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Étude mathématique de quelques systèmes quantiques ouverts

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Jury

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A Gwen, Lilian et ...

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Introduction

Les systèmes ouverts

Le cadre général de la plupart des résultats présentés dans ce mémoire est celui des Systèmes Quantiques Ouverts. Par opposition avec un système fermé (ou isolé), un système $\mathcal{S}$ est dit ouvert lorsqu'il est en contact/interagit avec un ou plusieurs autres systèmes souvent appelés réservoir(s) ou environnement. Un exemple typique d'un tel système ouvert est celui d'un atome en interaction avec le champ électromagnétique, mais on trouve de nombreux autres exemples dans divers domaines de la physique tels que la physique du solide, l'optique quantique,etc. Ils servent aussi de paradigme à la mécanique statistique quantique à et hors de l'équilibre.

Bien qu'un tel système puisse être considéré comme un gros système fermé, composé de plusieurs morceaux, ces derniers sont habituellement traités à des niveaux différents. Il faut d'abord penser au système $\mathcal{S}$ comme étant beaucoup plus petit que son environnement, par exemple nombre fini versus nombre infini de degrés de liberté. Ensuite, l'environnement étant habituellement très grand, il est souvent très difficile d'avoir des informations précises dessus et l'on est plutôt intéressé par comprendre les effets que celui-ci peut avoir sur la dynamique du petit système $\mathcal{S}$. On pensera par exemple à des phénomènes de dissipation.

Pour étudier les systèmes ouverts, deux approches distinctes ont été utilisées dans la littérature aussi bien mathématique que physique: l'approche dite *hamiltonienne* et celle dite *markovienne*.

L'approche hamiltonienne est plus fondamentale au sens où l'on regarde la dynamique du système total d'un point de vue microscopique: on s'intéresse à une description complète du système (petit système + environnement). Les deux parties du système sont décrites respectivement par des espaces de Hilbert $\mathcal{H}_{\mathcal{S}}$ et $\mathcal{H}_{\mathrm{env}}$ représentant les états du système, et des hamiltoniens $H_{\mathcal{S}}$ et H_{env}. L'espace des états du système total est alors donné par $\mathcal{H} = \mathcal{H}_{\mathcal{S}} \otimes \mathcal{H}_{\mathrm{env}}$ et la dynamique du système couplé décrite par un hamiltonien de la forme $H = H_{\mathcal{S}} \otimes \mathbb{1}_{\mathrm{env}} + \mathbb{1}_{\mathcal{S}} \otimes H_{\mathrm{env}} + H_{\mathrm{int}}$ où H_{int} décrit l'interaction entre les deux parties. Le but est alors de comprendre le comportement dynamique du système total en utilisant les outils habituels de la mécanique quantique (analyse spectrale et théorie de la diffusion) [BFS, DG1, DG2, JP1, JP2, MMS].

A l'inverse, dans l'approche markovienne, on abandonne l'idée de décrire le système dans son ensemble et on se concentre sur le petit système $\mathcal{S}$ uniquement, la philosophie étant que l'environnement est trop compliqué à décrire (dans l'approche hamiltonienne, seuls des environnements très simples tels que des champs libres ou des gaz idéaux de bosons et de fermions ont été traités avec succès), voir même est tel que l'on n'a pas accès aux informations le concernant. A la place, on considère uniquement la dynamique effective du système $\mathcal{S}$ qui est créée par

cet environnement. L'évolution de $\mathcal{S}$ est alors gouvernée par une équation maitresse quantique du type

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = \mathcal{L}(\rho) := -\mathrm{i}[H,\rho] + \frac{1}{2}\sum_j (2L_j^*\rho L_j - L_j^* L_j \rho - \rho L_j^* L_j).$$

Celle-ci engendre sur l'espace des états de $\mathcal{S}$ un semi-groupe d'applications complètement positives et préservant la trace. Il y a en général deux façons d'obtenir une telle dynamique effective: soit dans une certaine limite de la dynamique hamiltonienne du système total (comme la limite de couplage faible de van Hove [Da1, Da2, DJ2, DF]), ou bien comme étant le résultat de forces stochastiques agissant sur $\mathcal{S}$ (équations de Langevin quantique [HP]). Pour une introduction plus complète sur ce sujet, on pourra consulter [AJP].

Dans ces deux approches, les questions auxquelles on s'intéresse sont cependant similaires et concernent le comportement à grand temps du système: y a-t-il un (unique) état invariant (pour $\mathcal{S}$ ou pour le système total)? a-t-on convergence d'un état initial vers cet état invariant? à quelle vitesse? quelles sont les propriétés de cet état invariant?

Les systèmes en interactions répétées

La majeure partie de ce mémoire, les Chapitres 1 et 2, est consacrée à une classe particulière de systèmes ouverts qui est devenue récemment très populaire, tant à cause de nouvelles applications physiques que du fait de leur structure mathématique particulière: les système en interactions répétées (IR). Dans ces systèmes, l'environnement est constitué d'une suite de sous-systèmes indépendants $\mathcal{E}_1$, $\mathcal{E}_2, \ldots$ Le petit système $\mathcal{S}$ interagit avec $\mathcal{E}_1$ pendant un intervalle de temps $[0, \tau_1[$, puis avec $\mathcal{E}_2$ pendant un intervalle $[\tau_1, \tau_1 + \tau_2[$, et ainsi de suite. Pendant que $\mathcal{S}$ interagit avec le sous-système $\mathcal{E}_n$ les autres éléments de l'environnement évolue chacun indépendamment selon leur propre dynamique libre. L'évolution du système total $\mathcal{S} + \mathcal{E}_1 + \cdots$ est donc déterminée par la suite de temps $\tau_1, \tau_2, \ldots$, la dynamique propre de chacun des $\mathcal{E}_n$ ainsi que la dynamique couplée de chaque paire $\mathcal{S} + \mathcal{E}_n$. On peut noter que, puisque l'environnement est constitué d'une infinité de sous-système, il n'est pas nécessaire de prendre ces derniers "grands" afin d'avoir un gros environnement.

L'importance des systèmes quantiques avec interactions répépétées, d'un point de vue théorique aussi bien que pratique, est mise en évidence par des expériences d'interaction matière-lumière dans lesquelles des atomes interagissent avec des modes du champ électromagnétique quantifié. Dans cette situation le "petit" système $\mathcal{S}$ représente un (ou plusieurs) mode du champ dans une cavité et l'environnement représente un faisceau d'atomes $\mathcal{E}_n$ qui sont envoyés dans la cavité. De tels "Masers à un atome" dans lesquels le faisceau est réglé de façon à ce qu'à chaque instant *un seul atome* soit dans la cavité ont été réalisés en laboratoire [MWM, WVHW].

Dans les modèles en interactions répétées les plus simples, toutes les interactions sont identiques (on parlera alors d'interaction répétées idéales). Plus précisément, chaque $\mathcal{E}_n$ est une copie d'un même sous-système $\mathcal{E}$, $\tau_n \equiv \tau$, et les dynamiques des $\mathcal{E}_n$ et des paires $\mathcal{S} + \mathcal{E}_n$ ne dépendent pas de n et sont engendrées par des hamiltoniens $H_{\mathcal{E}}$, $H_{\mathcal{SE}}$. La structure particulière de systèmes IR les placent alors à mi-chemin entre les approches hamiltonienne et markovienne: ils sont à la fois hamiltoniens (avec une hamiltonien dépendant du temps) et markovien en temps discret (pour des temps $n\tau$ la dynamique effective de $\mathcal{S}$ est décrite par un semi-groupe discret

d'applications complètement positives, voir (7) ci-dessous). Ces modèles fournissent ainsi un cadre intéressant pour développer notre compréhension des systèmes ouverts.

Description mathématique des systèmes IR

On décrit maintenant de façon plus précise le cadre mathématique des systèmes avec interactions répétées, et en particulier on montre comment leur structure particulière permet d'obtenir une dynamique markovienne en temps discret pour $\mathcal{S}$ à partir de la dynamique hamiltonienne du système total.

Les différents éléments nécessaires pour décrire un système IR sont:

1. un espace de Hilbert $\mathfrak{h}_{\mathcal{S}}$ et un hamiltonien $h_{\mathcal{S}}$ décrivant le petit système $\mathcal{S}$ "seul",
2. des espaces de Hilbert $\mathfrak{h}_{\mathcal{E}_n}$ et des hamiltoniens $h_{\mathcal{E}_n}$ décrivant les sous-sytèmes $\mathcal{E}_n$,
3. une suite de temps d'interaction $(\tau_n)_n$ où $\tau_n \geq \tau > 0$ pour tout n et pour un certain τ. Le temps τ_n repréente la durée de l'interaction entre $\mathcal{S}$ et le sous-système $\mathcal{E}_n$,
4. des opérateurs v_n décrivant les interactions entre $\mathcal{S}$ et les sous-systèmes $\mathcal{E}_n$.

L'espace de Hilbert décrivant le système IR est alors

$$\mathfrak{h} := \mathfrak{h}_{\mathcal{S}} \otimes \mathfrak{h}_{\mathrm{env}}, \qquad \mathfrak{h}_{\mathrm{env}} := \bigotimes_{n\geq 1} \mathfrak{h}_{\mathcal{E}_n}.$$

On notera également $t_n := \tau_1 + \cdots + \tau_n$. Pendant l'intervalle de temps $[t_{n-1}, t_n)$, le système $\mathcal{S}$ interagit avec le n-ème sous-système, i.e. $\mathcal{E}_n$, et aucun autre. L'évolution complète du système est alors décrite par le hamiltonien

$$h(t) = h_{\mathcal{S}} + \sum_{n\geq 1} h_{\mathcal{E}_n} + \sum_{n\geq 1} \chi_n(t) v_n,$$

où χ_n est la fonction caractéristique de l'intervalle $[t_{n-1}, t_n)$. On utilisera aussi les notations suivantes:

$$h_n := h_{\mathcal{S}} + h_{\mathcal{E}_n} + v_n, \quad \text{and} \quad \tilde{h}_n := h_n + \sum_{k\neq n} h_{\mathcal{E}_k}.$$

En particulier $h(t) \equiv \tilde{h}_n$ pour $t \in [t_{n-1}, t_n)$. On a également omis les facteurs "identité" triviaux, par exemple $h_{\mathcal{S}}$ devrait être $h_{\mathcal{S}} \otimes \mathbb{1}_{\mathrm{env}}$.

Etant donnés un état initial ρ du système $\mathcal{S}$ au temps $t = 0$ (i.e. ρ est un opérateur positif de classe trace sur $\mathfrak{h}_{\mathcal{S}}$ avec trace 1), et une suite $(\rho_{\mathcal{E}_n})_n$ d'états initiaux pour les sous-systèmes $\mathcal{E}_n$, l'état du système IR total après n interactions est alors donné par

$$\rho^{\mathrm{tot}}(n) := \mathrm{e}^{-\mathrm{i}\tau_n \tilde{h}_n} \cdots \mathrm{e}^{-\mathrm{i}\tau_1 \tilde{h}_1} \left(\rho_{\mathcal{S}} \otimes \bigotimes_{k\geq 1} \rho_{\mathcal{E}_k} \right) \mathrm{e}^{\mathrm{i}\tau_1 \tilde{h}_1} \cdots \mathrm{e}^{\mathrm{i}\tau_n \tilde{h}_n}.$$

On s'intéresse surtout au système $\mathcal{S}$ (voir cependant la Section 1.3 pour des observables plus générales), c'est-à-dire à la valeur moyenne d'observables du type

$$O = O_{\mathcal{S}} \otimes \bigotimes_{k\geq 1} \mathbb{1}_{\mathcal{E}_k}.$$

On est donc intéressé par $\rho(n) := \mathrm{Tr}_{\mathfrak{h}_{\mathrm{env}}}(\rho^{\mathrm{tot}}(n))$, la matrice densité réduite sur $\mathcal{S}$. Celle-ci est définie comme étant l'unique état sur $\mathcal{S}$ tel que, pour toute observbles $O_{\mathcal{S}}$ du système $\mathcal{S}$,

$$\mathrm{Tr}_{\mathfrak{h}_{\mathcal{S}}}\left(\rho(n)O_{\mathcal{S}}\right) = \mathrm{Tr}_{\mathfrak{h}}\left(\rho^{\mathrm{tot}}(n) \times \left(O_{\mathcal{S}} \otimes \bigotimes_{k\geq 1} \mathbb{1}_{\mathcal{E}_k}\right)\right).$$

Pour obtenir l'état $\rho(n)$ du système $\mathcal{S}$ après n interactions on prend donc la trace partielle suivante

$$\rho(n) := \mathrm{Tr}_{\mathfrak{h}_{\mathrm{env}}}\left[\mathrm{e}^{-\mathrm{i}\tau_n\tilde{h}_n}\cdots\mathrm{e}^{-\mathrm{i}\tau_1\tilde{h}_1}\left(\rho\otimes\bigotimes_{k\geq 1}\rho_{\mathcal{E}_k}\right)\mathrm{e}^{\mathrm{i}\tau_1\tilde{h}_1}\cdots\mathrm{e}^{\mathrm{i}\tau_n\tilde{h}_n}\right]. \tag{1}$$

Bien entendu le calcul ci-dessus est un peu formel. En effet, un produit tensoriel infini d'espaces de Hilbert est défini via une suite stabilisante, i.e. une suite de vecteurs $(\psi_n)_n$ avec $\psi_n \in \mathfrak{h}_{\mathcal{E}_n}$. L'espace de Hilbert $\mathfrak{h}_{\mathrm{env}}$ est obtenu comme la complétion de l'espace vectoriel des combinaisons linéaires finies d'éléments de la forme $\otimes_{n\geq 1}\phi_n$, où $\phi_n \in \mathfrak{h}_{\mathcal{E}_n}$, $\phi_n = \psi_n$ excepté pour un nombre fini d'indices, et pour la norme correspondant au produit scalaire

$$\langle \otimes_n \varphi_n, \otimes_n \phi_n\rangle = \prod_n \langle \varphi_n, \phi_n\rangle_{\mathfrak{h}_{\mathcal{E}_n}}.$$

En général, le produit tensoriel infini $\bigotimes_{k\geq 1}\rho_{\mathcal{E}_k}$ n'a alors pas de sens. Il n'est cependant pas difficile de donner un sens à l'équation (1): au temps t_n, seuls les n premiers éléments de l'environnement ont déjà joué un rôle et on peut donc remplacer $\bigotimes_{k\geq 1}\rho_{\mathcal{E}_k}$ par $\rho_{\mathrm{env}}^{(n)} := \bigotimes_{k=1}^{n}\rho_{\mathcal{E}_k}$ et la trace partielle sur tout l'environnement par celle sur le produit tensoriel fini $\mathfrak{h}_{\mathrm{env}}^{(n)} := \bigotimes_{k=1}^{n}\mathfrak{h}_{\mathcal{E}_k}$, i.e.

$$\rho(n) = \mathrm{Tr}_{\mathfrak{h}_{\mathrm{env}}^{(n)}}\left[\mathrm{e}^{-\mathrm{i}\tau_n\tilde{h}_n}\cdots\mathrm{e}^{-\mathrm{i}\tau_1\tilde{h}_1}\left(\rho\otimes\bigotimes_{k=1}^{n}\rho_{\mathcal{E}_k}\right)\mathrm{e}^{\mathrm{i}\tau_1\tilde{h}_1}\cdots\mathrm{e}^{\mathrm{i}\tau_n\tilde{h}_n}\right]. \tag{2}$$

Remarque. *Une autre possibilité serait de définir le produit tensoriel infini "par rapport à la suite d'états $(\rho_{\mathcal{E}_n})_n$". Pour cela, on représente d'abord les états $\rho_{\mathcal{E}_n}$ comme des états vectoriels avec vecteurs représentatifs Ψ_n (en utilisant la représentation GNS), puis on considère la suite stabilisante $(\Psi_n)_n$. Cela conduit à la description "Liouvillienne" du système qui sera présentée de façon plus détaillée dans la Section 1.1.*

La structure très particulière des systèmes avec interactions répétées permet de réécrire $\rho(n)$ d'une façon beaucoup plus pratique. Les deux caractéristiques principales de ces systèmes sont:

1. Les différents sous-systèmes de l'environment n'interagissent pas directement entre eux (seulement via $\mathcal{S}$), i.e. $[h_{\mathcal{E}_k}, h_{\mathcal{E}_n}] = 0$ pour tous $k \neq n$,
2. Le système $\mathcal{S}$ interagit une et une seule fois avec chacun des sous-systèmes $\mathcal{E}_n$, et avec un seul à la fois, i.e. $[h_{\mathcal{E}_k}, h_n] = 0$ pour tous $k \neq n$.

On obtient alors la décomposition suivante qui sert à isoler la dynamique des sous-systèmes qui n'interagissent pas à un instant donné

$$\mathrm{e}^{-\mathrm{i}\tau_n\tilde{h}_n}\cdots\mathrm{e}^{-\mathrm{i}\tau_1\tilde{h}_1} = u_n^- \times \mathrm{e}^{-\mathrm{i}\tau_n h_n}\cdots\mathrm{e}^{-\mathrm{i}\tau_1 h_1} \times u_n^+, \tag{3}$$

où

$$u_n^- = \exp\left(-\mathrm{i}\sum_{k=1}^{n-1}(t_n - t_k)h_{\mathcal{E}_k}\right) \text{ et } u_n^+ = \exp\left(-\mathrm{i}\sum_{k=2}^{n} t_{k-1}h_{\mathcal{E}_k} - \mathrm{i}t_n\sum_{k>n} h_{\mathcal{E}_k}\right)$$

sont les propagateurs au temps t_n des sous-systèmes $\mathcal{E}_k$ après leur interaction avec $\mathcal{S}$, respectivement avant leur interaction. En insérant (3) dans (2) on obtient

$$\rho(n) = \mathrm{Tr}_{\mathfrak{h}_{\mathrm{env}}^{(n)}}\left[\mathrm{e}^{-\mathrm{i}\tau_n h_n}\cdots\mathrm{e}^{-\mathrm{i}\tau_1 h_1}\left(\rho\otimes\bigotimes_{k=1}^{n}\rho_{\mathcal{E}_k}(t_{k-1})\right)\mathrm{e}^{\mathrm{i}\tau_1 h_1}\cdots\mathrm{e}^{\mathrm{i}\tau_n h_n}\right],$$

où $\rho_{\mathcal{E}_k}(t_{k-1}) = \mathrm{e}^{-\mathrm{i}t_{k-1}h_{\mathcal{E}_k}}\rho_{\mathcal{E}_k}\mathrm{e}^{\mathrm{i}t_{k-1}h_{\mathcal{E}_k}}$ est l'état du k-ème sous-système juste avant qu'il n'interagisse avec $\mathcal{S}$. Cette expression est bien entendu beaucoup plus simple si l'état initial $\rho_{\mathcal{E}_k}$ est invariant sous la dynamique libre de $\mathcal{E}_k$, un état thermique par exemple, ce qui sera souvent le cas par la suite.

Il est maintenant facile de voir que l'évolution de $\mathcal{S}$ est markovienne: l'état $\rho(n)$ ne dépend que de $\rho(n-1)$ et de la n-ème interaction. Plus précisément, on a

$$\rho(n) = \mathcal{L}_n(\rho(n-1)), \tag{4}$$

où

$$\mathcal{L}_n(\rho) := \mathrm{Tr}_{\mathfrak{h}_{\mathcal{E}_n}}\left[\mathrm{e}^{-\mathrm{i}\tau_n h_n}\,\rho\otimes\rho_{\mathcal{E}_n}(t_{n-1})\,\mathrm{e}^{\mathrm{i}\tau_n h_n}\right]. \tag{5}$$

Definition 1. *L'application $\mathcal{L}_n$ définie sur $\mathcal{B}^1(\mathfrak{h}_{\mathcal{S}})$ est appelée* application de dynamique réduite *au temps n.*

N.B.: $\mathcal{B}^1(\mathfrak{h}_{\mathcal{S}})$ est l'espace des opérateurs de classe trace sur $\mathfrak{h}_{\mathcal{S}}$.

Les propriétés suivantes des applications de dynamique réduite découlent directement de leur définition.

Proposition 1. *Une application de dynamique réduite $\mathcal{L}$ est une application contractante, complètement positive et qui préserve la trace.*

Une conséquence immédiate du fait que $\mathcal{L}$ préserve la trace est que 1 est toujours valeur propre de l'application duale $\mathcal{L}^*$ (pour la dualité $\mathcal{B}^1(\mathfrak{h}_{\mathcal{S}})/\mathcal{B}(\mathfrak{h}_{\mathcal{S}})$) avec état propre l'opérateur identité.

L'application $\mathcal{L}_n$ décrit la dynamique effective de $\mathcal{S}$ sous l'influence du n-ème sous-système. Pour tout état initial ρ du système $\mathcal{S}$, (4) implique que

$$\rho(n) = \mathcal{L}_n\circ\mathcal{L}_{n-1}\circ\cdots\circ\mathcal{L}_1(\rho). \tag{6}$$

Dans le cas particulier d'interactions idéales, et si les $\rho_{\mathcal{E}_n}$ sont invariants pour la dynamique libre des $\mathcal{E}_n$, on a $\mathcal{L}_n \equiv \mathcal{L}$ pour tout n et (6) s'écrit simplement

$$\rho(n) = \mathcal{L}^n(\rho). \tag{7}$$

L'application $\mathcal{L}$ est le générateur d'un semi-groupe discret d'applications complètement positives et préservant la trace sur l'espace des états de $\mathcal{S}$. En d'autres termes, la structure particulière des systèmes en interactions répétées conduit à une description effective de la dynamique de $\mathcal{S}$ semblable à celle de l'approche markovienne, en partant de la description hamiltonienne du système total et sans être dans un quelconque régime limite.

L'étude du comportement à temps long de $\mathcal{S}$ se réduit alors à l'analyse de l'application de dynamique réduite $\mathcal{L}$ définie par (5). On peut également noter que, dans le cas où $\mathfrak{h}_{\mathcal{S}}$ est de dimension finie, le fait que $\mathcal{L}$ préserve la trace entraine que 1 est également valeur propre de $\mathcal{L}$ et il y a donc toujours au moins un état invariant lorsque $\mathcal{L}_n \equiv \mathcal{L}$. Par contre, lorsque $\mathfrak{h}_{\mathcal{S}}$ est de dimension infinie $\mathcal{L}$ peut avoir ou non un état invariant (cf Sections 2.1 et 2.2).

L'étude générale des systèmes en interactions répétées fait l'objet du Chapitre 1 dans lequel plusieurs situations seront considérées: interactions idéales dans la Section 1.3, avec un alea supplémentaire dans la Section 1.4, et lorsque $\mathcal{S}$ est couplé à un réservoir additionnel dans la Section 1.5. On illustrera les divers résultats à travers un exemple simple dans la Section 1.6. Les résultats présentés dans ce chapitre proviennent des articles [BJM1, BJM3, BJM4].

