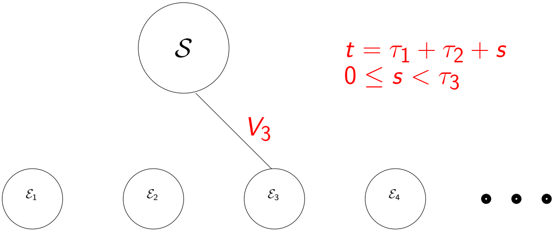

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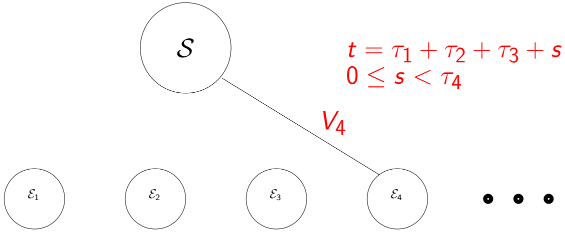
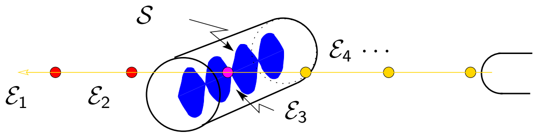


# Motivation

Physics: One-atom maser (Walther et al '85, Haroche et al '92)

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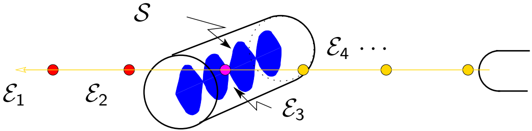


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ideal RIQS as simple models (Vogel et al '93, Wellens et al '00)
random RIQS: some fluctuation in the various parameters (temperature, interaction time, etc).

# The repeated interaction dynamics.

Data:

1. Full Hamiltonian: $H_n = H_{\mathcal{S}} \otimes \mathbb{1}_{\mathcal{E}_n} + \mathbb{1}_{\mathcal{S}} \otimes H_{\mathcal{E}_n} + V_n$.

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After 1 interaction, the state of the total system is

$$\rho_1^{\text{tot}} := \qquad \mathrm{e}^{-i\tau_1 H_1} \left( \rho \otimes \bigotimes_{k \geq 1} \rho_{\mathcal{E}_k} \right) \mathrm{e}^{i\tau_1 H_1}$$

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After 2 interactions, the state of the total system is

$$\rho_2^{\text{tot}} := \mathrm{e}^{-i\tau_2 H_2}\mathrm{e}^{-i\tau_1 H_1}\left(\rho \otimes \bigotimes_{k\geq 1} \rho_{\mathcal{E}_k}\right)\mathrm{e}^{i\tau_1 H_1}\mathrm{e}^{i\tau_2 H_2}$$

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After $n$ interactions, the state of the total system is

$$\rho_n^{\mathrm{tot}} := \mathrm{e}^{-i\tau_n H_n} \cdots \mathrm{e}^{-i\tau_2 H_2} \mathrm{e}^{-i\tau_1 H_1} \left( \rho \otimes \bigotimes_{k \geq 1} \rho_{\mathcal{E}_k} \right) \mathrm{e}^{i\tau_1 H_1} \mathrm{e}^{i\tau_2 H_2} \cdots \mathrm{e}^{i\tau_n H_n}.$$

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Long time behaviour:

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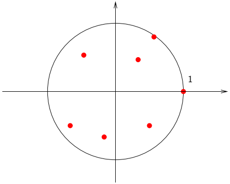
$\Longrightarrow$ We shall understand $\mathcal{L}_n \circ \cdots \circ \mathcal{L}_1$ as $n \to \infty$.

# Spectrum of a RDM

The $\mathcal{L}_n$ are completely positive and trace preserving maps on $\mathcal{J}_1(\mathcal{H}_\mathcal{S})$.

General case:

$\mathrm{Spec}(\mathcal{L}_n) \subset \{z \in \mathbb{C} \,|\, |z| \leq 1\}$,
1 is an eigenvalue.