Deux modèles spécifiques de systèmes en interactions répétées sont présentés dans le Chapitre 2. Dans la Section 2.1, on considère un modèle pour l'expérience du "Maser à un atome" mentionnée ci-dessus, dans laquelle le système $\mathcal{S}$ représente un mode du champ électromagnétique dans un cavité interagissant avec un faisceau d'atomes à 2 niveaux via leur moment électrique dipolaire et dans l'approximation des ondes tournantes (c'est le hamiltonien de Jaynes-Cummings, voir l'équation (2.1). On étudie ici le problème du retour à l'équilibre: peut-on thermaliser un mode du champ dans une cavité au moyen d'atomes à 2 niveaux si ces derniers sont intialement à l'équilibre thermique? Dans la Section 2.2 le petit système est un électron sans spin, dans l'approximation des liaisons fortes (i.e. sur $\ell^2(\mathbb{Z})$), soumis à un champ électrique constant, et qui va interagir avec une chaîne d'atomes à 2 niveaux à l'équilibre thermique. Lorsque l'électron est seul, les oscillations de Bloch empêchent un courant de s'établir dans le système et on montre que l'interaction avec cet environnement thermique supprime ces oscillations et conduisent à un courant stationnaire. Les résultats présentés dans ces deux sections proviennent respectivement des articles [BP] et [BDP].

Systèmes IR dans divers régimes limites

Dans tous les travaux décrits dans ce mémoire la seule limite considérée est celle des temps longs: tous les paramètres du modèle sont fixes et on s'intéresse au comportement du système lorsque le nombre n d'interactions tend vers l'infini. Mis à part ce régime de temps long, les systèmes avec interactions répétées ont également été étudié dans la limite des temps continus, c'est-à-dire lorsque le temps d'interaction τ tend vers zéro (avec un changement d'échelle approprié dans l'interaction) [AJ2, APa, Pe1, Pe2], ainsi que dans la limite habituelle du couplage faible [AJ1].

Le premier travail dans cette direction est [APa]. Les auteurs montrent que dans la limite des temps continus, et à température nulle (les sous-systèmes $\mathcal{E}_n$ sont supposés être initialement dans un état pur), la dynamique effective de $\mathcal{S}$ converge vers un semi-groupe continu d'applications complètement positives associé à une équation de Langevin quantique dans lequel l'environnement est décrit à l'aide de bruits quantiques (à température nulle). En particulier, ce travail permet de "justifier" des équations du type Langevin comme étant des limites continues

de certaines évolutions hamiltoniennes. Ce travail a ensuite été étendu dans [AJ2] au cas de la température positive conduisant ainsi à des "bruits quantiques thermiques".

La limite des temps continus dans les systèmes avec interactions répétées a également été utilisée dans le contexte des mesures continues. Des expériences récentes d'optique quantique [G-al], basées sur des mesures indirectes sur l'environnement, ont montré l'évolution aléatoire de l'état d'un système ouvert quantique et en particulier des sauts quantiques. Les modèles correspondants sont décrits par des équations différentielles stochastiques appelées *équations de Schrödinger stochastiques* ou *équations de Belavkin*, et leurs solutions des *trajectoires quantiques*. Dans le cadre des interactions répétées l'idée est d'effectuer une mesure sur les sous-systèmes $\mathcal{E}_n$ juste après leur interaction avec $\mathcal{S}$. De tels systèmes sont alors un analogue discret des processus de mesure continue. Dans les travaux [Pe1, Pe2], l'auteur montre que les processus à temps discret obtenus par des interactions répétées suivies de mesures convergent dans la limite des temps continus vers des solutions d'équations de Schrödinger stoschastiques.

Finalement, la dynamique effective du petit système $\mathcal{S}$ dans des modèles avec interactions répétées a également été étudiée dans certains régimes limites reliés à la limite de couplage faible de van Hove, et dans lesquels non seulement le temps d'interaction τ peut tendre vers zéro, mais où la constante de couplage λ entre également en jeu [AJ1]. Le résultat de telles limites est une évolution effective markovienne en temps continu, gouvernée par certains générateurs de Lindblad dépendant de l'interaction et du régime limite considéré. En particulier, les auteurs montrent que n'importe quel générateur de Lindblad peut être obtenu à partir d'un modèle simple d'interactions répétées (les sous-systèmes sont identiques et de dimension finie) dans le régime $\tau \to 0$ et $\lambda = 1/\sqrt{\tau}$.

Hamiltoniens sur l'espace de Fock symmétrique

Dans le troisième chapitre de ce mémoire on s'intéresse à des hamiltoniens linéaires ou quadratiques sur l'espace de Fock symmétrique (décrivant l'environnement), et interagissant éventuellement avec un petit système confiné. Dans ce chapitre on se place dans l'approche hamiltonienne des systèmes ouverts et on étudie les propriétés du hamiltonien du système total. Les résultats présentés proviennent des articles [BD1, BD2]. Formellement, les deux classes de hamiltoniens auxquels on s'intéresse peuvent s'écrire sous la forme

$$H_1 = H_{\mathcal{S}} \otimes \mathbb{1} + \mathbb{1} \otimes \int h(k) a^*(k) a(k) \mathrm{d}k + \int v(k) \otimes a^*(k) \mathrm{d}k + \int v(k)^* \otimes a(k) \mathrm{d}k,$$

agissant sur $\mathcal{H} = \mathcal{H}_{\mathcal{S}} \otimes \Gamma_+(\mathfrak{h})$ où $\mathfrak{h} = L^2(\mathbb{R}^d; \mathbb{C}^n)$, et

$$H_2 = \int h(k) a^*(k) a(k) \mathrm{d}k + \frac{1}{2} \int \big(v(k,k') a^*(k) a^*(k') + \bar{v}(k,k') a(k) a(k') \big) \mathrm{d}k \mathrm{d}k' + c,$$

agissant sur $\mathcal{H} = \Gamma_+(\mathfrak{h})$ où $\mathfrak{h} = L^2(K, \mathrm{d}k)$ avec $(K, \mathrm{d}k)$ un espace mesuré. Les hamiltoniens du type H_1 seront appelés hamiltoniens de *spin-boson généralisés* (ils sont parfois appelés hamiltoniens de Pauli-Fierz [DG1, DJ1, GGM, Ge]) et on les étudie dans la Section 3.1. Les hamiltoniens du type H_2 seront eux appelés hamiltoniens de *Bogoliubov* et étudiés dans la Section 3.2.

Ces hamiltoniens sont utilisés en tant que versions simplifiées de hamiltoniens apparaissant dans divers contextes en physique quantique telles que l'approximation dipolaire de la QED non-relativiste. Dans les deux cas, notre objectif est d'étudier ces hamiltoniens sous les conditions les plus générales possibles. Par exemple, l'opérateur d'"interaction" v sera souvent défini uniquement au sens des formes. En ce qui concerne les hamiltoniens de spin-boson généralisés, on étend un résultat de type Théorème HVZ concernant le spectre essentiel et prouvé dans [DG1] ainsi qu'un théorème d'existence d'état fondamental obtenu dans [Ge], en considérant des hypothéses plus générales. Pour les hamiltoniens de Bogoliubov, à cause du terme quadratique dans l'"interaction", donner un sens aux opérateurs de la forme H_2 (en tant qu'opérateurs auto-adjoints) n'est déjà pas si évident lorsque l'on considère des v assez généraux. Par exemple, nous verrons que dans certaines situations il est nécessaire d'introduire un contre-terme infini dans la définition de H_2, c'est-à-dire que la constante c subit une renormalisation infinie.

Un peu de matrices aléatoires

Dans le Chapitre 4 on présente enfin deux résultats concernant les matrices aléatoires. Bien qu'a priori pas directement reliées aux systèmes ouverts quantiques celles-ci apparaissent naturellement lorsque l'on étudie des systèmes en interactions répétées dans un contexte alétoire. Cet alea peut avoir des origines diverses comme des temps d'interaction aléatoires, des états initiaux aléatoires pour les sous-systèmes $\mathcal{E}_n$ (via leur température par exemple), etc. Dans tous les cas, chaque interaction est alors décrite par une *application de dynamique réduite aléatoire* $\mathcal{L}(\omega)$. L'équation (6) montre que l'étude de la limite à temps long de la dynamique réduite revient à comprendre le produit infini d'applications aléatoires $\mathcal{L}(\omega)$ vérifiant certaines propriétés. Au moins lorsque le petit système est de dimension finie cela correspond bien à un produit de matrices aléatoires qui sera l'objet de la Section 4.1. Les résultats concernant la convergence de ces produits de matrices aléatoires proviennent de l'article [BJM2].

Finalement, dans la Section 4.2 on présente brièvement un autre résultat sur les matrices aléatoires obtenu dans [BG] qui est cependant disjoint des autres travaux présentés ici. Ce résultat concerne l'invertibilité de grandes matrices alétoires dans le cas où les coefficients de celles-ci sont des variables indépendantes mais pas nécessairement identiquement distribuées.

Introduction

Open Systems

The general framework of most of the results presented in this thesis is that of Open Quantum Systems. By opposition with a closed/isolated system, a quantum system $\mathcal{S}$ is called *open* when it is in contact (interacts) with one or several other systems often called reservoir(s) or environment. A typical example of an open quantum system is that of an atom interacting with the quantized electromagnetic field, but there are numerous example coming from various branches of physics such as solid state phyics, quantum optics, etc. They also are the basic paradigms of (non-)equilibrium quantum statistical mechanics.

Although such a system could be considered as a bigger bipartite (or several partite) closed system, the various parts are considered on a different level. First, one should think of the system $\mathcal{S}$ as much smaller than its environment, e.g. finite versus infinite number of degrees of freedom. Second, since the environment is usually very big, it is often very hard to get precise information on it, and one is rather interested in understanding the effects of this huge environment on the dynamics of the small system $\mathcal{S}$, like dissipation phenomena.

Two distinct approaches to the study of open systems have been used in the literature, both in mathematics and in physics: the *Hamiltonian* approach and the *Markovian* approach.

The Hamiltonian approach is more fundamental in the sense that one considers a complete description of the microscopic dynamics of the entire system: small system + environment. Both parts are described by their state space $\mathcal{H}_{\mathcal{S}}$ and $\mathcal{H}_{\rm env}$ and Hamiltonians $H_{\mathcal{S}}$ and $H_{\rm env}$. The coupled system is then described by the total state space $\mathcal{H} = \mathcal{H}_{\mathcal{S}} \otimes \mathcal{H}_{\rm env}$ and a Hamiltonian of the form $H = H_{\mathcal{S}} \otimes \mathbb{1}_{\rm env} + \mathbb{1}_{\mathcal{S}} \otimes H_{\rm env} + H_{\rm int}$ where $H_{\rm int}$ describes the interaction between the two parts. The goal is then to understand the behaviour of the total system under the dynamics generated by H, using the traditional tools of quantum mechanics (spectral analysis and scattering theory), see e.g. [BFS, DG1, DG2, JP1, JP2, MMS].

On the other hand, in the markovian approach, one gives up the idea of describing the entire system and concentrates on the small system $\mathcal{S}$ only. The philosophy behind this is that the environment is too complicated to describe (in the Hamiltonian approach only very simple environment, like free fields or ideal bose and fermi gases, have been succesfully considered) or even one does not have access to it. Instead, one considers only the effective dynamics on the system $\mathcal{S}$ which is induced by the environment. This evolution is governed by a quantum master

equation of the form

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = \mathcal{L}(\rho) := -\mathrm{i}[H,\rho] + \frac{1}{2}\sum_j (2L_j^*\rho L_j - L_j^* L_j \rho - \rho L_j^* L_j),$$

which defines a semi-group of completely positive, trace preserving maps on the state space of $\mathcal{S}$. There are two ways to get such an effective dynamics: either as a scaling limit of the microscopic (Hamiltonian) dynamics of the coupled system (e.g. the van Hove weak coupling limit [Da1, Da2, DJ2, DF]), or as the result of driving the system $\mathcal{S}$ with stochastic forces (quantum Langevin equation [HP]). For a more complete introduction to the subject, we refer to [AJP].

In both approaches the questions one is interested in are however similar and concern the large time behaviour of the system: is there a (unique) invariant state (for $\mathcal{S}$ or the entire system)? do initial states converge towards this invariant state? at what speed? what are the properties of the invariant state?

Repeated interaction systems

The main part of this thesis, Chapters 1 and 2, will be devoted to a particular class of open systems which, motivated by several new physical applications as well as by their attractive mathematical structure, has recently become very popular in the literature: the so-called repeated interaction (RI) systems. In RI systems the environment consists in a sequence of independent subsystems $\mathcal{E}_1$, $\mathcal{E}_2, \ldots$ The small system $\mathcal{S}$ interacts with $\mathcal{E}_1$ during some time interval $[0, \tau_1[$, then with $\mathcal{E}_2$ during an interval $[\tau_1, \tau_1 + \tau_2[$, etc. While $\mathcal{S}$ interacts with $\mathcal{E}_n$, the other elements of the environment evolve freely according to their intrinsic (uncoupled) dynamics. Thus, the evolution of the joint system $\mathcal{S} + \mathcal{E}_1 + \cdots$ is completely determined by the sequence $\tau_1, \tau_2, \ldots$, the individual dynamics of each $\mathcal{E}_n$ and the coupled dynamics of each pair $\mathcal{S} + \mathcal{E}_n$. Note that since the environment consists in an infinite number of subsystems one does not need to take each of them extended in order to have a "large" environment.

The theoretical and practical importance of repeated interaction quantum systems is exemplified by systems of radiation-matter coupling, where atoms interact with modes of the quantized electromagnetic field. In this setting, the system $\mathcal{S}$ describes one or several modes of the field in a cavity and the environment represents a beam of atoms $\mathcal{E}_n$ that are injected into the cavity. So-called "One-Atom Masers", where the beam is tuned in such a way that at each given moment a *single atom* is inside a microwave cavity have been experimentally realized in laboratories [MWM, WVHW].

In the simplest RI models all the interactions are identical (we shall speak of ideal repeated interactions). Namely, each $\mathcal{E}_n$ is a copy of some $\mathcal{E}$, $\tau_n \equiv \tau$, and the dynamics of $\mathcal{E}_n$ and $\mathcal{S} + \mathcal{E}_n$ are independent of n, generated by some Hamiltonians $H_\mathcal{E}$, $H_{\mathcal{SE}}$. The particular structure of RI systems then makes them at the same time Hamiltonian (with a time-dependent Hamiltonian) and Markovian (for discrete times $n\tau$, the effective dynamics of $\mathcal{S}$ is described by a discrete semigroup of completely positive maps, see (15)). For that reason, we believe that these models provide a useful framework to develop our understanding of various aspects of the quantum statistical mechanics of open systems.

Mathematical description of RI systems

We now describe more precisely the mathematical setup of repeated interaction systems, and explain how their particular structure allows one to derive a Markovian, discrete-time, dynamics for $\mathcal{S}$ from the Hamiltonian dynamics of the entire system.

The various elements needed to describe a RI system are:

1. the Hilbert space $\mathfrak{h}_{\mathcal{S}}$ and Hamiltonian $h_{\mathcal{S}}$ describing the small system $\mathcal{S}$ "alone",
2. Hilbert spaces $\mathfrak{h}_{\mathcal{E}_n}$ and Hamiltonians $h_{\mathcal{E}_n}$ describing the various subsytems $\mathcal{E}_n$,
3. a sequence of duration times $(\tau_n)_n$ where $\tau_n \geq \tau > 0$ for any n and some given τ. The time τ_n is the amount of time the system $\mathcal{S}$ interacts with the subsytem $\mathcal{E}_n$,
4. operators v_n describing the interactions between $\mathcal{S}$ and the subsystems $\mathcal{E}_n$.

The Hilbert space describing the RI system is then

$$\mathfrak{h} := \mathfrak{h}_{\mathcal{S}} \otimes \mathfrak{h}_{\mathrm{env}}, \qquad \mathfrak{h}_{\mathrm{env}} := \bigotimes_{n\geq 1} \mathfrak{h}_{\mathcal{E}_n}.$$

We also denote $t_n := \tau_1 + \cdots + \tau_n$. During the time interval $[t_{n-1}, t_n)$, the system $\mathcal{S}$ interacts with the n-th subsystem, i.e. $\mathcal{E}_n$, and with none of the others. The full evolution of the system is thus described by the Hamiltonian

$$h(t) = h_{\mathcal{S}} + \sum_{n\geq 1} h_{\mathcal{E}_n} + \sum_{n\geq 1} \chi_n(t) v_n, \tag{8}$$

where χ_n is the characteristic function of the interval $[t_{n-1}, t_n)$. We will use the following notation:

$$h_n := h_{\mathcal{S}} + h_{\mathcal{E}_n} + v_n, \quad \text{and} \quad \tilde{h}_n := h_n + \sum_{k\neq n} h_{\mathcal{E}_k}.$$

Note that $h(t) \equiv \tilde{h}_n$ when $t \in [t_{n-1}, t_n)$. We have also omitted trivial factors $\mathbb{1}$, e.g. $h_{\mathcal{S}}$ should be $h_{\mathcal{S}} \otimes \mathbb{1}_{\mathrm{env}}$.

Given any initial state ρ for the system $\mathcal{S}$ at time $t = 0$ (i.e. ρ is a positive trace class operator on $\mathfrak{h}_{\mathcal{S}}$ with trace one), and a sequence $(\rho_{\mathcal{E}_n})_n$ of initial states for the subsystems $\mathcal{E}_n$, the state of the total repeated interaction system after n interactions is thus given by

$$\rho^{\mathrm{tot}}(n) := \mathrm{e}^{-\mathrm{i}\tau_n \tilde{h}_n} \cdots \mathrm{e}^{-\mathrm{i}\tau_1 \tilde{h}_1} \left(\rho_{\mathcal{S}} \otimes \bigotimes_{k\geq 1} \rho_{\mathcal{E}_k} \right) \mathrm{e}^{\mathrm{i}\tau_1 \tilde{h}_1} \cdots \mathrm{e}^{\mathrm{i}\tau_n \tilde{h}_n}.$$

We are mainly interested in the system $\mathcal{S}$ (see however Section 1.3 for more general observables), i.e. in expectation values of observables of the form

$$O = O_{\mathcal{S}} \otimes \bigotimes_{k\geq 1} \mathbb{1}_{\mathcal{E}_k}.$$

Therefore, we are rather interested in $\rho(n) := \mathrm{Tr}_{\mathfrak{h}_{\mathrm{env}}}(\rho^{\mathrm{tot}}(n))$, the reduced density matrix on $\mathcal{S}$. It is defined as the unique state on $\mathfrak{h}_{\mathcal{S}}$ such that, for any observable $O_{\mathcal{S}}$ on $\mathcal{S}$,

$$\mathrm{Tr}_{\mathfrak{h}_{\mathcal{S}}} \left(\rho(n) O_{\mathcal{S}} \right) = \mathrm{Tr}_{\mathfrak{h}} \left(\rho^{\mathrm{tot}}(n) \times \left(O_{\mathcal{S}} \otimes \bigotimes_{k\geq 1} \mathbb{1}_{\mathcal{E}_k} \right) \right).$$

To obtain the state $\rho(n)$ of the system $\mathcal{S}$ after these n interactions we thus take the following partial trace:

$$\rho(n) := \mathrm{Tr}_{\mathfrak{h}_{\mathrm{env}}} \left[\mathrm{e}^{-\mathrm{i}\tau_n \tilde{h}_n} \cdots \mathrm{e}^{-\mathrm{i}\tau_1 \tilde{h}_1} \left(\rho \otimes \bigotimes_{k\geq 1} \rho_{\mathcal{E}_k} \right) \mathrm{e}^{\mathrm{i}\tau_1 \tilde{h}_1} \cdots \mathrm{e}^{\mathrm{i}\tau_n \tilde{h}_n} \right]. \tag{9}$$

Of course, the above calculation is a little bit formal. Indeed, in order to define a countable tensor product of Hilbert spaces one should specify a stabilizing sequence, i.e. a sequence of vectors $(\psi_n)_n$ where $\psi_n \in \mathfrak{h}_{\mathcal{E}_n}$. The Hilbert space $\mathfrak{h}_{\mathrm{env}}$ is then obtained by taking the completion of the vector space of finite linear combinations of the form $\otimes_{n\geq 1}\phi_n$, where $\phi_n \in \mathfrak{h}_{\mathcal{E}_n}$, $\phi_n = \psi_n$ except for finitely many indices, in the norm induced by the inner product

$$\langle \otimes_n \varphi_n, \otimes_n \phi_n \rangle = \prod_n \langle \varphi_n, \phi_n \rangle_{\mathfrak{h}_{\mathcal{E}_n}}.$$

In general, the infinite tensor product $\bigotimes_{k\geq 1} \rho_{\mathcal{E}_k}$ then does not make sense. It is however easy to make sense of the formal expression (9). Indeed, at time t_n only the n first elements of the environment have played a role so that we can replace $\bigotimes_{k\geq 1} \rho_{\mathcal{E}_k}$ by $\rho_{\mathrm{env}}^{(n)} := \bigotimes_{k=1}^{n} \rho_{\mathcal{E}_k}$ and the partial trace over the environment by the partial trace over the finite tensor product $\mathfrak{h}_{\mathrm{env}}^{(n)} := \bigotimes_{k=1}^{n} \mathfrak{h}_{\mathcal{E}_k}$, i.e.

$$\rho(n) = \mathrm{Tr}_{\mathfrak{h}_{\mathrm{env}}^{(n)}} \left[\mathrm{e}^{-\mathrm{i}\tau_n \tilde{h}_n} \cdots \mathrm{e}^{-\mathrm{i}\tau_1 \tilde{h}_1} \left(\rho \otimes \bigotimes_{k=1}^{n} \rho_{\mathcal{E}_k} \right) \mathrm{e}^{\mathrm{i}\tau_1 \tilde{h}_1} \cdots \mathrm{e}^{\mathrm{i}\tau_n \tilde{h}_n} \right]. \tag{10}$$

Remark. *Another possibility would be to define the infinite tensor product "with respect to the sequence of states $(\rho_{\mathcal{E}_n})_n$". For that purpose one should first represent the states $\rho_{\mathcal{E}_n}$ as vector states with vector Ψ_n (using the GNS representation), and then consider the stabilizing sequence $(\Psi_n)_n$. This then leads to the "Liouvillian" description of the RI system which will be presented in details in Section 1.1.*

The very particular structure of the repeated interaction systems allows us to rewrite $\rho(n)$ in a much more convenient way. The two main characteristics of these systems are:

1. The various subsystems of the environment do not interact directly (only via $\mathcal{S}$), i.e. $[h_{\mathcal{E}_k}, h_{\mathcal{E}_n}] = 0$ for any $k \neq n$,
2. The system $\mathcal{S}$ interacts only once with each subsystem $\mathcal{E}_n$, and with only one at a time, i.e. $[h_{\mathcal{E}_k}, h_n] = 0$ for any $k \neq n$.

We therefore have the following decomposition which serves to isolate the dynamics of the subsystems which do not interact at a given time

$$\mathrm{e}^{-\mathrm{i}\tau_n \tilde{h}_n} \cdots \mathrm{e}^{-\mathrm{i}\tau_1 \tilde{h}_1} = u_n^- \times \mathrm{e}^{-\mathrm{i}\tau_n h_n} \cdots \mathrm{e}^{-\mathrm{i}\tau_1 h_1} \times u_n^+, \tag{11}$$

where

$$u_n^- = \exp\left(-\mathrm{i} \sum_{k=1}^{n-1} (t_n - t_k) h_{\mathcal{E}_k} \right) \text{ and } u_n^+ = \exp\left(-\mathrm{i} \sum_{k=2}^{n} t_{k-1} h_{\mathcal{E}_k} - \mathrm{i} t_n \sum_{k>n} h_{\mathcal{E}_k} \right)$$

are respectivley the propagators at time t_n of the subsystems $\mathcal{E}_k$ after their interaction with $\mathcal{S}$, and the one before their interaction. Inserting (11) into (10) we get

$$\rho(n) = \mathrm{Tr}_{\mathfrak{h}_{\mathrm{env}}^{(n)}} \left[\mathrm{e}^{-\mathrm{i}\tau_n h_n} \cdots \mathrm{e}^{-\mathrm{i}\tau_1 h_1} \left(\rho \otimes \bigotimes_{k=1}^{n} \rho_{\mathcal{E}_k}(t_{k-1}) \right) \mathrm{e}^{\mathrm{i}\tau_1 h_1} \cdots \mathrm{e}^{\mathrm{i}\tau_n h_n} \right],$$

where $\rho_{\mathcal{E}_k}(t_{k-1}) = \mathrm{e}^{-\mathrm{i}t_{k-1}h_{\mathcal{E}_k}} \rho_{\mathcal{E}_k} \mathrm{e}^{\mathrm{i}t_{k-1}h_{\mathcal{E}_k}}$ is the state of the k-th subsytem when it begins to interact with $\mathcal{S}$. Of course, this formula is simpler if $\rho_{\mathcal{E}_k}$ is invariant under the free dynamics of $\mathcal{E}_k$, e.g. a thermal state, which will often be the case.

It is now easy to see that the evolution of $\mathcal{S}$ is Markovian: the state $\rho(n)$ only depends on the state $\rho(n-1)$ and the n-th interaction. More precisely, one can write

$$\rho(n) = \mathcal{L}_n(\rho(n-1)), \tag{12}$$

where

$$\mathcal{L}_n(\rho) := \mathrm{Tr}_{\mathfrak{h}_{\mathcal{E}_n}} \left[\mathrm{e}^{-\mathrm{i}\tau_n h_n} \, \rho \otimes \rho_{\mathcal{E}_n}(t_{n-1}) \, \mathrm{e}^{\mathrm{i}\tau_n h_n} \right]. \tag{13}$$

Definition 2. *The map $\mathcal{L}_n$, from $\mathcal{B}^1(\mathfrak{h}_{\mathcal{S}})$ to itself, is called the* reduced dynamics map *(RDM) at time n.*

Note: $\mathcal{B}^1(\mathfrak{h}_{\mathcal{S}})$ denotes the space of trace class operators on $\mathfrak{h}_{\mathcal{S}}$.

The following properties of a reduced dynamics map follow directly from its definition.

Proposition 2. *A RDM $\mathcal{L}$ is a contracting, completely positive and trace preserving map.*

As a corollary of the trace preserving property, 1 is always an eigenvalue of the dual map $\mathcal{L}^*$ (for the $\mathcal{B}^1(\mathfrak{h}_{\mathcal{S}})/\mathcal{B}(\mathfrak{h}_{\mathcal{S}})$ duality) with eigenstate the identity operator.

The map $\mathcal{L}_n$ describes the effective evolution of $\mathcal{S}$ under the influence of the n-th subsystem. Using (12), we therefore have for any initial state ρ of the small system $\mathcal{S}$

$$\rho(n) = \mathcal{L}_n \circ \mathcal{L}_{n-1} \circ \cdots \circ \mathcal{L}_1(\rho). \tag{14}$$

In the particular case of ideal interactions, and if the $\rho_{\mathcal{E}_n}$ are invariant for the dynamics of $\mathcal{E}_n$, we then have $\mathcal{L}_n \equiv \mathcal{L}$ for all n and (14) becomes simply

$$\rho(n) = \mathcal{L}^n(\rho). \tag{15}$$

The map $\mathcal{L}$ is the discrete-time generator of a semi-group of completely positive, trace preserving maps on the state space of $\mathcal{S}$. In other words, the particular structure of RI systems leads to an effective description of $\mathcal{S}$ as in the markovian approach, starting from an Hamiltonian description and without any further scaling limit.

The study of the large time behaviour of $\mathcal{S}$ reduces to the analysis of the RDM $\mathcal{L}$ defined in (13). Note that, in the case where $\mathfrak{h}_{\mathcal{S}}$ has finite dimension, the fact that $\mathcal{L}$ is trace preserving implies that 1 is also an eigenvalue of $\mathcal{L}$ so that there is always an invariant state when $\mathcal{L}_n \equiv \mathcal{L}$. However, if $\mathfrak{h}_{\mathcal{S}}$ has infinite dimension, $\mathcal{L}$ may have an invariant state or not (see Sections 2.1 and 2.2).

The general analysis of repeated interaction systems will be the purpose of Chapter 1 where various situations will be considered: ideal interactions in Section 1.3, where some randomness is added in Section 1.4, and when $\mathcal{S}$ is connected to an additional reservoir in Section 1.5. We also present an instructive and illustrative (toy) example in Section 1.6. The results presented in this chapter are taken from the articles [BJM1, BJM3, BJM4]. Two more concrete models of repeated interaction systems will then be presented in Chapter 2. In Section 2.1 we consider a mathematical model for the "One-Atom Maser" experiment we have mentioned, where the small system $\mathcal{S}$ is one mode of the quantized electromagnetic field in a cavity and interacts with a beam of 2-level atoms through its electric dipole moment and in the rotating wave approximation (this is the well-known Jaynes-Cummings Hamiltonian, cf equation (2.1), and we consider the problem of thermal relaxation: is it possible to thermalize a mode of a QED cavity by means of 2-level atoms if the latter are initially at thermal equilibrium? In Section 2.2 the small system consists in a spinless electron in the single band tight-binding approximation, and subject to an homogeneous static electric field, which will interact with a chain of two-level atoms in thermal equilibrium. For the electron alone Bloch oscillations prevent a current from being set up in the system and we show that the interaction with this thermal environment will suppress the Bloch oscillations and lead to a steady current. The results presented in these two sections come respectively from the articles [BP] and [BDP].

RI systems in various limiting regimes

In all the works described in this thesis, the only limit which is considered is the large time limit. In other words, all the parameters are fixed and we consider a large number n of interactions, eventually infinite. Besides this usual large time regime, repeated interaction quantum systems have also been used in the continuous time limit, i.e. when the interaction time τ goes to zero (with approriate rescaling in the interaction Hamiltonian) [AJ2, APa, Pe1, Pe2], and also in the usual weak coupling limit [AJ1].

The first work in this direction is [APa]. It is shown than in the continuous time limit, and at zero temperature (the subsystems $\mathcal{E}_n$ are assumed to be initially in a pure state), the effective dynamics of $\mathcal{S}$ converges towards a continuous semigroup of completely positive maps associated to a quantum Langevin equation and where the environment is described by means of quantum noises (at zero temperature). In particular, it justifies Langevin-type equations as continuous limits of certain Hamiltonian evolution. This work has been further extended in [AJ2] to the positive temperature situation and leading to "thermal quantum noises".

The continuous limit of repeated interaction systems have also been used in the framework of continuous measurement. Some recent experiments in quantum optics [G-al], and based on an indirect measurement on the environment, have shown the random evolution of the state of an open quantum system and in particular quantum jumps. The corresponding models are described by stochastic differential equations usually called *Stochastic Schrödinger equations* or *Belavkin equations*, and their solutions *quantum trajectories*. In the framework of repeated interactions, the idea is to make measurements on the subsystems $\mathcal{E}_n$ right after their interaction with $\mathcal{S}$. Such systems are then a discrete analogue of the continuous measurement procedure. It as been shown in [Pe1, Pe2] that the discrete-time processes described by repeated interactions and measurement converge in the continuous time limit to solutions of Stochastic Schrödinger equations.

Finally, the effective evolution of the small system $\mathcal{S}$ in repeated interaction models has also been investigated in certain regimes related to the van Hove weak coupling limit and where not only the interaction time τ can go to zero but the coupling constant λ is involved as well [AJ1]. The result is a continuous Markovian effective evolution, driven by certain Lindblad generators depending on the interaction and on the asymptotic regimes considered. In particular, it is shown that any Lindblad generator can be derived from a simple (the subsystems are identical and finite dimensional) repeated interaction system in the regime $\tau \to 0$ and $\lambda = 1/\sqrt{\tau}$.

Hamiltonians on the bosonic Fock space

In the third Chapter of this thesis we consider linear or quadratic Hamiltonians on the bosonic Fock space (describing the environment), and possibly interacting with a confined small system. In this chapter, we stick to the Hamiltonian approach of open quantum systems and study properties of the Hamiltonian of the full system. The results presented here come from the articles [BD1, BD2]. Formally, the two classes of Hamiltonians we consider can be written as:

$$H_1 = H_{\mathcal{S}} \otimes \mathbb{1} + \mathbb{1} \otimes \int h(k) a^*(k) a(k) \mathrm{d}k + \int v(k) \otimes a^*(k) \mathrm{d}k + \int v(k)^* \otimes a(k) \mathrm{d}k,$$

acting on $\mathcal{H} = \mathcal{H}_{\mathcal{S}} \otimes \Gamma_+(\mathfrak{h})$ where $\mathfrak{h} = L^2(\mathbb{R}^d; \mathbb{C}^n)$, and

$$H_2 = \int h(k) a^*(k) a(k) \mathrm{d}k + \frac{1}{2} \int \big(v(k,k') a^*(k) a^*(k') + \bar{v}(k,k') a(k) a(k') \big) \mathrm{d}k \mathrm{d}k' + c,$$

acting on $\mathcal{H} = \Gamma_+(\mathfrak{h})$ where $\mathfrak{h} = L^2(K, \mathrm{d}k)$ for some measure space $(K, \mathrm{d}k)$. Hamiltonians of the type H_1 will be called *generalized spin-boson* Hamiltonians (they are also sometimes called Pauli-Fierz Hamiltonians [DG1, DJ1, GGM, Ge]) and are studied in Section 3.1. Hamiltonians of the type H_2 will be called *Bogoliubov* Hamiltonians and considered in Section 3.2.

These Hamiltonians are used as simplified versions of Hamiltonians arising in various contexts of quantum physics (the dipole approximation of non-relativistic QED is of this form). In both cases, our goal will be to analyze these Hamiltonians under as general conditions as possible. For example, the "coupling" operators v will often be defined only as unbounded quadratic forms. For the generalized spin-boson Hamiltonians, we extend the HVZ-type theorem of [DG1] about the essential spectrum and the theorem about the existence of a ground state from [Ge] to a larger class of Hamiltonians. For Bogoliubov Hamiltonians, due to the quadratic terms in the "interaction part", already giving a sense to operators of the form H_2 is not so obvious if we allow for general v's. For example, we shall see that there are situations where one needs to introduce an infinite counterterm in the definition of H_2, i.e. the constant c undergoes an infinite renormalization.

A little bit of random matrices

In Chapter 4 we finally present two results concerning random matrices. Although a priori not directly related to open quantum systems they appear naturally when studying repeated interaction systems in a random context. This randomness can have various origins like random

interaction times, random initial states of the subsystems $\mathcal{E}_n$ (through a random temperature for example), etc. In any case, each interaction is then described by a *random reduced dynamics map* $\mathcal{L}(\omega)$. From (14), one can see that the study of the large time limit for the reduced dynamics amounts to understand the infinite product of random maps $\mathcal{L}(\omega)$ satisfying certain properties. At least when the small system is finite dimensional, this indeed corresponds to a product of random matrices and will be the purpose of Section 4.1. The results concerning the convergence of products of random matrices come from the article [BJM2].

Finally, in Section 4.2 we briefly present another result about random matrices obtained in [BG] which is however disconnected from the other works presented here. This result concerns the singularity of large random matrices in the case where its entries are independent but not necessarily identically distributed.

Chapter 1

Repeated interaction systems in quantum mechanics

We start this chapter by giving another way to describe RI systems, using a "Liouvillian" formalism. We will start from the standard algebraic C^* description of quantum statistical mechanics and then explain how to rewrite things in a Hilbert space setting. This formalism will be important when dealing with leaky RIS (when an additional reservoir is added). It also allows one to consider rather general subsystems $\mathcal{E}$ of the environment (see the examples in [BJM1, BJM3]).

Contrary to the usual context of open systems, in RI systems the total Hamiltonian is (piecewise constant) time-dependent as we can see from (8). Hence the energy of the full system is not necessarily constant. It is constant during each interaction (where the total Hamiltonian is constant) but we may have energy changes when one switches from an interaction to the next one. In other words, the switch from one interaction to the other may require some external work. In Section 1.2 we show how to define an "external work" observable. We also consider entropy production in RI systems and relate it to the external work.

In Sections 1.3 to 1.5 we then analyze general RI systems in various contexts: ideal (or identical) interactions in Section 1.3, where we add some randomness in Section 1.4 and with some additional leak in Section 1.5. Unless explicitly mentioned, we will work in this Liouvillian formalism and we assume that $\mathcal{S}$ is a finite dimensional system, i.e. $\dim(\mathfrak{h}_{\mathcal{S}}) < +\infty$. Finally in Section 1.6 we analyze in detail a simple concrete (though instructive) example. Other examples may be found in [BJM1, BJM3, BJM4].

1.1 Liouvillian description of RI systems

In this section we give an alternative description of RI systems using the language of algebraic quantum statistical mechanics, and starting from the C^*- dynamical system formalism. There are several reason for that:

1. This allows for more general systems, e.g. take the subsystems $\mathcal{E}_n$ to be thermal reservoirs described by infinitely extended Fermi gas.

2. Include an extra-reservoir $\mathcal{R}$ with which $\mathcal{S}$ will interact (leaky RIS) and get a unified description of the full model.

3. Even if it is not our main concern, show how to construct the full system, including the infinite tensor product.

We first briefly recall some basic concepts of algebraic quantum statistical mechanics that we need here. We refer to e.g. [BR, P] for a more complete introduction to the subject. A C^*- dynamical system is a pair $(\mathfrak{A}, \alpha^t)$ where $\mathfrak{A}$ is a C^*- algebra (describing the observables of the physical system under consideration) and $t \mapsto \alpha^t$ is a strongly continuous group of $*$-automorphisms of $\mathfrak{A}$ (describing the evolution of the observables). A state of the system is described by a positive linear functional ϱ on $\mathfrak{A}$ satisfying $\varrho(\mathbb{1}) = 1$. Following [JP2], a triple $(\mathfrak{A}, \alpha^t, \varrho)$, where ϱ is an invariant state (i.e. $\varrho \circ \alpha^t \equiv \varrho$), is called a quantum dynamical system. As an example, if a quantum system is described in the Hamiltonian formalism by a Hilbert space $\mathcal{H}$, Hamiltonian H and density matrix ρ, then the corresponding quantum dynamical system is the triple $(\mathcal{B}(\mathcal{H}), \alpha^t, \varrho)$ where $\alpha^t(A) = \mathrm{e}^{\mathrm{i}tH} A \, \mathrm{e}^{-\mathrm{i}tH}$ and $\varrho(A) = \mathrm{Tr}(\rho A)$.

Each component $\# = \mathcal{S}, \mathcal{E}_n$ of the RI system will be described by a quantum dynamical system $(\mathfrak{A}_\#, \alpha^t_\#, \varrho_\#)$. The "reference" states $\varrho_\#$ determine the macroscopic properties of the systems, e.g. they are KMS states at some inverse temperature $\beta_\#$. We also assume that they are faithful states, i.e. for any $A \in \mathfrak{A}_\#$, $\varrho_\#(A^*A) = 0 \Rightarrow A = 0$ (this would correspond to $\rho > 0$).

To analyze the (time) asymptotic behaviour of the system, we will use a spectral approach. For that purpose, it is convenient to have a "Hilbert space description" of the system. Such a description is easy to obtain via the GNS-representation[a] $(\mathcal{H}_\#, \pi_\#, \Psi_\#)$ of the algebras $\mathfrak{A}_\#$ associated to the states $\varrho_\#$. Since the $\varrho_\#$ are faithful, the $\pi_\#$ are injections and we can identify $\mathfrak{A}_\#$ and $\pi_\#(\mathfrak{A}_\#)$ (we will therefore simply write A for $\pi(A)$). We set $\mathfrak{M}_\# = \pi_\#(\mathfrak{A}_\#)'' \subset \mathcal{B}(\mathcal{H}_\#)$, where $''$ denotes the double commutant. The $\mathfrak{M}_\#$ form the von Neumann algebras of observables. Finally, by construction the representative vectors $\Psi_\#$ are cyclic for $\mathfrak{M}_\#$, and we assume that they are also seperating vectors for $\mathfrak{M}_\#$, i.e. $A\Psi_\# = 0 \Rightarrow A = 0$ for any $A \in \mathfrak{M}_\#$ (note that since $\varrho_\#$ is faithful, this is automatic when $A \in \pi_\#(\mathfrak{A}_\#)$). Typically, the $\Psi_\#$ describe the equilibrium states at some fixed temperature $T_\# > 0$.

The free dynamics $\alpha^t_\#$ of each constituent is implemented in the GNS-representation by self-adjoint operators $L_\#$ called Liouvillians, i.e. the Heisenberg evolution of an observable $A \in \mathfrak{M}_\#$ at time t is given by $\mathrm{e}^{\mathrm{i}tL_\#} A \mathrm{e}^{-\mathrm{i}tL_\#}$. In other words we have $\pi_\#(\alpha^t_\#(A)) = \mathrm{e}^{\mathrm{i}tL_\#} \pi_\#(A) \mathrm{e}^{-\mathrm{i}tL_\#}$. Since the $\varrho_\#$ were invariant states, one can also chose the Liouville operators $L_\#$ so that $L_\# \Psi_\# = 0$ (actually such an $L_\#$ is unique).

As an illustration of the above formalism, let's see how all this work for finite systems. Consider a quantum system described by the Hilbert space $\mathfrak{h} = \mathbb{C}^n$, the Hamiltonian h and the invariant state $\rho = \sum \rho_j |\psi_j\rangle\langle\psi_j|$ where $\{\psi_j\}$ is an orthonormal basis of eigenvectors of h. The corresponding quantum dynamical system is described by the algebra of observables $\mathfrak{A} = M_n(\mathbb{C})$, the dynamics $\alpha^t(A) = \mathrm{e}^{\mathrm{i}th} A \, \mathrm{e}^{-\mathrm{i}th}$ and state $\varrho(A) = \mathrm{Tr}(\rho A)$. In the GNS representation, the Hilbert space, the observable algebra and the Liouville operator are then

[a] We recall that the GNS (Gelfand-Naimark-Segal) representation of a C^*- algebra $\mathfrak{A}$ associated to a state ϱ is a triple $(\mathcal{H}, \pi, \Psi)$ where $\mathcal{H}$ is a Hilbert space, π a $*$- algebra morphism from $\mathfrak{A}$ to $\mathcal{B}(\mathcal{H})$, and Ψ a unit vector in $\mathcal{H}$ such that $\{\pi(A)\Psi, \ A \in \mathfrak{A}\}$ is dense in $\mathcal{H}$ and $\varrho(A) = \langle \Psi, \pi(A)\Psi \rangle$ for any $A \in \mathfrak{A}$.

given by

$$\mathcal{H} = \mathfrak{h} \otimes \mathfrak{h}, \qquad \mathfrak{M} = \mathcal{B}(\mathfrak{h}) \otimes \mathbb{1}, \qquad L = h \otimes \mathbb{1} - \mathbb{1} \otimes h,$$

and the representative vector Ψ by $\Psi = \sum \sqrt{\rho_j}\, \psi_j \otimes \psi_j$. (The morphism π is defined as $\pi(A) = A \otimes \mathbb{1}$.)

Each component of the RI system is thus now described by a von Neumann algebra (of observables) $\mathfrak{M}_\#$ acting on the Hilbert space $\mathcal{H}_\#$, a self-adjoint operator $L_\#$ on $\mathcal{H}_\#$ which implements the dynamics and a unit vector $\Psi_\# \in \mathcal{H}_\#$ which represents some reference invariant state. The Hilbert space $\mathcal{H}_{\rm env}$ for the environment is then the infinite tensor product of factors $\mathcal{H}_{\mathcal{E}_n}$, taken with respect to the stabilizing sequence $(\Psi_{\mathcal{E}_n})_n$. The vector $\Psi_{\rm env} = \otimes_{n\geq 1}\Psi_{\mathcal{E}_n}$ is the reference vector for the environment, and the algebra of observables $\mathfrak{M}_{\rm env}$ of the environment is the von Neumann algebra $\mathfrak{M}_{\rm env} = \otimes_{n\geq 1}\mathfrak{M}_{\mathcal{E}_n}$ acting on $\mathcal{H}_{\rm env}$, which is obtained by taking the weak closure of finite linear combinations of operators $\otimes_{n\geq 1}A_n$, where $A_n \in \mathfrak{M}_{\mathcal{E}_n}$ and $A_n = \mathbb{1}_{\mathcal{H}_{\mathcal{E}_n}}$ except for finitely many indices.

In summary, the non-interacting system is described by a von Neumann algebra $\mathfrak{M} = \mathfrak{M}_{\mathcal{S}} \otimes \mathfrak{M}_{\rm env}$, acting on the Hilbert space $\mathcal{H} = \mathcal{H}_{\mathcal{S}} \otimes \mathcal{H}_{\rm env}$, and its dynamics is generated by the (free) Liouvillian

$$L_0 = L_{\mathcal{S}} + \sum_{n\geq 1} L_{\mathcal{E}_n}.$$

The operators governing the couplings between $\mathcal{S}$ and $\mathcal{E}_n$ are given by operators

$$V_n \in \mathfrak{M}_{\mathcal{S}} \otimes \mathfrak{M}_{\mathcal{E}_n}.$$

(If the system is initially given in the Hamiltonian formalism, then $V_n = \pi_{\mathcal{S}} \otimes \pi_{\mathcal{E}_n}(v_n)$.) The evolution of the interacting system is thus generated by the Liouvillian

$$L(t) = L_0 + \sum_{n\geq 1} \chi_n(t) V_n.$$

In the same way as for the Hamiltonian description, we will denote

$$L_n := L_{\mathcal{S}} + L_{\mathcal{E}_n} + V_n, \quad \text{and} \quad \widetilde{L}_n = L_n + \sum_{k\neq n} L_{\mathcal{E}_k},$$

so that $L(t) \equiv \widetilde{L}_n$ when $t \in [t_{n-1}, t_n)$. We will also denote by $U(t,0)$ the associated propagator, i.e. for $t \in [t_n, t_{n+1})$ one has

$$U(t,0) = \mathrm{e}^{-\mathrm{i}(t-t_n)\widetilde{L}_{n+1}} \mathrm{e}^{-\mathrm{i}\tau_n \widetilde{L}_n} \cdots \mathrm{e}^{-\mathrm{i}\tau_1 \widetilde{L}_1}.$$

Finally we denote by

$$\alpha^t_{\rm RI}(A) := U(t,0)^* A U(t,0)$$

the evolution of an observable $A \in \mathfrak{M}$ at time t.