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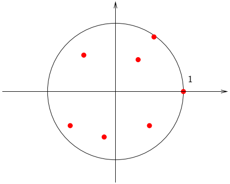
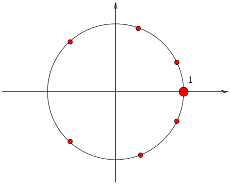



Uncoupled case:

If $V_n = 0$, $\mathcal{L}_n(\cdot) = \mathrm{e}^{-i\tau_n H_\mathcal{S}} \cdot \mathrm{e}^{i\tau_n H_\mathcal{S}}$,
$\Rightarrow \mathrm{Spec}(\mathcal{L}_n) = \{\mathrm{e}^{i\tau_n(\lambda_k - \lambda_l)}\}, \lambda_k \in \mathrm{Spec}(\mathcal{H}_\mathcal{S})$,
1 is degenerate ($\dim(\mathcal{H}_\mathcal{S})$ times).



# Ideal RIQS: $\mathcal{L}_n \equiv \mathcal{L}$

Assumption (E):

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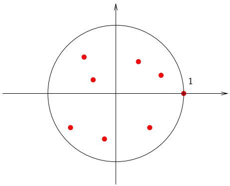

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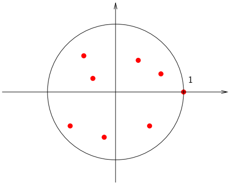


**Theorem**

*If (E) is satisfied, there exist $C, \alpha > 0$ s.t. for any initial state $\rho$*

$$\|\mathcal{L}^n(\rho) - \rho_+\|_1 \leq C\mathrm{e}^{-\alpha n}, \qquad \forall n \in \mathbb{N},$$

*where $\rho_+$ is the (unique) invariant state of $\mathcal{L}$.*

Note that $\rho_+$ does not depend on the initial state of $\mathcal{S}$.

# A simple example: spin-spin model

- $\mathcal{S}$ and $\mathcal{E}_n$ are 2-level systems, i.e. $\mathcal{H}_\mathcal{S} = \mathcal{H}_{\mathcal{E}_n} \equiv \mathcal{H}_\mathcal{E} = \mathbb{C}^2$, with energy levels $\{0, E_\mathcal{S}\}$, resp. $\{0, E_\mathcal{E}\}$, i.e. $H_\# = \begin{pmatrix} 0 & 0 \\ 0 & E_\# \end{pmatrix}$.

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- $V_n = \lambda(a_\mathcal{S} \otimes a_n^* + a_\mathcal{S}^* \otimes a_n)$ where $a_\# = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

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- $V_n = \lambda(a_\mathcal{S} \otimes a_n^* + a_\mathcal{S}^* \otimes a_n)$ where $a_\# = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.
- $\rho_{\mathcal{E}_n}$ is a Gibbs state, i.e. $\rho_{\mathcal{E}_n} = \rho_{\beta_n,\mathcal{E}} := \mathrm{e}^{-\beta_n H_\mathcal{E}}/\mathrm{Tr}(\mathrm{e}^{-\beta_n H_\mathcal{E}})$ with $\beta_n \equiv \beta$.

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Explicit computation: $\mathcal{L}$ satisfies (E) iff $\tau \notin T\mathbb{N}$ with $T = 2\pi/\sqrt{(E_\mathcal{S} - E_\mathcal{E})^2 + 4\lambda^2}$ (non-resonance condition).

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**Proposition**

*If $\tau \notin T\mathbb{N}$, $\lim_{n\to\infty} \mathrm{Tr}(\rho_n A_\mathcal{S}) = \rho_{\beta^*,\mathcal{S}}(A_\mathcal{S})$ (exponentially fast) where $\beta^* = \beta E_\mathcal{E}/E_\mathcal{S}$.*

# Random RIQS: $\mathcal{L} = \mathcal{L}(\omega)$

Fluctuations w.r.t. ideal situation: $\mathcal{L} = \mathcal{L}(\omega_0)$ random variable with values in RDM (CP, trace preserving maps on $\mathcal{H}_\mathcal{S}$) over a probability space $(\Omega_0, \mathcal{F}, \mathrm{p})$.