As in the Hamiltonian description, we now explain how to reduce the analysis of expectation values of observables on $\mathcal{S}$ to the product of "Reduced Dynamics Operators" acting on $\mathcal{H}_{\mathcal{S}}$ only. In order not to muddle the essence of the argument, let us assume that the initial state of the

entire system is given by the vector $\Psi_0 = \Psi_{\mathcal{S}} \otimes \Psi_{\rm env}$ (see [BJM1, BJM3] for more details). If $A_{\mathcal{S}} \in \mathfrak{M}_{\mathcal{S}}$ we thus want to calculate

$$\begin{aligned} \langle A_{\mathcal{S}} \rangle(n) &:= \langle \Psi_0, \alpha_{\rm RI}^{t_n}(A_{\mathcal{S}} \otimes \mathbb{1}_{\rm env})\Psi_0 \rangle \\ &= \langle \Psi_0, {\rm e}^{{\rm i}\tau_n \widetilde{L}_n} \cdots {\rm e}^{{\rm i}\tau_1 \widetilde{L}_1}\, A_{\mathcal{S}} \otimes \mathbb{1}_{\rm env}\, {\rm e}^{-{\rm i}\tau_1 \widetilde{L}_1} \cdots {\rm e}^{-{\rm i}\tau_n \widetilde{L}_n} \Psi_0 \rangle. \end{aligned} \tag{1.1}$$

The first step consists in the following decomposition which serves to isolate the dynamics of the elements $\mathcal{E}$ which do not interact at a given time, and which is the equivalent of (11):

$${\rm e}^{-{\rm i}\tau_n \widetilde{L}_n} \cdots {\rm e}^{-{\rm i}\tau_1 \widetilde{L}_1} = U_m^- {\rm e}^{-{\rm i}\tau L_n} \cdots {\rm e}^{-{\rm i}\tau L_1} U_n^+,$$

where

$$U_n^- = \exp\left(-{\rm i} \sum_{k=1}^{n-1} (t_n - t_k) L_{\mathcal{E}_k} \right), \quad U_n^+ = \exp\left(-{\rm i} \sum_{k=2}^{n} t_{k-1} L_{\mathcal{E}_k} - {\rm i} t_n \sum_{k>n} L_{\mathcal{E}_k} \right).$$

One easily sees that $U_n^+ \Psi_0 = \psi_0$ and that U_n^- commutes with $A_{\mathcal{S}} \otimes \mathbb{1}_{\rm env}$, so that (1.1) can be written as

$$\langle A_{\mathcal{S}} \rangle(n) = \langle \Psi_0, {\rm e}^{{\rm i}\tau_1 L_1} \cdots {\rm e}^{{\rm i}\tau_n L_n} A_{\mathcal{S}} \otimes \mathbb{1}_{\rm env}\, {\rm e}^{-{\rm i}\tau_n L_n} \cdots {\rm e}^{-{\rm i}\tau_1 L_1} \Psi_0 \rangle. \tag{1.2}$$

The second step is to replace, for all n, the Liouvillean L_n by another (non selfadjoint) generator K_n of the interacting dynamics, called a C-Liouville operator, which satisfies the following additional property:

$$K_n\, \Psi_{\mathcal{S}} \otimes \Psi_{\mathcal{E}_n} = 0, \tag{1.3}$$

i.e. it "kills" the reference vector. The C-Liouville operator has been introduced in [JP2] to study non-equilibrium steady states (NESS).

Remark. For the existence of such a generator, we refer to e.g. [JP2]. One can also get an explicit expression for it in terms of the Liouvillean and the modular data of the pair $(\mathfrak{M}_{\mathcal{S}} \otimes \mathfrak{M}_{\mathcal{E}_n}, \Psi_{\mathcal{S}} \otimes \Psi_{\mathcal{E}_n})$ [AJP, BR, JP2].

Since the operators K_n are also generators of the dynamics, and using (1.3), (1.2) becomes

$$\langle A_{\mathcal{S}} \rangle(n) = \langle \Psi_0, {\rm e}^{{\rm i}\tau_1 K_1} \cdots {\rm e}^{{\rm i}\tau_n K_n} (A_{\mathcal{S}} \otimes \mathbb{1}_{\rm env}) \Psi_0 \rangle. \tag{1.4}$$

The last step is to use the independance of the various elements of the environment and to rewrite (1.4) in terms of a product of "reduced dynamics operators" M_n. Let

$$P := \mathbb{1}_{\mathcal{H}_{\mathcal{S}}} \otimes |\Psi_{\rm env}\rangle\langle \Psi_{\rm env}| \tag{1.5}$$

denote the orthogonal projection onto $\mathcal{H}_{\mathcal{S}} \otimes \mathbb{C}\Psi_{\rm env} \cong \mathcal{H}_{\mathcal{S}}$. If B is an operator acting on $\mathcal{H}$ then we identify PBP as an operator acting on $\mathcal{H}_{\mathcal{S}}$. Note that $P\Psi_0 = \Psi_0$, hence

$$\langle A_{\mathcal{S}} \rangle(n) = \langle \Psi_0, P {\rm e}^{{\rm i}\tau_1 K_1} \cdots {\rm e}^{{\rm i}\tau_n K_n} P (A_{\mathcal{S}} \otimes \mathbb{1}_{\rm env}) \Psi_0 \rangle.$$

The structure of RI systems gives

$$P {\rm e}^{{\rm i}\tau_1 K_1} \cdots {\rm e}^{{\rm i}\tau_n K_n} P = (P {\rm e}^{{\rm i}\tau_1 K_1} P) \times (P {\rm e}^{{\rm i}\tau_2 K_2} P) \times \cdots \times (P {\rm e}^{{\rm i}\tau_n K_n} P),$$

(this is nothing but the Markov property). Hence, introducing $M_j := P\mathrm{e}^{\mathrm{i}\tau_j K_j}P$ (considered as an operator acting on $\mathcal{H}_\mathcal{S}$), we finally get

$$\langle A_\mathcal{S}\rangle(n) = \langle \Psi_\mathcal{S}, M_1 \cdots M_n A_\mathcal{S}\Psi_\mathcal{S}\rangle. \tag{1.6}$$

We have thus reduced the analysis to the one of the product of the operators $M_1 \cdots M_n$.

At first sight, and despite the fact that the operators M_n give the desired reduction procedure, their definition may look quite obscure. Actually they are nothing but the GNS version of the dual $\mathcal{L}_n^*$ of the RDM's $\mathcal{L}_n$ as we shall now explain, see (1.7). We suppose that the RI system is given in the Hamiltonian formalism. Let $A, B \in \mathfrak{A}_\mathcal{S} = \mathcal{B}(\mathfrak{h}_\mathcal{S})$. We consider the quantity $\langle \pi_\mathcal{S}(B)\Psi_\mathcal{S}, M\pi_\mathcal{S}(A)\Psi_\mathcal{S}\rangle$ (we drop the index n to simplify notation). One can then write

$$\begin{aligned}
\langle \pi_\mathcal{S}(B)\Psi_\mathcal{S}, M\pi_\mathcal{S}(A)\Psi_\mathcal{S}\rangle &= \langle \pi_\mathcal{S}(B)\otimes \mathbb{1}\,\Psi_\mathcal{S}\otimes\Psi_{\mathrm{env}}, \mathrm{e}^{\mathrm{i}\tau K}\pi_\mathcal{S}(A)\otimes\mathbb{1}\,\Psi_\mathcal{S}\otimes\Psi_{\mathrm{env}}\rangle \\
&= \langle \pi(B\otimes\mathbb{1})\Psi_\mathcal{S}\otimes\Psi_{\mathrm{env}}, \mathrm{e}^{\mathrm{i}\tau L}\pi(A\otimes\mathbb{1})\mathrm{e}^{-\mathrm{i}\tau L}\Psi_\mathcal{S}\otimes\Psi_{\mathrm{env}}\rangle \\
&= \langle \Psi_\mathcal{S}\otimes\Psi_{\mathrm{env}}, \pi(B^*\otimes\mathbb{1})\pi(\mathrm{e}^{\mathrm{i}\tau h}\,A\otimes\mathbb{1}\,\mathrm{e}^{-\mathrm{i}\tau h})\Psi_\mathcal{S}\otimes\Psi_{\mathrm{env}}\rangle \\
&= \mathrm{Tr}\left(\rho_\mathcal{S}\otimes\rho_{\mathrm{env}}\times B^*\otimes\mathbb{1}\times \mathrm{e}^{\mathrm{i}\tau h}\,A\otimes\mathbb{1}\,\mathrm{e}^{-\mathrm{i}\tau h}\right) \\
&= \mathrm{Tr}\left(\rho_\mathcal{S}B^*\otimes\rho_{\mathrm{env}}\times \mathrm{e}^{\mathrm{i}\tau h}\,A\otimes\mathbb{1}\,\mathrm{e}^{-\mathrm{i}\tau h}\right) \\
&= \mathrm{Tr}\left(\rho_\mathcal{S}B^*\mathcal{L}^*(A)\right) \\
&= \langle \Psi_\mathcal{S}, \pi_\mathcal{S}(B^*\mathcal{L}^*(A))\Psi_\mathcal{S}\rangle \\
&= \langle \pi_\mathcal{S}(B)\Psi_\mathcal{S}, \pi_\mathcal{S}(\mathcal{L}^*(A))\Psi_\mathcal{S}\rangle.
\end{aligned}$$

Since $\Psi_\mathcal{S}$ is a cyclic vector this proves that, for any $A \in \mathfrak{A}_\mathcal{S}$,

$$M : \pi_\mathcal{S}(A)\Psi_\mathcal{S} \mapsto \pi_\mathcal{S}(\mathcal{L}^*(A))\Psi_\mathcal{S}. \tag{1.7}$$

Of course, the properties of a RDM $\mathcal{L}$ immediately translate into properties of M

Proposition 1.1. *The operator M is a contraction on the Banach space $\mathcal{C} = \{A\Psi_\mathcal{S} \mid A \in \mathfrak{A}_\mathcal{S}\}$ endowed with the norm $|||\phi||| = |||A\Psi_\mathcal{S}||| := \|A\|$. Moreover 1 is an eigenvalue for M with corresponding eigenvector $\Psi_\mathcal{S}$.*

These two properties correspond respectively to the contracting and trace preserving properties of $\mathcal{L}$. Note also that when the small system has finite dimension the Banach space $\mathcal{C}$ is simply $\mathcal{H}_\mathcal{S}$.

1.2 External work and entropy production in RI systems

External work.

As already mentioned, since the total Hamiltonian/Liouvillian of a repeated interaction system is time-dependent, the total energy is not necessarily constant: to switch from an interaction to the other may require some external work. To define the corresponding "external work" observable, we first come back to the usual Hamiltonian formalism described in the Introduction.

Since the total system is infinite (there are infinitely many subsystems $\mathcal{E}_n$), the total energy makes no sense. However, energy variation does. Formally, the total energy at time t is simply

$$u(t,0)^* h(t) u(t,0),$$

where $u(t,0)$ is the propagator between time 0 and t, i.e. if $t \in [t_n, t_{n+1})$ then

$$u(t,0) = \mathrm{e}^{-\mathrm{i}(t-t_n)\tilde{h}_{n+1}} \mathrm{e}^{-\mathrm{i}\tau_n \tilde{h}_n} \cdots \mathrm{e}^{-\mathrm{i}\tau_1 \tilde{h}_1}.$$

The change of energy between time t and time t' is therefore

$$\Delta E(t',t) = u(t',0)^* \tilde{h}(t') u(t',0) - u(t,0)^* \tilde{h}(t) u(t,0).$$

Now, for $t_{n-1} \leq t < t_n \leq t' < t_{n+1}$, it is easy to see that

$$\Delta E(t',t) = u(t_n,0)^* (v_{n+1} - v_n) u(t_n,0) =: w(n).$$

The observable $w(n)$ is the work observable at time t_n. If $\mathcal{S}$ is initially in the state ρ, one therefore has

$$\begin{aligned} \delta E(n) &:= \mathrm{Tr}_{\mathfrak{h}}\big(\rho \otimes \bigotimes_{k\geq 1} \rho_{\mathcal{E}_k} \times w(n)\big) = \mathrm{Tr}_{\mathfrak{h}}\big(\rho^{\mathrm{tot}}(n) \times (v_{n+1} - v_n)\big) \\ &= \mathrm{Tr}_{\mathfrak{h}_{\mathcal{S}} \otimes \mathfrak{h}_{\mathcal{E}_{n+1}}} \left[\rho(n) \otimes \rho_{\mathcal{E}_{n+1}}(t_n)\, v_{n+1}\right] \\ &\qquad - \mathrm{Tr}_{\mathfrak{h}_{\mathcal{S}} \otimes \mathfrak{h}_{\mathcal{E}_n}} \left[\rho(n-1) \otimes \rho_{\mathcal{E}_n}(t_{n-1})\, \mathrm{e}^{\mathrm{i}\tau_n h_n} v_n \mathrm{e}^{-\mathrm{i}\tau_n h_n}\right], \end{aligned} \tag{1.8}$$

and the mean work per time unit, i.e. the power delivered to the system, is therefore (if it exists),

$$\Delta W := \lim_{n\to\infty} \frac{1}{t_n} \sum_{k=1}^{n} \delta E(k).$$

When turning to the Liouvillian description, it is now natural to define the work observable as

$$W(n) := \pi(w(n)) = U(t_n,0)^* \,(V_{n+1} - V_n)\, U(t_n,0) = \alpha_{\mathrm{RI}}^{t_n}(V_{n+1} - V_n) \tag{1.9}$$

If ϱ is the initial state of the (entire) system, the power delivered to the system is therefore (if it exists)

$$\Delta W = \lim_{n\to\infty} \frac{1}{t_n} \sum_{k=1}^{n} \varrho(W(n)). \tag{1.10}$$

Entropy production

If ϱ and ϱ_0 are two normal states[b] on $\mathfrak{M}$, the relative entropy of Araki of the state ϱ with respect to ϱ_0 is denoted by $\mathrm{Ent}(\varrho|\varrho_0)$[c]. We here adopt the same convention as in [BR, JP2], so

[b] A state ϱ on a von Neumann algebra $\mathfrak{M}$ is normal if it is σ-weakly continuous

[c] For finite systems, and if ϱ and ϱ_0 are given by density matrices ρ and ρ_0 respectively, then $\mathrm{Ent}(\varrho|\varrho_0) = -\mathrm{Tr}(\rho \log \rho - \rho \log \rho_0)$.

that $\mathrm{Ent}(\varrho|\varrho_0) \leq 0$. The reference state ϱ_0 will naturally be the vector state on $\mathfrak{M}$ determined by the vector $\Psi_0 = \Psi_{\mathcal{S}} \otimes \Psi_{\mathrm{env}}$.

We are interested in the mean entropy production per time unit, i.e.

$$\Delta S := \lim_{n\to\infty} -\frac{1}{t_n}\left[\mathrm{Ent}(\varrho \circ \alpha_{\mathrm{RI}}^{t_n}|\varrho_0) - \mathrm{Ent}(\varrho|\varrho_0)\right],$$

and its relation to the external work. (Note that since the relative entropy is negative, the entropy production is indeed a positive quantity.)

Remark 1.1. *For a thermodynamic interpretation of the entropy and its relation to the total work, when we will deal with entropy we will always assume that all the reference states $\varrho_\#$ are $(\beta_\#, \alpha_\#^t)$–KMS states for some inverse temperatures $\beta_\#$.*

The analysis of the entropy production relies on the so-called *entropy production formula* [JP3] (see 1.11) which we recall here for the sake of completeness. Consider a quantum dynamical system $(\mathfrak{A}, \alpha^t, \omega)$. We moreover assume that ω is a $(-1, \sigma_\omega^t)$–KMS state for some C^*-dynamics σ_ω^t with generator δ_ω. Let $V \in \mathrm{Dom}(\delta_\omega)$ and consider the perturbed dynamics α_V^t defined in the natural way:

$$\alpha_V^t(A) := \alpha^t(A) + \sum_{n\geq 1} \mathrm{i}^n \int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \cdots \int_0^{t_{n-1}} \mathrm{d}t_n [\alpha^{t_n}(V), [\cdots [\alpha^{t_1}(V), A]\cdots],$$

(if δ_α is the generator of α^t, the one of α_V^t is $\delta_\alpha + \mathrm{i}[V, \cdot]$). Then for any state η:

$$\mathrm{Ent}(\eta \circ \alpha_V^t|\omega) - \mathrm{Ent}(\eta|\omega) = -\int_0^t \eta \circ \alpha_V^s(\delta_\omega(V))\mathrm{d}s. \tag{1.11}$$

In the particular case of a composite system ($\mathfrak{A} = \otimes_k \mathfrak{A}_k$ and $\alpha^t = \otimes_k \alpha_k^t$) where the reference state ω is of the form $\omega = \otimes_k \omega_k$ and where the ω_k are (β_k, α_k^t)-KMS states, one can take $\sigma^t = \otimes_k \alpha_k^{-\beta_k t}$. In the GNS-representation, if the α_k^t are implemented by Liouvillians L_k then σ^t is generated by $L = -\sum_k \beta_k L_k$, so that $\delta_\omega(V)$ becomes $-\mathrm{i}\sum_k \beta_k [L_k, V]$.

In our RI setting (1.11) translates as follows:

$$\begin{aligned}
&\mathrm{Ent}(\varrho \circ \alpha_{\mathrm{RI}}^{t_n}|\varrho_0) - \mathrm{Ent}(\varrho|\varrho_0) \\
&= \sum_{k=1}^n \left[\beta_{\mathcal{E}_k} \varrho\left(\alpha_{\mathrm{RI}}^{t_k}(V_k) - \alpha_{\mathrm{RI}}^{t_{k-1}}(V_k)\right) + (\beta_{\mathcal{E}_k} - \beta_{\mathcal{S}})\varrho\left(\alpha_{\mathrm{RI}}^{t_k}(L_{\mathcal{S}}) - \alpha_{\mathrm{RI}}^{t_{k-1}}(L_{\mathcal{S}})\right)\right] \\
&= -\sum_{k=1}^n \beta_{\mathcal{E}_k} \varrho\left(W(k)\right) + \sum_{k=1}^n \beta_{\mathcal{E}_k} \varrho\left(\alpha_{\mathrm{RI}}^{t_k}(V_{k+1}) - \alpha_{\mathrm{RI}}^{t_{k-1}}(V_k)\right) \\
&\qquad + \sum_{k=1}^n (\beta_{\mathcal{E}_k} - \beta_{\mathcal{S}})\varrho\left(\alpha_{\mathrm{RI}}^{t_k}(L_{\mathcal{S}}) - \alpha_{\mathrm{RI}}^{t_{k-1}}(L_{\mathcal{S}})\right),
\end{aligned} \tag{1.12}$$

Note that $L_{\mathcal{S}}$ is a priori not an observable ($L_{\mathcal{S}} \notin \mathfrak{M}$) and $\alpha^t_{\rm RI}(L_{\mathcal{S}})$ neither. However the differences $\alpha^{t_k}_{\rm RI}(L_{\mathcal{S}}) - \alpha^{t_{k-1}}_{\rm RI}(L_{\mathcal{S}})$ are observables. This follows from the fact that $\mathrm{e}^{\mathrm{i}\tau_k L_k} L_{\mathcal{S}}\, \mathrm{e}^{-\mathrm{i}\tau_k L_k} - L_{\mathcal{S}} \in \mathfrak{M}_{\mathcal{S}} \otimes \mathfrak{M}_{\mathcal{E}_k}$, which in turn is proven by noting that

$$\mathrm{e}^{\mathrm{i}\tau_k L_k} L_{\mathcal{S}}\, \mathrm{e}^{-\mathrm{i}\tau_k L_k} - L_{\mathcal{S}} = \int_0^{\tau_k} \mathrm{e}^{\mathrm{i}tL_k}[\mathrm{i}L_k, L_{\mathcal{S}}]\mathrm{e}^{-\mathrm{i}tL_k}\mathrm{d}t = \int_0^{\tau_k} \mathrm{e}^{\mathrm{i}tL_k}[\mathrm{i}V_k, L_{\mathcal{S}}]\mathrm{e}^{-\mathrm{i}tL_k}\mathrm{d}t,$$

where $[\mathrm{i}V_k, L_{\mathcal{S}}] = -\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{\mathrm{i}tL_{\mathcal{S}}} V_k \mathrm{e}^{-\mathrm{i}tL_{\mathcal{S}}}|_{t=0} \in \mathfrak{M}_{\mathcal{S}} \otimes \mathfrak{M}_{\mathcal{E}_k}$.

1.3 Ideal RI systems

We start our analysis of RI systems with the simplest "ideal" case of identical interactions, i.e. $\mathcal{E}_n \equiv \mathcal{E}$, $\tau_n \equiv \tau$, etc. In this case, the reduced dynamics operators M_n do not depend on n and we are essentially led to the study of powers of M. Since M is a contraction on $\mathcal{H}_{\mathcal{S}}$ for the norm defined in Proposition 1.1 (recall that throughout this chapter $\mathcal{S}$ is a finite system so that the Banach space $\mathcal{C}$ on which M is a contraction is the full of $\mathcal{H}_{\mathcal{S}}$), its spectrum lies in the complex unit disk. Moreover, since 1 is an eigenvalue for M, it is also an eigenvalue for M^*. The idea is that M^* is the GNS version of the RDM $\mathcal{L}$. Hence this means that the system possesses at least one invariant state which is therefore a natural candidate for the limiting state of the system. The results of this Section come from [BJM1].

1.3.1 Asymptotic state

System observables

We first consider observables on the small system $\mathcal{S}$, i.e. $A = A_{\mathcal{S}} \otimes \mathbb{1}_{\rm env}$. As we argued in the previous section, the asymptotic behaviour of expectation values for such observables can be reduced to the analysis of M^n as n goes to infinity (at least if the initial state is the reference state). In all this section we will assume the following ergodicity assumption which is a kind of Fermi Golden Rule.

(E) The spectrum of M on the complex unit circle consists of the single eigenvalue $\{1\}$. This eigenvalue is simple (with corresponding eigenvector $\Psi_{\mathcal{S}}$).

Since $\mathcal{H}_{\mathcal{S}}$ is finite dimensional, M^* also has 1 as a non-degenerate eigenvalue. We denote by $\Psi^*_{\mathcal{S}}$ the unique corresponding eigenvector normalized by $\langle \Psi^*_{\mathcal{S}}, \Psi_{\mathcal{S}} \rangle = 1$. In particular, there exists $\gamma > 0$ such that $M^n = |\Psi_{\mathcal{S}}\rangle\langle\Psi^*_{\mathcal{S}}| + O(\mathrm{e}^{-\gamma n})$.

The reduction process described in Section 1.1 makes use of another generator of the interacting dynamics than the Liouvillian, the so-called C-Liouville operator. Its explicit form involves the modular data (J, Δ) of the pair $(\mathfrak{M}_{\mathcal{S}} \otimes \mathfrak{M}_{\mathcal{E}}, \Psi_{\mathcal{S}} \otimes \Psi_{\mathcal{E}})$, see e.g. [AJP, BR]. More precisely, it can be written as

$$K = L_{\mathcal{S}} + L_{\mathcal{E}} + V - J\Delta^{1/2} V \Delta^{-1/2} J.$$

In order to make it simple, we shall assume that

(H) $\Delta^{1/2} V \Delta^{-1/2} \in \mathfrak{M}_{\mathcal{S}} \otimes \mathfrak{M}_{\mathcal{E}}$.

This ensures that K generates a strongly continuous group $\mathrm{e}^{\mathrm{i}tK}$ of bounded operators on $\mathcal{H}_{\mathcal{S}} \otimes \mathcal{H}_{\mathcal{E}}$ (this assumption can certainly be relaxed, see [JP2]).

Theorem 1.2. *[BJM1] Suppose Assumption* **(E)-(H)** *are satisfied. Then there exists $\gamma > 0$ such that for any normal initial state ϱ on $\mathfrak{M}$ and any observable $A_{\mathcal{S}} \in \mathfrak{M}_{\mathcal{S}}$,*

$$\varrho\left(\alpha_{\mathrm{RI}}^{n\tau}(A_{\mathcal{S}} \otimes \mathbb{1}_{\mathrm{env}})\right) = \langle \Psi_{\mathcal{S}}^*, A_{\mathcal{S}} \Psi_{\mathcal{S}} \rangle + O(\mathrm{e}^{-\gamma n}), \qquad \forall n \in \mathbb{N}. \tag{1.13}$$

The state $\rho_{\mathcal{S},+}$, defined on $\mathfrak{M}_{\mathcal{S}}$ as

$$\rho_{\mathcal{S},+}(A_{\mathcal{S}}) = \langle \Psi_{\mathcal{S}}^*, A_{\mathcal{S}} \Psi_{\mathcal{S}} \rangle, \tag{1.14}$$

is the (unique) asymptotic state of the RI system.

Note that the asymptotic state does not depend on the initial state of $\mathcal{S}$ (since $\mathcal{H}_{\mathcal{S}}$ is finite dimensional any initial state of $\mathcal{S}$ is normal), and in particular it does not depend on the choice of the reference vector $\Psi_{\mathcal{S}}$.

Remark 1.2. *If the ergodic assumption* **(E)** *is not satisfied then the limit $\lim_{n\to\infty} M^n$ still exists, in a weaker sense. Namely, if there are eigenvalues different from 1 on the circle, then the limit exists in the ergodic mean sense, $\frac{1}{N}\sum_{n=0}^{N-1} M^n = |\Psi_{\mathcal{S}}\rangle\langle\Psi_{\mathcal{S}}^*| + O\left(\frac{1}{N}\right)$. Further, if 1 is a degenerate eigenvalue of M then the limit is the projection π onto $\ker(M - Id)$ along $\mathrm{Ran}(M - Id) = \ker(M^* - Id)^{\perp}$. This reflects in Theorem 1.2 in the following way. If 1 is non degenerate, but there are other eigenvalues on the circle, then Theorem 1.2 holds with (1.13) replaced by*

$$\left| \frac{1}{N} \sum_{n=0}^{N-1} \varrho\left(\alpha_{\mathrm{RI}}^{n\tau}(A_{\mathcal{S}} \otimes \mathbb{1}_{\mathrm{env}})\right) - \rho_{\mathcal{S},+}(A_{\mathcal{S}}) \right| \leq \frac{C}{N}.$$

If on the other hand 1 is degenerate, $\rho_{\mathcal{S},+}$ is not anymore the unique invariant state. Hence one can still prove that $\varrho \circ \alpha_{\mathrm{RI}}^{n\tau}$ has a limit but the latter will depend on the initial state ϱ.

Instantaneous observables

Besides observables on the system $\mathcal{S}$, there are other observables of interest. To investigate the external work, one has to consider the quantity (see (1.9)-(1.10))

$$\varrho\left(\alpha_{\mathrm{RI}}^{n\tau}(V_{n+1} - V_n)\right).$$

The observable of interest, $V_{n+1} - V_n$, is not strictly speaking an obervable because it varies with time: at time $n\tau$ it lives on $\mathcal{S}$ and the subsystems $\mathcal{E}_n$ and $\mathcal{E}_{n+1}$. One may also think of $\mathcal{S}$ as being fixed in space and of the environment as a beam passing through $\mathcal{S}$ so that when $t \in [n\tau, (n+1)\tau)$ the subsystem $\mathcal{E}_{n+1}$ is located near $\mathcal{S}$. A detector placed in the vicinity of $\mathcal{S}$ can measure at this moment in time observables of $\mathcal{S}$ and of the $(n+1)$-th element in the beam,

i.e. an "instantaneous observable" of the form $A_{\mathcal{S}} \otimes \vartheta_{n+1}(A_{\mathcal{E}})$, where $A_{\mathcal{S}} \in \mathfrak{M}_{\mathcal{S}}$, $A_{\mathcal{E}} \in \mathfrak{M}_{\mathcal{E}}$, and $\vartheta_m : \mathfrak{M}_{\mathcal{E}} \to \mathfrak{M}_{\rm env}$ is defined by

$$\vartheta_m(A_{\mathcal{E}}) = \mathbb{1}_{\mathcal{E}} \otimes \cdots \mathbb{1}_{\mathcal{E}} \otimes A_{\mathcal{E}} \otimes \mathbb{1}_{\mathcal{E}} \cdots \tag{1.15}$$

where the $A_{\mathcal{E}}$ on the right side of (1.15) acts on the m-th factor in the environment. Recall also the processes leading to continuous measurements: one makes a measure on the subsystems $\mathcal{E}_n$ right after their interactions with $\mathcal{S}$, i.e. at time $n\tau$ one measures an observable of the form $\mathbb{1}_{\mathcal{S}} \otimes \vartheta_n(A_{\mathcal{E}})$.

More generally we may be interested in the expectation value of "observables" of the form

$$[A_{\mathcal{S}}; A^{(j)}](t) := A_{\mathcal{S}} \otimes_{j=-\ell}^{r} \vartheta_{m(t)+j+1}(A^{(j)}), \tag{1.16}$$

where $A_{\mathcal{S}} \in \mathfrak{M}_{\mathcal{S}}$, $A^{(-\ell)}, \ldots, A^{(0)}, \ldots, A^{(r)} \in \mathfrak{M}_{\mathcal{E}}$, where $m(t)$ is the integer part of t/τ, and ϑ_k is given in (1.15). The parameters $\ell, r \geq 0$ are not displayed in the l.h.s. in (1.16). $A_{\mathcal{S}}$ represents an observable measured on the small system, the $A^{(0)}$ is the "instantaneous" observable, measured in the element $m(t)+1$ of the chain (the one in contact with $\mathcal{S}$ at time t), while the $A^{(j)}$ with negative and positive index are the quantities measured in the elements preceding and following the $(m(t)+1)$-th.

Theorem 1.3. *[BJM1] Suppose Assumption* **(E)-(H)** *are satisfied. Then there exists $\gamma > 0$ such that for any normal initial state ϱ on $\mathfrak{M}$ and any instantaneous observable $[A_{\mathcal{S}}; A^{(j)}]$,*

$$\left| \varrho \left(\alpha_{\rm RI}^{n\tau} \left([A_{\mathcal{S}}; A^{(j)}](n\tau) \right) \right) - \varrho_+([A_{\mathcal{S}}; A^{(j)}]) \right| \leq C \mathrm{e}^{-\gamma n},$$

where

$$\begin{aligned} \varrho_+([A_{\mathcal{S}}; A^{(j)}]) &= \langle \Psi_{\mathcal{S}}^* \otimes \Psi_{\rm env}, \alpha_{\rm RI}^{\ell\tau} \Big(A_{\mathcal{S}} \otimes A^{(-\ell)} \otimes \cdots \otimes A^{(r)} \Big) \Psi_{\mathcal{S}} \otimes \Psi_{\rm env} \rangle \\ &= \rho_{\mathcal{S},+} \left(P \, \alpha_{\rm RI}^{\ell\tau} \Big(A_{\mathcal{S}} \otimes A^{(-\ell)} \otimes \cdots \otimes A^{(r)} \Big) P \right), \end{aligned}$$

with $\rho_{\mathcal{S},+}$ as in (1.14). (Recall that P is the projection onto $\mathcal{H}_{\mathcal{S}}$.)

1.3.2 External work and entropy production

Equations (1.9) and (1.10) show that the mean work per time unit in the system reduces to the asymptotic behaviour of the instantaneous observable $V_{n+1} - V_n$ (at time $n\tau$, it acts on $\mathcal{S}$ and the n-th and $(n+1)$-th subsystems of the environment). As a direct consequence of Theorem 1.3 we have

Proposition 1.4. *[BJM1] Suppose Assumption* **(E)-(H)** *hold. Then for any normal initial state ϱ, the power delivered to the system is*

$$\begin{aligned} \Delta W &= \frac{1}{\tau} \rho_{\mathcal{S},+} \left(PVP - P\alpha_{\rm RI}^{\tau}(V)P \right) \\ &= \frac{1}{\tau} \langle \Psi_{\mathcal{S}}^* \otimes \Psi_{\mathcal{E}}, \left(V - \mathrm{e}^{\mathrm{i}\tau L} V \mathrm{e}^{-\mathrm{i}\tau L} \right) \Psi_{\mathcal{S}} \otimes \Psi_{\mathcal{E}} \rangle, \end{aligned}$$

where $L = L_{\mathcal{S}} + L_{\mathcal{E}} + V$ acts on $\mathcal{H}_{\mathcal{S}} \otimes \mathcal{H}_{\mathcal{E}}$.

Remark 1.3. *One actually proves that*

$$\varrho\left(W(n)\right) = \rho_{\mathcal{S},+}\left(PVP - P\alpha_{\mathrm{RI}}^{\tau}(V)P\right) + O(\mathrm{e}^{-\gamma n}).$$

Moreover, in the invariant state $\rho_{\mathcal{S},+}$ the external work is of course constant, i.e. for $\varrho = \rho_{\mathcal{S},+} \otimes \rho_{\mathrm{env}}$ where $\rho_{\mathrm{env}} = \langle \Psi_{\mathrm{env}}, \cdot\, \Psi_{\mathrm{env}} \rangle$, one has $\varrho(W(n)) \equiv \rho_{\mathcal{S},+}\left(PVP - P\alpha_{\mathrm{RI}}^{\tau}(V)P\right)$.

We now turn to the entropy production. For that purpose we thus assume that the reference states are KMS-states at some inverse temperatures $\beta_{\mathcal{S}}$, resp. $\beta_{\mathcal{E}}$. The entropy production formula (1.12) then simplifies as

$$\begin{aligned}
&\mathrm{Ent}(\varrho \circ \alpha_{\mathrm{RI}}^{t_n} | \varrho_0) - \mathrm{Ent}(\varrho|\varrho_0) \\
&= \; -\beta_{\mathcal{E}} \sum_{k=1}^{n} \varrho\left(W(k)\right) \,+\, \beta_{\mathcal{E}}\, \varrho\left(\alpha_{\mathrm{RI}}^{n\tau}(V_{n+1}) - V_1\right) + (\beta_{\mathcal{E}} - \beta_{\mathcal{S}})\varrho\left(\alpha_{\mathrm{RI}}^{n\tau}(L_{\mathcal{S}}) - L_{\mathcal{S}}\right).
\end{aligned}$$

As a conseqence, we have the following

Proposition 1.5. *[BJM1] Suppose Assumption* **(E)-(H)** *hold, and that $[V, L_{\mathcal{S}}]$, $[V, L_{\mathcal{E}}] \in \mathfrak{M}_{\mathcal{S}} \otimes \mathfrak{M}_{\mathcal{E}}$. Then for any normal initial state ϱ, one has*

$$\Delta S := \lim_{n\to\infty} -\frac{1}{t_n}\left[\mathrm{Ent}(\varrho \circ \alpha_{\mathrm{RI}}^{t_n}|\varrho_0) - \mathrm{Ent}(\varrho|\varrho_0)\right] = \beta_{\mathcal{E}} \Delta W.$$

Since the environment is in thermal equilibrium (all the subsystems have the same temperature), the above formula is nothing but the 2nd law of thermodynamics. Together with the positivity of entropy production this proposition also shows that the external work is positive as well. The only issue is therefore whether these quantities are strictly positive or not. One can answer this question only for concrete systems, several examples are given in [BJM1]. We will see in Section 1.5, that the situation is different if $\mathcal{S}$ is coupled to an extra reservoir $\mathcal{R}$.

Remark 1.4. *The assumptions on $[V, L_{\mathcal{S}}]$ and $[V, L_{\mathcal{E}}]$ ensure that, with the notation of (1.11), $V \in \mathrm{Dom}(\delta_\omega)$ so that we can apply the entropy production formula (1.12).*

1.4 Random RI systems

In this section, we turn to a more general situation where the various interactions are not identical. Of course, if one considers arbitrary interactions it is hopeless to expect any convergence (even in the ergodic mean) to some invariant state. As we mentioned in Section 1.1, see (1.6), the asymptotic behaviour of the system is essentially described by the product of reduced dynamics operators: $M_1 \cdots M_n$. If the M_n's are more or less arbitrary, anything can happen. We shall consider here the case where the interactions are random (but still independent identically distributed). Closely related results can be found in [NPe]. This randomness may have various origins: the interaction time, the reference state of the $\mathcal{E}_n$'s (via e.g. the temperature), the subsystems $\mathcal{E}_n$ themselves (and hence the interaction operators),... All these parameters are eventually encoded in the RDO and our assumption will be that the sequence $(M_n)_n$ will be

independent and identically distributed (i.i.d.). We are thus reduced to the analysis of a product of i.i.d. random matrices and we can apply the results presented in Section 4.1.

To motivate this analysis, consider the "One-Atom Maser" experiment where a beam of atoms interacts with modes of the quantized electromagnetic field. It is clear that in actual experiments, neither the interaction time τ_n nor the reference states of the subsystems $\mathcal{E}_n$ can be exactly the same for all n! Typically, the interaction time will be *random* (because of the random velocities of the atoms in the beam, see [BRH, FJM]), given e.g. by a Gaussian distribution around some mean value, and the state of the incoming atoms will be random as well, for instance determined by a temperature that fluctuates slightly around a mean temperature (in experiments, the atoms are ejected from an atom oven, then they are cooled down to a wanted temperature before entering the cavity). One could also imagine that the subsystems $\mathcal{E}_n$ themselves are not all the same (e.g. different kind of atoms, or maybe some impurities).

Another motivation is to consider a non-equilibrium situation. In the general setup of open quantum systems one gets a non-equilibrium situation when the environment is made of several reservoirs, each of them being in an equilibrium state but with different intensive thermodynamic parameters (different temperatures for instance). Then one expects the joint system $\mathcal{S}+\mathcal{R}_1+\cdots$ to relax towards a non-equilibrium steady states (NESS). Such states have been constructed in [R, AH, JP2, AP, MMS, CNZ]. They carry currents, have non vanishing entropy production rate,... These transport properties were investigated in [FMU, CJM, AJPP, N]. The linear response theory (Green-Kubo formula, Onsager reciprocity relations, central limit theorem) was developed in [FMU],[JOP1]-[JOP4],[JPP].

In the framework of RI systems, we can create a non-equilibrium situation by imposing the initial state of the subsystems $\mathcal{E}_n$ to be for example thermal equilibrium states at different temperatures. In other words, we assume that the system $\mathcal{S}$ interacts with K "reservoirs" at a priori different temperatures, i.e. for any $m \in \mathbb{N}$, $\Psi_{\mathcal{E}_{mK+1}} = \Psi_{\beta_1}$, $\Psi_{\mathcal{E}_{mK+2}} = \Psi_{\beta_2}$, etc... where Ψ_β is the representating vector for the KMS-state of $\mathcal{E}$ at inverse temperature β. (One could imagine a "One-Atom Maser" where the field in the cavity is coupled to K beams at different temperatures.) However, the particular structure imposed here leads to a lack of symmetry and in particular the system is not at all time reversal invariant (reservoir 2 always interacts right after reservoir 1 while the inverse is not true). As we will see in the toy example of Section 1.6, a direct consequence is that Onsager reciprocity relations do not hold. One way to restore symmetry is then to chose the temperature of the n-th subsystem in a random way from the set $\{\beta_1, \ldots, \beta_K\}$, each temperature having probability $\frac{1}{K}$ to occur (see Section 1.6.5). The results of this Section come from [BJM3].

1.4.1 Asymptotic state

Let $(\Omega, \mathcal{F}, \mathrm{p})$ be a probability space. To describe the stochastic dynamic process at hand, we introduce the standard probability measure $\mathrm{d}\mathbb{P}$ on $\tilde{\Omega} := \Omega^{\mathbb{N}^*}$,

$$\mathrm{d}\mathbb{P} = \Pi_{j\geq 1}\mathrm{dp}_j, \quad \text{where} \quad \mathrm{dp}_j \equiv \mathrm{dp}, \ \ \forall j \in \mathbb{N}^*.$$

We denote by $\tilde{\omega} = (\omega_n)_n$ the elements of $\tilde{\Omega}$. As we already mentioned, we will assume that the various interactions are independent and identically distributed (i.i.d.). This is precisely the meaning of the following randomness assumption

(R1) The reduced dynamics operators M_k are i.i.d. random operators. We write $M_k = M(\omega_k)$, where $M : \Omega \to \mathcal{B}(\mathcal{H}_\mathcal{S})$ is an operator valued random variable.

Throughout this section, we will assume, without further mentioning it, that Assumption **(H)** is satisfied, i.e. $\Delta^{1/2} V_m \Delta^{-1/2} \in \mathfrak{M}_\mathcal{S} \otimes \mathfrak{M}_{\mathcal{E}_m}$, and that **(R1)** is satisfied. To indicate the randomness, we will also write $\alpha^t_{\mathrm{RI},\tilde\omega}$ instead of α^t_{RI}. Finally let $\mathcal{M}_{(E)}$ be the set of RDO's which satisfy the ergodic assumption **(E)**.

We first consider expectation values of observables on the small system $\mathcal{S}$. The following theorem shows that the RI system relaxes almost surely in the ergodic mean towards a deterministic asymptotic state, and is a direct consequence of Theorem 4.3 of Section 4.1 on the product of random matrices and of Proposition 1.1.

Theorem 1.6. *[BJM3] Suppose that* $\mathrm{p}(M(\omega) \in \mathcal{M}_{(E)}) > 0$. *Then there exists a set* $\Omega_0 \subset \tilde\Omega$, *s.t.* $\mathbb{P}(\Omega_0) = 1$, *and s.t. for any* $\tilde\omega \in \Omega_0$, *any normal state* ϱ *and any* $A_\mathcal{S} \in \mathfrak{M}_\mathcal{S}$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \varrho\left(\alpha^{t_n(\tilde\omega)}_{\mathrm{RI},\tilde\omega}(A_\mathcal{S})\right) = \langle \Psi^*, A_\mathcal{S}\Psi_\mathcal{S}\rangle =: \rho_+(A_\mathcal{S}),$$

where Ψ^* *is the unique invariant vector of* $\mathbb{E}(M^*(\omega))$ *normalized by* $\langle \Psi^*, \Psi_\mathcal{S}\rangle = 1$, *and* $t_n(\tilde\omega) = \tau(\omega_1) + \cdots + \tau(\omega_n)$.

As for ideal interactions, and having in mind the application to the study of external work and entropy production, we also consider instantaneous observables. Since we will be interested in observables like $V_{n+1} - V_n$ (see (1.9)), not only these observables are instantaneous, but they can be random as well. The philosophy of instantaneous observables is that we measure the same physical observable but the system on which it is measured varies with time, namely at time t the observable $A^{(j)}$ is measured on the subsystem $\mathcal{E}_{j+m(t)+1}$ (see (1.16)). Hence, we shall replace (1.16) by

$$[A_\mathcal{S}; A^{(j)}](t, \tilde\omega) := A_\mathcal{S} \otimes_{j=-\ell}^{r} \vartheta_{m(t)+j+1}(A^{(j)}(\omega_{m(t)+j+1})).$$

Of course, in the case where the subsystems $\mathcal{E}_n$ are not identical, i.e. $\mathcal{E}_n = \mathcal{E}(\omega_n)$, one should have in mind that $A^{(j)}(\omega) \in \mathfrak{M}_{\mathcal{E}(\omega)}$.

Most of the analysis (essentially the reduction procedure to observables on $\mathcal{S}$ only) is done in a deterministic way, i.e. for fixed $\tilde\omega \in \tilde\Omega$. However, once the reduction is performed, we really turn to probability theory and we need that the relevant quantities define *bona fide* random variables. In view of Theorem 1.3, we introduce the following observable:

$$N : \tilde\Omega \ni \tilde\omega \mapsto P(\tilde\omega)\alpha^{t_\ell(\tilde\omega)}_{\mathrm{RI},\tilde\omega}\Big(A_\mathcal{S} \otimes A^{(-\ell)}(\omega_1) \otimes \cdots \otimes A^{(r)}(\omega_{\ell+r+1})\Big)P(\tilde\omega) \in \mathcal{B}(\mathcal{H}_\mathcal{S}),$$

where $P(\tilde\omega)$ is defined as in (1.5).

(R2) The observable $N(\tilde\omega)$ is a well-defined random variable.

Note that actually, $N(\tilde{\omega})$ only depends on $(\omega_1, \ldots, \omega_{\ell+r+1})$.

Theorem 1.7. *[BJM3] Suppose that* $\mathrm{p}(M(\omega) \in \mathcal{M}_{(E)}) > 0$. *Then there exists a set* $\Omega_1 \subset \tilde{\Omega}$, *s.t.* $\mathbb{P}(\Omega_1) = 1$, *and s.t. for any* $\tilde{\omega} \in \Omega_1$, *any normal state* ϱ *and any instantaneous observable* $[A_{\mathcal{S}}; A^{(j)}]$ *satisfying* **(R2)**,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \varrho\left(\alpha_{\mathrm{RI},\tilde{\omega}}^{t_n(\tilde{\omega})}([A_{\mathcal{S}}; A^{(j)}](n\tau, \tilde{\omega}))\right) = \rho_+(\mathbb{E}(N)),$$

where ρ_+ *is as in Theorem 1.6.*

1.4.2 External work and Entropy production

The following result on external work is a direct consequence of (1.9)-(1.10) and Theorem 1.7.

Proposition 1.8. *[BJM3] Suppose that* $\mathrm{p}(M(\omega) \in \mathcal{M}_{(E)}) > 0$. *Then*

$$\begin{aligned} \Delta W &:= \lim_{n\to\infty} \frac{1}{t_n(\tilde{\omega})} \sum_{k=1}^{n} \varrho\left(\alpha_{\mathrm{RI},\tilde{\omega}}^{t_k(\tilde{\omega})}(V(\omega_{k+1}) - V(\omega_k))\right) \\ &= \frac{1}{\mathbb{E}(\tau)} \varrho_+\Big(\mathbb{E}\big[P(V - \mathrm{e}^{\mathrm{i}\tau L} V \mathrm{e}^{-\mathrm{i}\tau L})P\big]\Big), \quad \mathbb{P}-a.s. \end{aligned} \tag{1.17}$$

Note that not only the external work exists and does not depend on the initial state, but is a deterministic quantity.