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Product of i.i.d. RDMs: $\Omega = \Omega_0^{\mathbb{N}^*}$, $\mathrm{d}\mathbb{P} = \prod_{n\geq 1} \mathrm{dp}$ and $\omega = (\omega_n)_{n\geq 1}$.
$\Rightarrow$ Understand $\Phi(n,\omega) = \mathcal{L}(\omega_n) \circ \cdots \circ \mathcal{L}(\omega_1)$.

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$\Rightarrow$ For any $n$, $\rho_n^\omega = (\Phi(n,\omega))(\rho) = \rho_+(\omega_n)$.

Consequence: unless $\rho_+(\omega_0) \equiv \rho_+$, no convergence in the usual sense (local fluctuations), but in the ergodic mean

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \rho_n^\omega = \mathbb{E}(\rho_+), \quad a.e.\ \omega.$$

# Random RIQS: $\mathcal{L} = \mathcal{L}(\omega)$

## Theorem

*If* $\mathrm{p}(\mathcal{L}(\omega_0)$ *satisfies (E))* $> 0$*, then*

1. $\mathbb{E}(\mathcal{L})$ *satisfies (E),*
2. *For any* $\rho \in \mathcal{J}_1(\mathcal{H}_\mathcal{S})$*,* $\displaystyle\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} (\Phi(n,\omega))(\rho) = \rho_+, \quad a.e.\ \omega \in \Omega,$
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*If* $\mathrm{p}(\mathcal{L}(\omega_0)$ *satisfies (E))* $> 0$ *and there exists* $\rho_+$ *s.t.* $\mathcal{L}(\omega_0)(\rho_+) = \rho_+$ *for a.e.* $\omega_0$*, i.e. there is a deterministic invariant state, then*

1. $\mathbb{E}(\mathcal{L})$ *satisfies (E),*
2. *There exists* $\alpha > 0$ *s.t. for any* $\rho \in \mathcal{J}_1(\mathcal{H}_\mathcal{S})$ *and for a.e.* $\omega \in \Omega$*, there exists* $C(\omega) > 0$

$$\|(\Phi(n,\omega))(\rho) - \rho_+\|_1 \le C(\omega)\mathrm{e}^{-\alpha n}, \quad \forall n \in \mathbb{N}.$$

# Back to the example

Recall:

1. $\mathcal{L}$ satisfies (E) iff $\tau \notin T\mathbb{N}$ with $T = 2\pi/\sqrt{(E_{\mathcal{S}} - E_{\mathcal{E}})^2 + 4\lambda^2}$,
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1. the interaction time is random: $\tau_n = \tau(\omega_n)$,
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*2) Suppose $\tau_n \equiv \tau \notin T\mathbb{N}$ and $\beta(\omega)$ is a random variable. Then for any $\rho \in \mathcal{J}_1(\mathcal{H}_\mathcal{S})$,*

$$\lim_{N\to\infty} \sum_{n=1}^{N} \rho_n^\omega = \mathbb{E}(\rho_{\beta^*(\omega),\mathcal{S}}).$$

# Energy variation

During the $n$-th interaction the energy is constant, formally given by

$$\mathrm{Tr}\left(\rho_{n-1}^{\mathrm{tot}} H_n\right) = \mathrm{Tr}\left(\rho_n^{\mathrm{tot}} H_n\right).$$

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When one switches from interaction $n$ to interaction $n+1$, there is an energy jump:

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&= \mathrm{Tr}_{\mathcal{S},\mathcal{E}_{n+1}}\left((\mathcal{L}_n \circ \cdots \circ \mathcal{L}_1(\rho) \otimes \rho_{\mathcal{E}_{n+1}})V_{n+1}\right) - \mathrm{Tr}\left(\rho_{n-1}^{\mathrm{tot}}(\mathrm{e}^{i\tau_n H_n} V_n \mathrm{e}^{-i\tau_n H_n})\right)
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\end{aligned}$$