Similarly, if the reference state $\Psi_{\mathcal{E}}(\omega)$ is a $\beta_{\mathcal{E}}(\omega)$-KMS state, using (1.12) together with Theorem 1.7 we have

Proposition 1.9. *[BJM3] Suppose that* $\mathrm{p}(M(\omega) \in \mathcal{M}_{(E)}) > 0$. *Then*

$$\begin{aligned} \Delta S &:= \lim_{n\to\infty} -\frac{1}{t_n(\tilde{\omega})} \left[\mathrm{Ent}(\varrho \circ \alpha_{\mathrm{RI},\tilde{\omega}}^{t_n(\tilde{\omega})} | \varrho_0) - \mathrm{Ent}(\varrho|\varrho_0)\right], \\ &= \frac{1}{\mathbb{E}(\tau)} \varrho_+\Big(\mathbb{E}\big[\beta_{\mathcal{E}}\, P(L_{\mathcal{S}} + V - \mathrm{e}^{\mathrm{i}\tau L}(L_{\mathcal{S}} + V)\, \mathrm{e}^{-\mathrm{i}\tau L})P\big]\Big), \quad \mathbb{P}-a.s. \end{aligned} \tag{1.18}$$

As for the external work, the entropy production is also deterministic.

A priori, the link between external work and entropy production is not as clear as in the ideal case. Actually, one can prove that $\varrho_+[\mathbb{E}(P(L_{\mathcal{S}} - \alpha^{\tau}(L_{\mathcal{S}}))P)] = 0$ (see [BJM3]). Hence, we may rewrite (1.17) as

$$\Delta W = \frac{1}{\mathbb{E}(\tau)} \varrho_+\Big(\mathbb{E}\big[P(L_{\mathcal{S}} + V - \mathrm{e}^{\mathrm{i}\tau L}(L_{\mathcal{S}} + V)\, \mathrm{e}^{-\mathrm{i}\tau L})P\big]\Big), \quad \mathbb{P}-a.s.$$

Comparing with (1.18), one can recognize a sort of 2nd law of thermodynamics. If in particular the inverse temperature $\beta_{\mathcal{E}}$ is not random (i.e. we are in equlibrium), then $\Delta S = \beta_{\mathcal{E}} \Delta W$ as expected. Of course, if $\beta_{\mathcal{E}}$ is not constant, we are in a non-equilibrium situation and the fact that there is no clear relation between ΔS and ΔW should not come as a surprise.

1.5 Leaky RI systems

In this section, we consider the situation where, besides the repeated interactions with the subsystems $\mathcal{E}_k$, the system $\mathcal{S}$ also interacts with another reservoir $\mathcal{R}$ in a continuous way, i.e. the Liouvillian of the system is now of the form

$$L = L_{\mathcal{S}} + \sum_{n\geq 1} L_{\mathcal{E}_n} + \sum_{n\geq 1} \chi_n(t) V_{\mathcal{S}\mathcal{E}_n} + L_{\mathcal{R}} + V_{\mathcal{S}\mathcal{R}}, \tag{1.19}$$

where $L_{\mathcal{R}}$ is the generator of the free dynamics of the reservoir and $V_{\mathcal{S}\mathcal{R}}$ describes the interaction between the system $\mathcal{S}$ and the reservoir $\mathcal{R}$. Note that $\mathcal{R}$ is not directly coupled to the subsystems $\mathcal{E}_n$. We will also stick to the situation where the repeated interactions are identical.

The motivation to study such systems is twofold. First, it describes for example a "One-Atom Maser" in which one also takes into account some losses in the cavity, the latter being not completely isolated from the exterior world, e.g. from the laboratory [FJM]. The assumption "$\mathcal{R}$ is not directly coupled to the subsystems $\mathcal{E}_n$" is physically reasonable. Indeed, again for the "one-atom maser" experiment, the idea is that the atoms are ejected from an oven one by one just before they interact with the cavity and moreover the atom-field interaction time τ is typically much smaller than the damping time due to the presence of the heat reservoir. Therefore, the atoms do not have enough time to feel the effects of the reservoir before and during their interaction with the field.

A second motivation is the study of non-equilibrium quantum systems. Suppose $\mathcal{S}$ is brought into contact with several reservoirs $\mathcal{R}_i$, each of them being in a thermal equilibrium state but with different intensive thermodynamic parameters. The interaction between $\mathcal{S}$ and the various reservoirs is most often "continuous", i.e. $\mathcal{S}$ and the $\mathcal{R}_i$ interact for all time (said differently, the generator of the interacting dynamics is time-independent). We have also considered in the previous section the case where the various reservoirs are all of the repeated interaction type (chosing e.g. reference states which are randomly distibuted with uniform distribution over a fixed set $\Psi_1, \ldots, \Psi_K$). In the system considered in this section we have a situation with two various reservoirs of different nature: one is described by a RI system and the other one interacts with $\mathcal{S}$ in a continuous way, and we want to understand the relative effects of these two reservoirs.

In a sense, one can consider this entire system as a RI system but where $\mathcal{S}$ has been replaced by $\mathcal{S} + \mathcal{R}$, i.e the "small" system becomes large as well. The general approach to RI systems, as described in Section 1.1, can therefore be used. However, the reduced dynamics operator M now acts on the space $\mathcal{H}_{\mathcal{S}} \otimes \mathcal{H}_{\mathcal{R}}$ and , as we shall see, its spectral properties are of course much more complicated. The results presented in this section come from [BJM4].

1.5.1 The additional reservoir

The reservoir $\mathcal{R}$ is a thermal reservoir of free Fermi particles at a temperature $T_{\mathcal{R}} > 0$, in the thermodynamic limit. Its description was originally given in the work by Araki and Wyss [AW] (see also [JP1]). We give directly the descritpion in the GNS-representation, and refer to e.g. [AW, JP1, BJM4] for a precise derivation starting from the usual description via C^*- dynamical systems.

The Hilbert space is the anti-symmetric Fock space

$$\mathcal{H}_{\mathcal{R}} = \Gamma_-(\mathfrak{h}_{\mathcal{R}}) := \bigoplus_{n\geq 0} \wedge^n \mathfrak{h}_{\mathcal{R}},$$

over the one-particle space

$$\mathfrak{h}_{\mathcal{R}} = L^2(\mathbb{R}, \mathfrak{G}), \tag{1.20}$$

where $\mathfrak{G}$ is an 'auxiliary space' (typically an angular part like $L^2(S^2)$). In this representation, the one-particle Hamiltonian $h_{\mathcal{R}}$ is the operator of multiplication by the radial variable (extended to negative values) $s \in \mathbb{R}$ of (1.20), i.e. for $\varphi \in L^2(\mathbb{R}, \mathfrak{G})$

$$(h_{\mathcal{R}}\varphi)(s) = s\varphi(s).$$

The Liouville operator is the second quantization of h,

$$L_{\mathcal{R}} = \mathrm{d}\Gamma(h_{\mathcal{R}}) := \bigoplus_{n\geq 0} \sum_{j=1}^{n} h_{\mathcal{R}}^{(j)},$$

where $h_{\mathcal{R}}^{(j)}$ is understood to act as $h_{\mathcal{R}}$ on the j-th factor of $\wedge^n \mathfrak{h}_{\mathcal{R}}$ and trivially on the other ones.

The von Neumann algebra $\mathfrak{M}_{\mathcal{R}}$ is the subalgebra of $\mathcal{B}(\mathcal{H}_{\mathcal{R}})$ generated by the thermal fermionic field operators (at inverse temperature $\beta_{\mathcal{R}}$), represented on $\mathcal{H}_{\mathcal{R}}$ by

$$\varphi(g_{\beta_{\mathcal{R}}}) = \frac{1}{\sqrt{2}}\big[a^*(g_{\beta_{\mathcal{R}}}) + a(g_{\beta_{\mathcal{R}}})\big].$$

Here, we define for $g \in L^2(\mathbb{R}_+, \mathfrak{G})$,

$$g_{\beta_{\mathcal{R}}}(s) = \sqrt{\frac{1}{\mathrm{e}^{-\beta_{\mathcal{R}} s}+1}} \begin{cases} g(s) & \text{if } s \geq 0 \\ \overline{g}(-s) & \text{if } s < 0. \end{cases}$$

Finally, we choose the reference state to be the thermal equilibrium state, represented by the vacuum vector of $\mathcal{H}_{\mathcal{R}}$,

$$\Psi_{\mathcal{R}} = \Omega.$$

1.5.2 Translation analyticity

As already mentioned, the reduced dynamics operator M is now defined as an operator on the larger space $\mathcal{H}_{\mathcal{S}} \otimes \mathcal{H}_{\mathcal{R}}$ and will have more complicated spectral properties. To understand why, let's switch off the interactions. Then, clearly $M = \mathrm{e}^{\mathrm{i}\tau L_{\mathcal{S}}} \otimes \mathrm{e}^{\mathrm{i}\tau L_{\mathcal{R}}}$ which, besides some eigenvalues, has continuous spectrum equal to the whole circle S^1. When turning on the interaction, this continuous spectrum survives. In order to seperate it from the eigenvalues, we use analytic spectral deformation methods, see e.g. [BFS, JP2, MMS, RS4], and will resort to perturbation theory. For that purpose, we therefore add coupling constants in the interaction, i.e. (1.19) becomes

$$L = L_{\mathcal{S}} + L_{\mathcal{R}} + \sum_{n\geq 1} L_{\mathcal{E}_n} + \lambda_{\mathcal{SR}} V_{\mathcal{SR}} + \lambda_{\mathcal{SE}} \sum_{n\geq 1} \chi_n(t) V_{\mathcal{SE}_n},$$

and the perturbation will be in term of the coupling constant $\lambda := (\lambda_{\mathcal{SR}}, \lambda_{\mathcal{SE}}) \in \mathbb{R}^2$. We will also write $V(\lambda) = \lambda_{\mathcal{SR}} V_{\mathcal{SR}} + \lambda_{\mathcal{SE}} V_{\mathcal{SE}}$ and denote by $M(\lambda)$ the reduced dynamics operator.

As in Section 1.3, if Δ denotes the modular operator of the pair $(\mathfrak{M}_{\mathcal{S}} \otimes \mathfrak{M}_{\mathcal{R}} \otimes \mathfrak{M}_{\mathcal{E}}, \Psi_{\mathcal{S}} \otimes \Psi_{\mathcal{R}} \otimes \Psi_{\mathcal{E}})$ we shall assume that

(H') $\Delta^{1/2} V(\lambda) \Delta^{-1/2} \in \mathfrak{M}_{\mathcal{S}} \otimes \mathfrak{M}_{\mathcal{R}} \otimes \mathfrak{M}_{\mathcal{E}}$.

Moreover, since we will be using analytic spectral deformation methods on the factor $\mathcal{H}_{\mathcal{R}}$ of $\mathcal{H}$, we need to make a regularity assumption on the interaction. Let $\mathbb{R} \ni \theta \mapsto T(\theta) \in \mathcal{B}(\mathcal{H}_{\mathcal{R}})$ be the unitary group defined by

$$T(\theta) = \Gamma(e^{-\theta \partial_s}) \text{ on } \Gamma_-(L^2(\mathbb{R}, \mathfrak{G})),$$

where for any $f \in L^2(\mathbb{R}, \mathfrak{G})$,

$$(e^{-\theta \partial_s} f)(s) = f(s - \theta),$$

i.e. we use the generator of translation. In the following, we will abuse notation and (for simplicity) also write $T(\theta)$ for $\mathbb{1}_{\mathcal{S}} \otimes T(\theta) \otimes \mathbb{1}_{\mathcal{E}}$ and $\mathbb{1}_{\mathcal{S}} \otimes T(\theta) \otimes \mathbb{1}_{\mathrm{env}}$. Note that $T(\theta)$ commutes with all observables acting trivially on $\mathcal{H}_{\mathcal{R}}$, in particular with $P_{\mathcal{SR}} = \mathbb{1}_{\mathcal{S}} \otimes \mathbb{1}_{\mathcal{R}} \otimes |\Psi_{\mathrm{env}}\rangle\langle\Psi_{\mathrm{env}}|$. Also, we have $T(\theta)\Psi_{\mathcal{R}} = \Psi_{\mathcal{R}}$ for all θ. The spectral deformation technique relies on making the parameter θ complex.

(A) The coupling operator $W_{\mathcal{SR}} := V_{\mathcal{SR}} - J\Delta^{1/2} V_{\mathcal{SR}} \Delta^{-1/2} J$ is translation analytic in a strip $\kappa_{\theta_0} = \{z \,:\, 0 < \mathrm{Im} z < \theta_0\}$ and strongly continuous on the real axis. More precisely, there is a $\theta_0 > 0$ such that the map

$$\mathbb{R} \ni \theta \mapsto T^{-1}(\theta) W_{\mathcal{SR}} T(\theta) = W_{\mathcal{SR}}(\theta) \in \mathfrak{M}_{\mathcal{S}} \otimes \mathfrak{M}_{\mathcal{R}},$$

admits an analytic continuation into $\theta \in \kappa_{\theta_0}$ which is strongly continuous as $\mathrm{Im}\theta \downarrow 0$, and which satisfies

$$\sup_{0 \leq \mathrm{Im}\theta < \theta_0} \|W_{\mathcal{SR}}(\theta)\| < \infty.$$

The reduced dynamics operator will also be deformed as

$$M_\theta(\lambda) := T(\theta)^{-1} M(\lambda) T(\theta).$$

The ergodicity assumption **(E)** will now be written for this deformed operator $M_\theta(\lambda)$. More precisely, we will assume that the following Fermi Golden Rule condition holds

(FGR) There is a $\theta_1 \in \kappa_{\theta_0}$ and a $\lambda_0 > 0$ (depending on θ_1 in general) such that, for all λ with $0 < |\lambda| < \lambda_0$, $M_{\theta_1}(\lambda))$ satisfies **(E)**.

Of course, an important issue in the analysis of concrete models is the verification of this *Fermi Golden Rule* assumption **(FGR)**. Let us denote the eigenvalues of $h_{\mathcal{S}}$ by $E_1, \cdots, E_d$. When $\theta \in \kappa_{\theta_0}$ and $\lambda_{\mathcal{SR}} = \lambda_{\mathcal{SE}} = 0$. Then

$$M_\theta(0) = \mathrm{e}^{\mathrm{i}\tau(L_{\mathcal{S}}+L_{\mathcal{R}}+\theta N)} = \mathrm{e}^{\mathrm{i}\tau L_{\mathcal{S}}} \otimes \mathrm{e}^{\mathrm{i}\tau L_{\mathcal{R}}} \mathrm{e}^{\mathrm{i}\tau\theta N},$$

where N denotes the number operator on $\Gamma_-(\mathfrak{h}_{\mathcal{R}})$, and hence

$$\mathrm{sp}(M_\theta(0)) = \{\mathrm{e}^{\mathrm{i}\tau(E_j - E_k)}\}_{j,k\in\{1,\cdots,d\}} \cup \{\mathrm{e}^{\mathrm{i}l}\mathrm{e}^{-\tau j \mathrm{Im}\theta},\ l \in \mathbb{R}\}_{j\in\mathbb{N}^*}.$$

The effect of the analytic translation is to push the continuous spectrum of $M_\theta(0)$ onto circles with radii $\mathrm{e}^{-\tau j \mathrm{Im}\theta}$, $j = 1, 2, \ldots$, centered at the origin. Hence the discrete spectrum of $M_\theta(0)$, lying on the unit circle, is separated from the continuous spectrum by a distance $1 - e^{-\tau \mathrm{Im}\theta}$. Analytic perturbation theory in the parameters $\lambda_{\mathcal{SR}}$, $\lambda_{\mathcal{SE}}$ guarantees that the discrete and continuous spectra stay separated for small coupling. As a consequence a verification of **(FGR)** for concrete models, like the one of Section 1.6, is done via (perturbative) analysis *only of the discrete eigenvalues* of $M_\theta(\lambda)$.

1.5.3 Asymptotic state

As in Section 1.3 we will consider what we called *instantaneous observables*, i.e. "observables" of the form

$$[A_{\mathcal{SR}}; A^{(j)}](t) := A_{\mathcal{SR}} \otimes_{j=-\ell}^{r} \vartheta_{m(t)+j+1}(A^{(j)}),$$

where $A_{\mathcal{SR}} \in \mathfrak{M}_{\mathcal{S}} \otimes \mathfrak{M}_{\mathcal{R}}$ and $A^{(-\ell)}, \ldots, A^{(0)}, \ldots, A^{(r)} \in \mathfrak{M}_{\mathcal{E}}$.

Definition 1.10. *An observable A is called analytic if the map $\theta \to T(\theta)^{-1} A \Psi_0$, where $\Psi_0 = \Psi_{\mathcal{S}} \otimes \Psi_{\mathcal{R}} \otimes \Psi_{\mathrm{env}}$, has an analytic extension to $\theta \in \kappa_{\theta_0}$ which is continuous on the real axis.*

Note that for an instantaneous obervable $[A_{\mathcal{SR}}; A^{(j)}](t)$, since T acts on $\mathcal{H}_{\mathcal{R}}$ only, this is equivalent to $T(\theta)^{-1} A_{\mathcal{SR}} \Psi_0$ having such an extension.

Theorem 1.11. *[BJM4] Assume that assumptions* **(H')**, **(A)** *and* **(FGR)** *are satisfied. Then there is a $\lambda_0 > 0$ s.t. if $0 < |\lambda| < \lambda_0$, the following holds. There exists a state $\rho_{+,\lambda}$ on $\mathfrak{M}_{\mathcal{S}} \otimes \mathfrak{M}_{\mathcal{R}}$ such that for any normal initial state ϱ on $\mathfrak{M}$, and any analytic instantaneous observable $[A_{\mathcal{SR}}; A^{(j)}](t)$,*

$$\lim_{n\to\infty} \varrho\Big(\alpha_{\mathrm{RI}}^{n\tau}\big([A_{\mathcal{SR}}; A^{(j)}](n\tau)\big)\Big) = \rho_{+,\lambda}\Big(P_{\mathcal{SR}}\, \alpha_{\mathrm{RI}}^{\ell\tau}\big(A_{\mathcal{SR}} \otimes_{j=-\ell}^{r} A^{(j)}\big) P_{\mathcal{SR}}\Big).$$

In particular, if $A_{\mathcal{SR}} \in \mathfrak{M}_{\mathcal{S}} \otimes \mathfrak{M}_{\mathcal{R}}$, one has

$$\lim_{n\to\infty} \rho\big(\alpha_{\mathrm{RI}}^{n\tau}(A_{\mathcal{SR}})\big) = \rho_{+,\lambda}\big(A_{\mathcal{SR}}\big).$$

Moreover, for analytic $A_{\mathcal{SR}} \in \mathfrak{M}_{\mathcal{S}} \otimes \mathfrak{M}_{\mathcal{R}}$, we have the representation

$$\rho_{+,\lambda}(A_{\mathcal{SR}}) = \langle \psi_{\theta_1}^*(\lambda) | T(\theta_1)^{-1} A_{\mathcal{SR}} \Psi_{\mathcal{S}} \otimes \Psi_{\mathcal{R}} \rangle,$$

where $\psi_{\theta_1}^*(\lambda)$ is the unique invariant vector of the adjoint operator $[M_{\theta_1}(\lambda)]^*$, normalized as $\langle \psi_{\theta_1}^*(\lambda) | \Psi_{\mathcal{S}} \otimes \Psi_{\mathcal{R}} \rangle = 1$.

1.5.4 Thermodynamic properties

Besides the external work and the entropy production, the presence of two environments/reservoirs induces other quantities of interest, namely the heat fluxes. Starting from the Hamiltonian formalism, one defines the variation of energy in the system $\mathcal{S}$, the environment $\mathrm{env} = \mathcal{E}_1 + \mathcal{E}_2 + \cdots$ and the reservoir $\mathcal{R}$ between time $n\tau$ and $(n+1)\tau$ as

$$\begin{aligned}
\delta E^{\mathcal{S}}(n) &:= u((n+1)\tau,0)^* h_{\mathcal{S}} u((n+1)\tau,0) - u(n\tau,0)^* h_{\mathcal{S}} u(n\tau),\\
\delta E^{\mathcal{R}}(n) &:= u((n+1)\tau,0)^* h_{\mathcal{R}} u((n+1)\tau,0) - u(n\tau,0)^* h_{\mathcal{R}} u(n\tau),\\
\delta E^{\mathrm{env}}(n) &:= u((n+1)\tau,0)^* h_{\mathcal{E}_{n+1}} u((n+1)\tau,0) - u(n\tau,0)^* h_{\mathcal{E}_n} u(n\tau).
\end{aligned}$$

For the energy variation in the environment, recall that between time $n\tau$ and $(n+1)\tau$ only the $(n+1)$-th subsystem interacts with $\mathcal{S}$ and can thus exchange energy. When turning to the Liouvillian description one thus defines $\Delta E^{\#}(n) = \pi(\delta E^{\#}(n))$ for $\# = \mathcal{S}, \mathcal{R}, \mathrm{env}$. As a consequence of Theorem 1.11, we have the following

Proposition 1.12. *[BJM4] If* **(H')**, **(A)** *and* **(FGR)** *are satisfied and if the commutators* $[V_{\mathcal{SR}}, L_{\mathcal{S}}]$ *and* $[V_{\mathcal{SR}}, L_{\mathcal{R}}]$ *define analytic observables, then for any normal initial state* ϱ

$$\Delta E^{\#} := \lim_{n\to\infty} \frac{1}{n\tau} \sum_{k=1}^{n} \varrho(\Delta E^{\#}(n)) = \frac{1}{\tau} \rho_{+,\lambda}(P_{\mathcal{SR}} j^{\#} P_{\mathcal{SR}}), \qquad \# = \mathcal{R}, \mathrm{env},$$

where

$$j^{\mathrm{env}} = \mathrm{i} \int_0^\tau \alpha_{\mathrm{RI}}^t([\lambda_{\mathcal{SE}} V_{\mathcal{SE}}, L_{\mathcal{E}}])\, \mathrm{d}t, \qquad j^{\mathcal{R}} = \mathrm{i} \int_0^\tau \alpha_{\mathrm{RI}}^t([\lambda_{\mathcal{SR}} V_{\mathcal{SR}}, L_{\mathcal{R}}])\, \mathrm{d}t.$$

and $\Delta E^{\mathcal{S}} = \lim_{n\to\infty} \frac{1}{n\tau} \sum_{k=1}^{n} \varrho(\Delta E^{\mathcal{S}}(n)) = 0$. *The external work is*

$$\Delta W = \frac{1}{\tau} \rho_{+,\lambda}\Big(P_{\mathcal{SR}} V(\lambda) P_{\mathcal{SR}} - P_{\mathcal{SR}} \alpha_{\mathrm{RI}}^{\tau}(V(\lambda)) P_{\mathcal{SR}} \Big),$$

and we have

$$\Delta W = \Delta E^{\mathcal{R}} + \Delta E^{\mathrm{env}}.$$

If moreover, the reference states are KMS*-states at inverse temperatures* $\beta_{\mathcal{S}}$, $\beta_{\mathcal{R}}$ *and* $\beta_{\mathcal{E}}$, *the entropy production* ΔS *exists and satisfies*

$$\Delta S = \beta_{\mathcal{E}} \Delta E^{\mathrm{env}} + \beta_{\mathcal{R}} \Delta E^{\mathcal{R}}.$$

As expected, the energy gain in the system (due to the external work) is shared between the reservoir $\mathcal{R}$ and the environment. The details of how the energy variations are shared between the subsystems depends on the particulars of the model considered, see Section 1.6.4 for an explicit example. Moreover, note that contrary to what happened in Section 1.3, the external work might be positive or negative.

1.6 A toy example

We end this chapter with the analysis of a specific example, which is the simplest non-trivial example of RI system, namely all the subsystems ($\mathcal{S}$ and the $\mathcal{E}_k$'s) are 2-level systems (or spin $\frac{1}{2}$). Our purpose is twofold: on one hand we want to illustrate the general results decribed in the preceeding sections, on the other hand this model is a toy version of the "One-Atom Maser" which we will consider in Section 2.1 and it is therefore an instructive playground to further analysis of this (and other) more realistic model (e.g. in the non-equilibrium situation).

We describe the model in the Hamiltonian formalism. For most of our purpose it will be sufficient and we shall stick to it. Only when we add an extra reservoir (to illustrate the leaky situation) we will turn to the Liouvillian formalism.

1.6.1 Description of the model

The Hilbert spaces for $\mathcal{S}$ and the $\mathcal{E}_m$ are copies of $\mathbb{C}^2$. Let $E, E_0 > 0$ be the "excited" energy level of $\mathcal{S}$ and of $\mathcal{E}$, respectively. Accordingly, the Hamiltonians are given by

$$h_{\mathcal{S}} = \begin{pmatrix} 0 & 0 \\ 0 & E \end{pmatrix} \quad \text{and} \quad h_{\mathcal{E}} = \begin{pmatrix} 0 & 0 \\ 0 & E_0 \end{pmatrix}.$$

We will denote by $|0\rangle$, resp. $|1\rangle$, the ground state, resp. excited state, of $\mathcal{S}$ or $\mathcal{E}$. If we denote by a/a^*, resp. b/b^*, the annihilation/creation operators for $\mathcal{S}$, resp. $\mathcal{E}$, i.e.

$$a = b = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad a^* = b^* = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \tag{1.21}$$

we can then write

$$h_{\mathcal{S}} = Ea^*a, \quad \text{and} \quad h_{\mathcal{E}} = E_0 b^* b.$$

The interaction operator is

$$v(\lambda) = \frac{\lambda}{2}(a \otimes b^* + a^* \otimes b).$$

It induces an exchange process between $\mathcal{S}$ and the subsystem $\mathcal{E}_k$ it is coupled to: $\mathcal{S}$ flips from its ground state to its excited state while $\mathcal{E}_k$ flips the other way around, or vice versa (the parameter λ is just a coupling constant). Note that v has the particular feature that it commutes with the total number operator $N^{\mathrm{tot}} = a^*a \otimes \mathbb{1} + \mathbb{1} \otimes b^*b$.

It remains to specify the reference states of the subsystems $\mathcal{E}_k$. They will be thermal states at some inverse temperature β_k:

$$\rho_{\mathcal{E}_k} = \frac{\mathrm{e}^{-\beta_k h_{\mathcal{E}}}}{\mathrm{Tr}\left(\mathrm{e}^{-\beta_k h_{\mathcal{E}}}\right)} = Z_{\beta_k}^{-1} \mathrm{e}^{-\beta_k h_{\mathcal{E}}} =: \rho_{\beta_k}.$$

The calulation of the RDM $\mathcal{L}_\beta$ associated to a subsystem $\mathcal{E}$ at inverse temperature β and interacting with $\mathcal{S}$ for an amount of time τ is a straightforward calculation since $h = h_{\mathcal{S}} + h_{\mathcal{E}} + v(\lambda)$ can be easily diagonalized. One gets

$$\mathcal{L}_\beta(\rho) = \sum_{\sigma,\sigma'=0,1} V_{\sigma'\sigma} \rho V_{\sigma'\sigma}^*$$

where the operators $V_{\sigma'\sigma}$ are given by

$$V_{00} = \frac{1}{\sqrt{Z_\beta}}\,\mathrm{e}^{-\mathrm{i}\tau\frac{E+E_0}{2}N}\,C(N)^*, \qquad V_{10} = \frac{1}{\sqrt{Z_\beta}}\,\mathrm{e}^{-\mathrm{i}\tau\frac{E+E_0}{2}N}\,S(1-N)\,a,$$

$$V_{01} = \frac{\mathrm{e}^{-\beta E_0/2}}{\sqrt{Z_\beta}}\,\mathrm{e}^{-\mathrm{i}\tau\frac{E+E_0}{2}N}\,S(N)\,a^*, \quad V_{11} = \frac{\mathrm{e}^{-\beta E_0/2}}{\sqrt{Z_\beta}}\,\mathrm{e}^{-\mathrm{i}\tau\frac{E+E_0}{2}N}\,C(1-N),$$

with $N = a^*a$ the number operator for $\mathcal{S}$,

$$C(N) = \cos\left(\frac{\nu\tau}{2}N\right) + \mathrm{i}\frac{\Delta}{\nu}\sin\left(\frac{\nu\tau}{2}N\right), \quad S(N) = \frac{\lambda}{\nu}\sin\left(\frac{\nu\tau}{2}N\right),$$

and where $\Delta = E - E_0$ and $\nu = \sqrt{\Delta^2 + \lambda^2}$. It is now easy to derive the spectral data of $\mathcal{L}_\beta$.

Proposition 1.13. *The eigenvalues of $\mathcal{L}_\beta$ are* 1, $e_\pm = \mathrm{e}^{\pm\mathrm{i}\tau\frac{E+E_0}{2}}\left(\cos\left(\frac{\nu\tau}{2}\right) \pm \mathrm{i}\frac{\Delta}{\nu}\sin\left(\frac{\nu\tau}{2}\right)\right)$, *and* $e_0 = 1 - \frac{\lambda^2}{\nu^2}\sin^2\left(\frac{\nu\tau}{2}\right)$. *Corresponding eigenstates are $\rho_{\mathcal{S},\beta^*} := \frac{\mathrm{e}^{-\beta^* h_\mathcal{S}}}{\mathrm{Tr}\left(\mathrm{e}^{-\beta^* h_\mathcal{S}}\right)}$ where $\beta^* = \frac{E_0}{E}\beta$, $|0\rangle\langle 1|$ (for e_+), $|1\rangle\langle 0|$ (for e_-) and $|0\rangle\langle 0| - |1\rangle\langle 1|$. Moreover, $|e_\pm|, |e_0| \leq 1$ with equality iff $\nu\tau \in 2\pi\mathbb{N}$.*

Note the renormalization factor $\frac{E_0}{E}$ in the inverse temperature of the invariant state. Note also that the ergodicity assumption **(E)** of Section 1.3 holds iff $\nu\tau \notin 2\pi\mathbb{N}$, which is a kind of non-resonant condition (a similar phenomenon holds for the "One-Atom Maser" model, see Section 2.1).

In the next sections we illustrate the general results about RI systems described in the previous sections. For simplicity, the initial states of the $\mathcal{E}_n$ will be the reference states $\rho_{\mathcal{E}_n}$.

1.6.2 Ideal case (or return to equilibrium)

Proposition 1.14. *Let $\tau_n \equiv \tau$ and $\beta_n \equiv \beta$. If $\nu\tau \notin 2\pi\mathbb{N}$, then for any initial state ρ of $\mathcal{S}$ there exists $C > 0$ such that*

$$\|\rho(n) - \rho_{\mathcal{S},\beta^*}\| \leq C\mathrm{e}^{-\gamma n}, \qquad \forall n \in \mathbb{N},$$

where $\beta^* = \frac{E_0}{E}\beta$ and $\gamma = -\log\left(\sqrt{1 - \frac{\lambda^2}{\nu^2}\sin^2\left(\frac{\nu\tau}{2}\right)}\right)$.

In other words, "return to equilibrium" holds at some explicit renormalized temperature. This renormalization can be understood as follows. The interaction is an exchange process in the *number* of excitation, but not in energy: each interaction involves an amount of energy E_0 for the environment but E for $\mathcal{S}$.

We can then compute the external work (and thus the entropy production) in the system. Since we are in the ideal case with invariant initial states on the $\mathcal{E}_n$, (1.8) is now simply

$$\delta E(n) = \lambda\mathrm{Tr}_{\mathfrak{h}_\mathcal{S}\otimes\mathfrak{h}_\mathcal{E}}\left[\rho(n)\otimes\rho_\beta\, v\right] - \lambda\mathrm{Tr}_{\mathfrak{h}_\mathcal{S}\otimes\mathfrak{h}_\mathcal{E}}\left[\rho(n-1)\otimes\rho_\beta\,\mathrm{e}^{\mathrm{i}\tau h}v\mathrm{e}^{-\mathrm{i}\tau h}\right].$$

Hence, using Proposition 1.14 and the fact that $\rho_{\mathcal{S},\beta^*}\otimes\rho_\beta$ commutes with h, we get

Proposition 1.15. *Let $\tau_n \equiv \tau$ and $\beta_n \equiv \beta$. If $\nu\tau \notin 2\pi\mathbb{N}$, then for any initial state ρ we have $\delta E(n) = O(\mathrm{e}^{-\gamma n})$. Hence $\Delta W = \Delta S = 0$.*

Remark 1.5. *That the entropy production and the external work vanish is specific to this model. If one changes the interaction, e.g. taking the full dipole interaction $(a + a^*) \otimes (b + b^*)$, this is not true anymore. Namely, one then has*

$$\lim_{n\to\infty} \delta E(n) = \frac{\lambda^2\tau^2 E}{2} \tanh\left(\frac{\beta E_0}{2}\right) \times \frac{\mathrm{sinc}^2\left(\frac{\nu\tau}{2}\right)\mathrm{sinc}^2\left(\frac{\mu\tau}{2}\right)}{\mathrm{sinc}^2\left(\frac{\nu\tau}{2}\right) + \mathrm{sinc}^2\left(\frac{\mu\tau}{2}\right)},$$

where $\mu = \sqrt{(E+E_0)^2 + \lambda^2}$ and $\mathrm{sinc}(x) = \frac{\sin(x)}{x}$. To switch from one element of the environment to the other therefore requires some non-trivial external work.

Remark 1.6. *It is easy to see that one can rewrite $\Delta W(N) := \sum_{n=1}^{N} \delta E(n)$ as*

$$\Delta W(N) = \sum_{n=1}^{N} \mathrm{Tr}_{\mathfrak{h}_\mathcal{S}\otimes\mathfrak{h}_\mathcal{E}} \left[\rho(n-1) \otimes \rho_\beta \left(\mathrm{e}^{\mathrm{i}\tau h} h_\mathcal{E} \mathrm{e}^{-\mathrm{i}\tau h} - h_\mathcal{E}\right)\right] + O(1), \tag{1.22}$$

where the $O(1)$ refers to the $N \to \infty$ limit. The sum in the right hand side is clearly the amount of energy which flows into the environment. In other words, Eq. (1.22) simply says that, up to some bounded term, the energy which is added to the system is spread in the environment (the system $\mathcal{S}$ is finite and can therefore not continuously gain or lose energy).

1.6.3 Random case

Proposition 1.16. *1.* Random interaction time. *Suppose that $\beta_n \equiv \beta$, and that $\tau(\omega) > 0$ is a random variable satisfying $\mathrm{p}(\nu\tau \notin 2\pi\mathbb{N}) \neq 0$. Then there exists a set $\Omega_0 \subset \tilde{\Omega}$ of probability one and a constant $\alpha > 0$ such that for all $\tilde{\omega} \in \Omega_0$, for all inital states ρ*

$$\|\rho(n, \tilde{\omega}) - \rho_{\mathcal{S},\beta^*}\| \leq C(\tilde{\omega})\mathrm{e}^{-\alpha n}, \qquad \forall n \in \mathbb{N}, \tag{1.23}$$

for some $C(\tilde{\omega}) > 0$ and with $\beta^* = \beta E_0/E$.

2. Random temperature of $\mathcal{E}$. *Suppose that $\beta(\omega)$ is a random variable, and that $\tau_n \equiv \tau > 0$ satisfies $\nu\tau \notin 2\pi\mathbb{N}$. Then there exists a set $\Omega_1 \subset \tilde{\Omega}$ of probability one s.t. for all $\tilde{\omega} \in \Omega_1$, for all initial states ρ,*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \rho(k, \tilde{\omega}) = \mathbb{E}\left[\rho_{\mathcal{S},\beta^*(\omega)}\right] = \rho_{\mathcal{S},\beta'},$$

with $\beta' := -E^{-1} \log\left(\mathbb{E}[Z^{-1}_{\mathcal{S},\beta(\omega)E_0/E}]^{-1} - 1\right)$.

Remark 1.7. *The better convergence in (1.23) than expected from Theorem 1.6 is due to the fact that the random RDM actually possesses a deterministic invariant state so that we can use Theorem 4.4 instead of Theorem 4.3.*

An application of Propositions 1.8 and 1.9 then gives

Proposition 1.17. *Suppose that* $\mathrm{p}\left(\nu\tau \notin 2\pi\mathbb{N}\right) \neq 0$*, and let* $\kappa = E_0(1-e_0)$*. Then*

$$\Delta W = \frac{1}{\mathbb{E}(\tau)}\mathrm{Cov}\left(\kappa, \frac{1}{1+\mathrm{e}^{-\beta E_0}}\right), \quad \mathbb{P}-a.s.,$$

$$\Delta S = \frac{1}{\mathbb{E}(\tau)}\mathrm{Cov}\left(\beta\kappa, \frac{1}{1+\mathrm{e}^{-\beta E_0}}\right), \quad \mathbb{P}-a.s..$$

Note that $\kappa = E_0(1-e_0) = E_0\frac{\lambda^2}{\nu^2}\sin^2\left(\frac{\nu\tau}{2}\right)$ is a random variable via E_0, τ or ν. We then specify to the two situations considered in Proposition 1.16:

1. If only τ is random, then $\Delta W = \Delta S = 0$. As in the ideal case, there is no external work for this system, and since we are still in an equilibrium situation there is no entropy production.

2. If only β is random, then $\Delta W = 0$ and $\Delta S = \frac{\kappa}{\tau}\mathrm{Cov}\left(\beta, \frac{1}{1+\mathrm{e}^{-\beta E_0}}\right) \geq 0$ with equality iff $\beta(\omega) = \mathbb{E}(\beta)$ a.s. In other words, there is still no external work to switch from an interaction to the other. However, as soon as there is some randomness in the temperature we are in a non-equilibrium situation and hence the entropy production is strictly positive.

1.6.4 Leaky case

For the reservoir, we consider a bath of non-interacting and non-relativistic fermions at inverse temperature $\beta_{\mathcal{R}}$. The one particle space is $L^2(\mathbb{R}^3, \mathrm{d}^3k)$ and the one-particle energy operator is the multiplication operator by $|k|^2$. The interaction between the small system $\mathcal{S}$ and the reservoir $\mathcal{R}$ is chosen of electric dipole type, i.e. of the form $v_{\mathcal{SR}} = (a + a^*) \otimes \varphi_{\mathcal{R}}(f)$ where $f \in \mathfrak{h}_{\mathcal{R}}$ is a form factor and $\varphi_{\mathcal{R}}(f) = \frac{1}{\sqrt{2}}(a_{\mathcal{R}}(f) + a^*_{\mathcal{R}}(f))$. We shall not explain here how to get a description similar to the one given in Section 1.5.1, and refer the reader to e.g. [AW, JP1, BJM4].

The situation where $\mathcal{S}$ is interacting with $\mathcal{R}$ alone has been treated in previous works, see e.g. [JP1]. Normal initial states approach the joint equilibrium state, i.e. the equilibrium state of the coupled system $\mathcal{S}+\mathcal{R}$ at temperature $\beta_{\mathcal{R}}^{-1}$, with speed $\mathrm{e}^{-n\tau\gamma_{\mathrm{th}}}$ (we consider discrete moments in time, $t = n\tau$, to compare with the repeated interaction situation). On the other hand, if $\mathcal{S}$ is coupled only to the subsystems $\mathcal{E}$, initial normal states approach the equilibrium state of $\mathcal{S}$ at inverse temperature $\beta^*_{\mathcal{E}}$ where $\beta^*_{\mathcal{E}} = \beta_{\mathcal{E}}\frac{E_0}{E}$, and with speed $\mathrm{e}^{-n\tau\gamma_{\mathrm{ri}}}$. The convergence rates are given by

$$\begin{aligned}
\gamma_{\mathrm{th}} &= \lambda^2_{\mathcal{SR}}\gamma^{(2)}_{\mathrm{th}} + O(\lambda^4_{\mathcal{SR}}), \qquad \text{with} \quad \gamma^{(2)}_{\mathrm{th}} = \frac{\pi}{4}\sqrt{E}\|f(\sqrt{E})\|^2,\\
\gamma_{\mathrm{ri}} &= -\frac{1}{\tau}\log\left(\sqrt{1-\frac{\lambda^2_{\mathcal{SE}}}{\nu^2}\sin^2\left(\frac{\nu\tau}{2}\right)}\right)\\
&= \lambda^2_{\mathcal{SE}}\gamma^{(2)}_{\mathrm{ri}} + O(\lambda^4_{\mathcal{SE}}), \qquad \text{with} \quad \gamma^{(2)}_{\mathrm{ri}} = \tau\,\mathrm{sinc}^2\left(\frac{\Delta\tau}{2}\right),
\end{aligned}$$

where $\mathrm{sinc}(x) = \sin(x)/x$ and $\|f(\sqrt{E})\|^2 := \int_{S^2} |f(\sqrt{E}\,\sigma)|^2 \mathrm{d}\sigma$.

In order to satisfy the translation analyticity hypothesis **(A)**, we need to make some assumption on the form factor f. Let $I(\delta) := \{z \in \mathbb{C},\, |\mathrm{Im}(z)| < \delta\}$. We denote by $H^2(\delta)$ the Hardy class of analytic functions $h : I(\delta) \to L^2(S^2, \mathrm{d}\sigma)$ which satisfy

$$\|h\|_{H^2(\delta)} := \sup_{|\theta|<\delta} \int_{\mathbb{R}} \|h(s+i\theta)\|^2 \mathrm{d}s < \infty.$$

(A') There is a $\delta > 0$ such taht $\mathrm{e}^{-\beta_{\mathcal{R}} s/2} f_0(s) \in H^2(\delta)$ where

$$f_0(s) : S^2 \ni \sigma \mapsto \frac{|s|^{1/4}}{2} \begin{cases} f(\sqrt{s}\,\sigma) & \text{if } s \geq 0, \\ \bar{f}(\sqrt{-s}\,\sigma) & \text{if } s < 0. \end{cases}$$

Remark 1.8. *The function f_0 appears in the representation of the interaction operator $V_{\mathcal{SR}}$ in the Liouvillian formalism, namely*

$$V_{\mathcal{SR}} = ((a + a^*) \otimes \mathbb{1}_{\mathbb{C}^2}) \otimes \varphi(f_{\beta_{\mathcal{R}}}) \in \mathfrak{M}_{\mathcal{S}} \otimes \mathfrak{M}_{\mathcal{R}}$$

where $f_{\beta_{\mathcal{R}}} \in \mathfrak{h} = L^2(\mathbb{R}, \mathrm{d}s; L^2(S^2, \mathrm{d}\sigma))$ is related to the initial form factor $f \in L^2(\mathbb{R}^3, \mathrm{d}^3k)$ by

$$\left(f_{\beta_{\mathcal{R}}}(s)\right)(\sigma) = \frac{1}{\sqrt{2}} \frac{|s|^{1/4}}{\sqrt{1 + \mathrm{e}^{-\beta_{\mathcal{R}} s}}} \begin{cases} f(\sqrt{s}\,\sigma) & \text{if } s \geq 0, \\ \bar{f}(\sqrt{-s}\,\sigma) & \text{if } s < 0. \end{cases}$$

Hence f_0 is $f_{\beta_{\mathcal{R}}=0}$.

Proposition 1.18. *Assume f satisfies* **(A')**, *$\|f(\sqrt{E})\| \neq 0$ and $\Delta\tau \notin 2\pi\mathbb{Z}$. Then the asymptotic state $\rho_{+,\lambda}$ is given by*

$$\rho_{+,\lambda} = \left(\gamma \rho_{\mathcal{S},\beta_{\mathcal{R}}} + (1-\gamma)\rho_{\mathcal{S},\beta^*_{\mathcal{E}}}\right) \otimes \rho_{\mathcal{R},\beta_{\mathcal{R}}} + O(\lambda),$$

where $\rho_{\#,\beta}$ is the Gibbs state of $\# (= \mathcal{S}, \mathcal{R})$ at inverse temperature β and where

$$\gamma = \frac{\lambda^2_{\mathcal{SR}} \gamma^{(2)}_{\mathrm{th}}}{\lambda^2_{\mathcal{SR}} \gamma^{(2)}_{\mathrm{th}} + \lambda^2_{\mathcal{SE}} \gamma^{(2)}_{\mathrm{ri}}}.$$

Remark 1.9. *The assumption $\Delta\tau \notin 2\pi\mathbb{Z}$ corresponds, in the perturbative regime, to the non-resonant condition $\nu\tau \notin 2\pi\mathbb{N}$.*

Remark 1.10. *The fact that the asymptotic state $\rho_{+,\lambda}$ is a convex combination of the two asymptotic states $\rho_{\mathcal{S},\beta_{\mathcal{R}}}$ and $\rho_{\mathcal{S},\beta^*_{\mathcal{E}}}$ holds only because the system $\mathcal{S}$ is a two-level system and is not true in general.*

An application of Proposition 1.12 then gives the following explicit values for the external work, the entropy production and the energy fluxes.

Proposition 1.19. *Assume f satisfies* **(A')**, $\|f(\sqrt{E})\| \neq 0$ *and* $\Delta\tau \notin 2\pi\mathbb{Z}$. *Then*

$$\begin{aligned}
\Delta E^{\rm env} &= \kappa E_0 \left(\mathrm{e}^{-\beta_{\mathcal{R}}} - \mathrm{e}^{-\beta^*_{\mathcal{E}} E}\right) + O(\lambda^3),\\
\Delta E^{\mathcal{R}} &= \kappa E \left(\mathrm{e}^{-\beta^*_{\mathcal{E}} E} - \mathrm{e}^{-\beta_{\mathcal{R}} E}\right) + O(\lambda^3),\\
\Delta W &= \kappa (E_0 - E) \left(\mathrm{e}^{-\beta_{\mathcal{R}} E} - \mathrm{e}^{-\beta^*_{\mathcal{E}} E}\right) + O(\lambda^3),\\
\Delta S &= \kappa (\beta^*_{\mathcal{E}} E - \beta_{\mathcal{R}} E) \left(\mathrm{e}^{-\beta_{\mathcal{R}} E} - \mathrm{e}^{-\beta^*_{\mathcal{E}} E}\right) + O(\lambda^3),
\end{aligned}$$

where $\kappa = Z^{-1}_{\mathcal{S},\beta_{\mathcal{R}}} Z^{-1}_{\mathcal{S},\beta^*_{\mathcal{E}}} \dfrac{\lambda^2_{\mathcal{SR}}\gamma^{(2)}_{\rm th}\ \lambda^2_{\mathcal{SE}}\gamma^{(2)}_{\rm ri}}{\lambda^2_{\mathcal{SR}}\gamma^{(2)}_{\rm th} + \lambda^2_{\mathcal{SE}}\gamma^{(2)}_{\rm ri}}$.