In the ideal case, this rewrites

$$\delta E_n = \mathrm{Tr}_{\mathcal{S},\mathcal{E}}\left((\mathcal{L}^n(\rho) \otimes \rho_{\mathcal{E}})V\right) - \mathrm{Tr}_{\mathcal{S},\mathcal{E}}\left((\mathcal{L}^{n-1}(\rho) \otimes \rho_{\mathcal{E}})(\mathrm{e}^{i\tau H} V \mathrm{e}^{-i\tau H})\right).$$

# Energy variation

In the ideal case, one easily gets

**Proposition**

*If Assumption (E) is satisfied,*

$$\mathrm{d}E_+ := \lim_{n\to\infty} \delta E_n = \mathrm{Tr}_{\mathcal{S},\mathcal{E}}\left(\rho_+ \otimes \rho_{\mathcal{E}}\left(V - \mathrm{e}^{i\tau H} V \mathrm{e}^{-i\tau H}\right)\right).$$

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In the random case we have, using

$$\begin{aligned} \delta E_n &= \mathrm{Tr}_{\mathcal{S},\mathcal{E}_{n+1}} \left( (\mathcal{L}_n \circ \cdots \circ \mathcal{L}_1(\rho) \otimes \rho_{\mathcal{E}_{n+1}}) V_{n+1} \right) \\ &\quad - \mathrm{Tr}_{\mathcal{S},\mathcal{E}_n} \left( (\mathcal{L}_{n-1} \circ \cdots \circ \mathcal{L}_1(\rho) \otimes \rho_{\mathcal{E}_n})(\mathrm{e}^{i\tau_n H_n} V_n \mathrm{e}^{-i\tau_n H_n}) \right), \end{aligned}$$

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*2) Random case: if $\mathrm{p}(\mathcal{L}(\omega_0)$ satisfies (E)$) > 0$, then*

$$\begin{aligned} \mathrm{d}S_+ &:= \lim_{n\to\infty} \frac{\mathrm{Ent}(\rho_n^{\mathrm{tot}}|\rho_0) - \mathrm{Ent}(\rho|\rho_0)}{n} \\ &= \mathbb{E}\left(\beta\, \mathrm{Tr}_{\mathcal{S},\mathcal{E}}\left(\rho_+ \otimes \rho_{\mathcal{E}}\left(V - \mathrm{e}^{i\tau H} V \mathrm{e}^{-i\tau H}\right)\right)\right). \end{aligned}$$

*In particular, if $\beta$ is not random we still have $\mathrm{d}S_+ = \beta \mathrm{d}E_+$.*

# Thermodynamics of the spin-spin example

Recall $T = 2\pi/\sqrt{(E_{\mathcal{S}} - E_{\mathcal{E}})^2 + 4\lambda^2}$, and let $\kappa := \frac{16\pi^2\lambda^2 E_{\mathcal{E}}}{T^2} \sin^2\left(\frac{\pi\tau}{T}\right)$.

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We compute explicitly

$$\mathrm{d}E_+ = \mathbb{E}\left(\frac{1}{1+\mathrm{e}^{-\beta E_{\mathcal{E}}}}\right)^{-1} \times \mathrm{Cov}\left(\kappa, \frac{1}{1+\mathrm{e}^{-\beta E_{\mathcal{E}}}}\right),$$

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2. if only $\beta$ is random, $\mathrm{d}E_+ = 0$ while

$$\mathrm{d}S_+ = \kappa\mathbb{E}\left(\frac{1}{1+\mathrm{e}^{-\beta E_\mathcal{E}}}\right)^{-1} \times \mathrm{Cov}\left(\beta, \frac{1}{1+\mathrm{e}^{-\beta E_\mathcal{E}}}\right) \geq 0$$

and vanishes iff $\beta(\omega) \equiv \beta$ a.s.