Remark 1.11. *1. The constant κ is positive and of order λ^2. Moreover it is zero if at least one of the two coupling constants vanishes (we are then in an equilibrium situation and there is no energy flux neither entropy production).*

2. The energy flux $\Delta E^{\rm env}$ is positive (energy flows into the environment) if and only if the reservoir temperature $T_{\mathcal{R}} = \beta_{\mathcal{R}}^{-1}$ is greater than the renormalized temperature $T^*_{\mathcal{E}} = (\beta^*_{\mathcal{E}})^{-1}$ of the environment, i.e. iff the reservoir is "hotter". A similar statement holds for the energy flux $\Delta E^{\mathcal{R}}$. Note that,as for return to equilibrium, it is not the temperature of the environment which plays a role but its renormalized value.

3. When both the reservoir and the environment are coupled to the system $\mathcal{S}$, i.e. $\lambda_{\mathcal{SR}}\lambda_{\mathcal{SE}} \neq 0$, the entropy production vanishes (at the main order) if and only if the two temperatures $T_{\mathcal{R}}$ and $T^*_{\mathcal{E}}$ are equal, i.e. if and only if we are in an equilibrium situation. Once again, it is not the initial temperature of the chain which plays a role but the renormalized one.

4. As mentioned at the end of Section 1.5, the external work can be either positive or negative depending on the parameters of the model.

1.6.5 Non-equilibrium situation

As we mentioned at the beginning of Section 1.4, one can create a non-equilibrium situation in the framework of RI systems if the initial state of the subsystems $\mathcal{E}_n$ are for example thermal equilibrium states at different temperatures, i.e. we assume that the system $\mathcal{S}$ interacts with K "reservoirs" at a priori different temperatures. More precisely, we take, for any $m \in \mathbb{N}$, $\rho_{\mathcal{E}_{mK+1}} = \rho_{\beta_1}$, $\rho_{\mathcal{E}_{mK+2}} = \rho_{\beta_2}$, etc... We denote by $\mathcal{L}_1$, $\mathcal{L}_2$,... the corresponding RDM's. The state of $\mathcal{S}$ at time n is therefore

$$\rho(n) = \left(\mathcal{L}_j \circ \mathcal{L}_{j-1} \circ \cdots \mathcal{L}_1 \circ \mathcal{L}_K \circ \cdots \circ \mathcal{L}_{j+1}\right)^m \left(\mathcal{L}_j \circ \cdots \circ \mathcal{L}_1(\rho)\right),\ n = mK + j. \quad (1.24)$$

Non-Equilibrium Steady State

Generically, in non-equilibrium situation, the usual limit $n \to \infty$ of $\rho(n)$ does not exist and one has to resort to limits in the ergodic mean. However here, in view of (1.24), the K subsequences $\rho(mK + j)$ will generically converge as m goes to infinity. This is due to the fact that the maps $\mathcal{L}^{\rm noneq}_j := \mathcal{L}_j \circ \mathcal{L}_{j-1} \circ \cdots \mathcal{L}_1 \circ \mathcal{L}_K \circ \cdots \circ \mathcal{L}_{j+1}$ have the same kind of spectrum

as $\mathcal{L}$ in the equilibrium case: they have a simple eigenvalue 1 while the other ones lie inside the open unit disk of $\mathbb{C}$. Indeed, the $\mathcal{L}_j$'s have almost the same spectral data. By Proposition 1.13, only their invariant states differ, otherwise they have the same eigenvalues e_0 and $e_\pm$ with the same eigenstates. Hence e_0^K and $e_\pm^K$ are eigenvalues of any of the $\mathcal{L}_j^{\mathrm{noneq}}$ (with eigenstates $|0\rangle\langle 0| - |1\rangle\langle 1|$, $|0\rangle\langle 1|$ and $|1\rangle\langle 0|$) and, provided $\nu\tau \notin 2\pi\mathbb{N}$, have modulus strictly less than 1. It thus remains to find their invariant state.

Theorem 1.20. *Suppose that the spins $\mathcal{E}_{mK+j}$ are initially at temperature β_j^{-1}, $j = 1, \ldots, K$ and assume $\nu\tau \notin 2\pi\mathbb{N}$. Then, the unique invariant state of $\mathcal{L}_j^{\mathrm{noneq}}$ is*

$$\rho_{\mathcal{S},+}^{(j)} = \frac{1-e_0}{1-e_0^K}\left(\rho_{\mathcal{S},\beta_j^*} + e_0\rho_{\mathcal{S},\beta_{j-1}^*} + \cdots + e_0^{K-1}\rho_{\mathcal{S},\beta_{j+1}^*}\right).$$

As a consequence, for any initial state ρ we have

$$\lim_{m\to\infty} \rho(mK+j) = \rho_{\mathcal{S},+}^{(j)}, \qquad j = 1, \ldots, K.$$

In particular, we have

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} \rho(n) = \rho_{\mathcal{S},+} = \frac{1}{K}\sum_{j=1}^{K} \rho_{\mathcal{S},+}^{(j)} = \frac{1}{K}\sum_{j=1}^{K} \rho_{\mathcal{S},\beta_j^*}.$$

The above theorem shows that the NESS $\rho_{\mathcal{S},+}$ is simply the average of the equilibrium states at the renormalized temperatures β_j^*, $j = 1, \ldots, K$.

The particular structure of repeated interaction systems leads to a stronger result than the usual convergence in the ergodic mean to the NESS $\rho_{\mathcal{S},+}$, namely the convergence of "complementary" subsequences to partial NESSes. Each of these partial NESSes only gives a part of the information on the large time limit of the system. However, as we will see, they appear naturally in the study of the heat fluxes.

Fluxes and entropy production

Obviously, the j-th beam can exchange energy only when it interacts with $\mathcal{S}$, that is in the time intervals $[(mK+j-1)\tau, (mK+j)\tau)$. During such an interval, and if the system $\mathcal{S}$ is in a state ρ at the beginning of the interaction, the amount of energy lost by the beam is

$$\begin{aligned}\Delta E_j &:= \mathrm{Tr}_{\mathfrak{h}_\mathcal{S}\otimes\mathfrak{h}_\mathcal{E}}\left[\rho\otimes\rho_{\beta_j}\ \left(h_\mathcal{E} - \mathrm{e}^{\mathrm{i}\tau h} h_\mathcal{E} \mathrm{e}^{-\mathrm{i}\tau h}\right)\right] \\ &= \frac{E_0\lambda^2}{Z_{\beta_j}\nu^2}\sin^2\left(\frac{\nu\tau}{2}\right)\mathrm{Tr}_\mathcal{S}\left[\rho\times\left(\mathrm{e}^{-\beta_j E_0}|0\rangle\langle 0| - |1\rangle\langle 1|\right)\right]. \qquad (1.25)\end{aligned}$$

Since any of the beams interacts with $\mathcal{S}$ for a fraction $1/K$ of time, it is reasonable to define the energy flux observable for the j-th beam as

$$\Phi_j := \frac{E_0}{K\tau Z_{\beta_j}} \times \underbrace{\frac{\lambda^2}{\nu^2}\sin^2\left(\frac{\nu\tau}{2}\right)}_{=(1-e_0)} \times \left(\mathrm{e}^{-\beta_j E_0}|0\rangle\langle 0| - |1\rangle\langle 1|\right), \qquad (1.26)$$

so that $\frac{\Delta E_j}{K\tau} = \mathrm{Tr}_{\mathcal{S}}\left[\rho\Phi_j\right]$.

The first idea would be to define the various energy fluxes in the NESS as $\mathrm{Tr}\left[\rho_{\mathcal{S},+}\Phi_j\right]$. It is actually not the right choice. Indeed, as we have seen there are several NESS'es: a "global" one and K "partial" ones. When $\mathcal{S}$ will start to interact with the j-th beam, i.e. when heat flux is possible through this beam, it is asymptotically in the partial NESS $\rho_{\mathcal{S},+}^{(j-1)}$. It is therefore more natural to define the flux in the j-th beam as

$$\phi_j := \mathrm{Tr}_{\mathcal{S}}\left[\rho_{\mathcal{S},+}^{(j-1)}\,\Phi_j\right].$$

It is easy to see that for any initial state ρ of $\mathcal{S}$, the total lost of energy in the j-th beam between time 0 and time $(mK+j)\tau$ is

$$\Delta E_j(mK+j) = \sum_{l=0}^{m} \mathrm{Tr}_{\mathcal{S}\otimes\mathcal{E}}\left[\rho(lK+j-1)\otimes\rho_{\beta_j}\,\left(h_{\mathcal{E}} - \mathrm{e}^{\mathrm{i}\tau h}h_{\mathcal{E}}\mathrm{e}^{-\mathrm{i}\tau h}\right)\right],$$

so that asymptotically the variation of energy per time unit in the j-th beam is indeed

$$\lim_{m\to\infty}\frac{\Delta E_j(mK+j)}{(mK+j)\tau} = \frac{1}{K\tau}\mathrm{Tr}_{\mathcal{S}\otimes\mathcal{E}}\left[\rho_{\mathcal{S},+}^{(j-1)}\otimes\rho_{\beta_j}\,\left(h_{\mathcal{E}} - \mathrm{e}^{\mathrm{i}\tau h}h_{\mathcal{E}}\mathrm{e}^{-\mathrm{i}\tau h}\right)\right] = \phi_j.$$

A simple calculation leads to

$$\phi_j = \frac{E_0(1-e_0)^2}{K\tau(1-e_0^K)}\sum_{k=1}^{K}(Z_{\beta_k}^{-1} - Z_{\beta_j}^{-1})e_0^{[j-k-1]}, \tag{1.27}$$

where $[j-k-1]$ is the residue class of $j-k-1$ mod K. One easily checks that $\sum_j \phi_j = 0$, i.e. (as in the equilibrium situation) no external work is added to the system.

One can then calculate the entropy production and get

$$\begin{aligned}\Delta S &= -\sum_{j=1}^{K}\beta_j\phi_j \qquad (1.28)\\ &= \frac{E_0(1-e_0)^2}{K\tau(1-e_0^K)}\sum_{l=0}^{K-1}\left(\sum_{j=1}^{K}\beta_j(Z_{\beta_j}^{-1} - Z_{\beta_{[j-l-1]}}^{-1})\right)e_0^l,\end{aligned}$$

which is stricly positive unless all the temperatures are the same. (One can see that each sum $\sum_{j=1}^{K}\beta_j(Z_{\beta_j}^{-1} - Z_{\beta_{[j-l-1]}}^{-1})$ is positive and non-zero unless all the temperatures appearing in the sum are the same.)

Remark 1.12. *As usual, in the case of 2 beams, the positivity of entropy production and the conservation of energy show that heat flows from the hot beam to the cold one.*

Linear response

The next step in the study of the non-equilibrium situation is the linear response theory, i.e. how the system will react to a small perturbation from the equilibrium.

Let $X_j := \beta - \beta_j$ denote the thermodynamical forces which drive the system out of equilibrium and $X = (X_1, \dots, X_K)$. Using (1.28) together with the conservation of energy, we may write

$$\mathrm{d}S_+(X) = \sum_{j=1}^{K} X_j \phi_j(X).$$

(We have emphasized here that the fluxes and the entropy production depend on the thermodynamical forces.) Note also that all the fluxes vanish when $X = 0$. Our concern here is with the kinetic coefficients

$$L_{jk} := \frac{\partial \phi_j}{\partial X_k} \lceil_{X=0}.$$

Taylor's formula then gives

$$\phi_j = \sum_{k=1}^{K} L_{jk} X_k + O(X^2), \quad \text{and} \quad \mathrm{d}S_+ = \sum_{j,k=1}^{K} L_{jk} X_j X_k + o(X^2).$$

The positivity of entropy production implies that the quadratic form $\sum_{j,k=1}^{K} L_{jk} X_j X_k$ is non-negative. However, this does not imply that the $K \times K$ matrix L is symmetric!

Our main concern here is not really with the explicit value of the kinetic coefficients (this is a trivial computation using (1.27)) but rather with the Green-Kubo formula and the Onsager reciprocity relations. The Onsager reciprocity relations assert that the matrix $L = (L_{jk})_{j,k}$ is symmetric. The Green-Kubo formula expresses the kinetic coefficients in terms of the *equilibrium* current-current correlation function, i.e. of $C_{jk}(n) := \mathrm{Tr}_{\mathcal{S}} \left[\left((\mathcal{L}_\beta^*)^n (\Phi_j) \times \Phi_k \right) \rho_{\mathcal{S},\beta^*} \right]$ where $\mathcal{L}_\beta^*$ is the dual of $\mathcal{L}_\beta$ (here we evolve an observable), namely

$$L_{jk} \stackrel{?}{=} \sum_{n=0}^{\infty} C_{jk}(n)\tau = \sum_{n=0}^{\infty} \mathrm{Tr}_{\mathcal{S}} \left[\left((\mathcal{L}_\beta^*)^n (\Phi_j) \times \Phi_k \right) \rho_{\mathcal{S},\beta^*} \right] \tau. \tag{1.29}$$

Remark 1.13. *In continuous time models the Green-Kubo formula is $L_{jk} = \frac{1}{2} \int_{-\infty}^{+\infty} C_{jk}(t)\mathrm{d}t$, Here we do not have the factor $\frac{1}{2}$ because the system is defined only for positive times, and the extra factor τ corresponds to our discrete time-step "$\mathrm{d}t$".*

From (1.27) one readily computes

$$L_{jk} = \frac{E_0^2 (1-e_0) \mathrm{e}^{-\beta E_0}}{K\tau (1+\mathrm{e}^{-\beta E_0})^2} \times \left(\delta_{jk} - \frac{1-e_0}{1-e_0^K} e_0^{[j-k-1]} \right). \tag{1.30}$$

In particular one can immediately see that the Onsager reciprocity relations do not hold! This should not come as a surprise since the system is not at all time reversal invariant. Indeed, the various beams interact with the spin $\mathcal{S}$ in a precise order : $1, 2, \dots, K, 1, 2, \dots$ It is therefore reasonnable to expect that a change of temperature in beam 1 will have a greater influence on the flux in beam 2 than on the one in beam K. In the same spirit, one expects that a change of

temperature in beam 1 will have a greater influence on the flux in beam 2 than a change in beam 2 on the flux in beam 1, i.e. $|L_{21}| > |L_{12}|$, and the larger K the greater should be the difference (beam 2 arrives right after beam 1 while we need to wait an amount of time $(K-2)\tau$ before beam 1 comes back after beam 2). More generically, the further beam j arrives after beam k the smaller L_{jk} should be. This is precisely what (1.30) tells us : for $j \neq k$, the factor $e_0^{[j-k-1]}$ shows that $|L_{jk}|$ decays exponentially with $j-k$. The only hope to have Onsager relations is thus to restore some symmetry in the system.

Although Green-Kubo formula is expected to hold only for time reversal invariant systems, for the sake of completeness we compute the current-current correlation function and compare it to the kinetic coefficients L_{jk}.

We may rewrite $C_{jk}(n)$ as $C_{jk}(n) = \mathrm{Tr}_{\mathcal{S}}\left[\Phi_j \mathcal{L}_\beta^n \left(\Phi_k \times \rho_{\mathcal{S},\beta^*}\right)\right]$, and hence get, using (1.26) together with Proposition 1.13,

$$\sum_{n=0}^{\infty} C_{jk}(n)\tau = \frac{E_0^2(1-e_0)\mathrm{e}^{-\beta E_0}}{K^2\tau Z_\beta^2}.$$

If we compare with (1.30) we get

$$L_{jk} = \left(\sum_{n=0}^{\infty} C_{jk}(n)\tau\right) \times K\left(\delta_{jk} - \frac{1-e_0}{1-e_0^K} e_0^{[j-k-1]}\right).$$

In a sense, Green-Kubo formula (1.29) holds up to this asymmetric prefactor $K\left(\delta_{jk} - \frac{1-e_0}{1-e_0^K} e_0^{[j-k-1]}\right)$.

Restoring Symmetry

One way to restore the lack of symmetry is to make the K beam arrive in a random way, with uniform probability distribution. Another way to say it is the following: the temperature of the n-th atom is randomly chosen from the set $\{\beta_1, \ldots, \beta_K\}$, each temperature having probability $\frac{1}{K}$ to occur. The calculation of the NESS and entropy production are thus particular cases of Section 1.6.3. One gets

Proposition 1.21. *Suppose $\beta(\omega)$ is a random variable uniformly distributed over $\{\beta_1, \cdots, \beta_K\}$ and assume $\nu\tau \notin 2\pi\mathbb{N}$. Then, for any initial state ρ of $\mathcal{S}$ we have*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \rho(n,\tilde{\omega}) = \mathbb{E}\left(\rho_{\mathcal{S},\beta(\omega)^*}\right) = \frac{1}{K}\sum_{j=1}^{K} \rho_{\mathcal{S},\beta_j^*} = \rho_{\mathcal{S},+}, \qquad \mathbb{P}-a.s, \tag{1.31}$$

and the entropy production is

$$\Delta S = \frac{E_0(1-e_0)}{\tau} \times \mathrm{Cov}\left(\beta(\omega), Z_{\beta(\omega)}^{-1}\right) = \frac{E_0(1-e_0)}{K^2\tau} \sum_{j,k=1}^{K} \beta_j (Z_{\beta_j}^{-1} - Z_{\beta_k}^{-1}).$$

We now turn to heat fluxes. The heat fluxes in the NESS are defined as

$$\varphi_j := \mathrm{Tr}_{\mathcal{S}}\left[\rho_{\mathcal{S},+}\, \Phi_j\right], \tag{1.32}$$

with Φ_j as in (1.26). One may wonder why this definition is the right choice in this random situation, while we argued that it was *not* in the previous "deterministic" situation. To understand why, let us come back to the total variation of energy in the j-th beam. Using (1.25), the variation of energy in the j-th beam between time $n\tau$ and $(n+1)\tau$ is

$$\Delta E_j^n(\tilde{\omega}) = \begin{cases} \frac{E_0\lambda^2}{Z_{\beta_j}\nu^2}\sin^2\left(\frac{\nu\tau}{2}\right)\mathrm{Tr}_{\mathcal{S}}\left[\rho(n,\tilde{\omega})\times\left(\mathrm{e}^{-\beta_j E_0}|0\rangle\langle 0| - |1\rangle\langle 1|\right)\right], & \text{if } \beta(\omega_{n+1}) = \beta_j, \\ 0, & \text{otherwise.} \end{cases}$$

Hence, if we define the "random j-th flux observable" $\Phi_j(\omega)$ as

$$\Phi_j(\omega) = \begin{cases} \frac{E_0}{\tau Z_{\beta_j}} \times \underbrace{\frac{\lambda^2}{\nu^2}\sin^2\left(\frac{\nu\tau}{2}\right)}_{=(1-e_0)} \times \left(\mathrm{e}^{-\beta_j E_0}|0\rangle\langle 0| - |1\rangle\langle 1|\right) & \text{if } \beta(\omega) = \beta_j, \\ 0, & \text{otherwise,} \end{cases}$$

we thus have

$$\frac{1}{\tau}\Delta E_j^n(\tilde{\omega}) = \mathrm{Tr}_{\mathcal{S}}[\Phi_j(\omega_{n+1})\rho(n,\tilde{\omega})].$$

Proceeding as in Section 1.4, we indeed get that $\mathbb{P}-a.s.$

$$\lim_{N\to\infty}\frac{\Delta E_j(N,\tilde{\omega})}{N\tau} \;=\; \mathrm{Tr}_{\mathcal{S}}\left[\mathbb{E}\left(\rho_{\mathcal{S},\beta(\omega)^*}\right)\times\mathbb{E}(\Phi_j(\omega))\right] = \mathrm{Tr}_{\mathcal{S}}[\rho_{\mathcal{S},+}\Phi_j].$$

Inserting (1.26) and (1.31) into (1.32) we find

$$\varphi_j = \frac{E_0(1-e_0)}{K^2\tau}\sum_{k=1}^{K}(Z_{\beta_k}^{-1} - Z_{\beta_j}^{-1}). \tag{1.33}$$

One of course easily checks that $\sum_{j=1}^{K}\varphi_j = 0$ and that $\Delta S = -\sum_{j=1}^{K}\beta_j\varphi_j$.

We finally turn to linear response theory, which was the reason why we wanted to restore symmetry. The current-current correlation function $C_{jk}(n)$ is exactly the same as before (it is calculated in equilibrium). It thus remains to compute the new kinetic coefficients L_{jk}. From (1.33) on gets

$$L_{jk} = \frac{E_0^2(1-e_0)\mathrm{e}^{-\beta E_0}}{K^2\tau Z_\beta^2}\left(K\delta_{jk} - 1\right).$$

which as to be compared with

$$\sum_{n=0}^{\infty}C_{jk}(n)\tau = \frac{E_0^2(1-e_0)\mathrm{e}^{-\beta E_0}}{K^2\tau Z_\beta^2}.$$

Again, Green-Kubo formula holds up to the prefactor $(K\delta_{jk}-1)$, but "Symmetry" has been restored in the system and indeed Onsager reciprocity relations hold: $L_{jk} = L_{kj}$.

Chapter 2

Applications of RI systems: two concrete models

In this chapter we present two concrete models of the repeated interaction type. These models show how repeated interaction systems can be used to adress some physically relevant situations, and are presented in Sections 2.1 and 2.2 respectively.

1. A specific model describing the "One-Atom Maser" experiment where $\mathcal{S}$ is the quantized electromagnetic field in a cavity through which a beam of atoms, the subsystems $\mathcal{E}_n$, is shot. Such systems play a fundamental role in the experimental and theoretical investigations of basic matter-radiation processes. They are also of practical importance in quantum optics and quantum state engineering [MWM, WVHW, WBKM, RH, VAS]. We consider here only the ideal case, i.e. the question of thermal relaxation: is it possible to thermalize a mode of a QED cavity by means of 2-level atoms if the latter are initially at thermal equilibrium? One particular feature here is that the Hilbert space of the small system $\mathcal{S}$ is *not* finite dimensional. There are very few models of open quantum systems in the literature with an infinite small system and for which return to equilibrium is proven. The RI structure of the model allows us to provide such a model. Moreover, one usually makes use of perturbation theory in the coupling constant (as in Section 1.6.4) to obtain information on the spectrum of the relevant operator. Here, we do not make use of any perturbation theory, i.e. our results do not restrict to small coupling constants. The results described here come from [BP].

2. A model where the system $\mathcal{S}$ describes a spinless electron in the single band tight-binding approximation and subject to an homogeneous static electric field. For the electron alone, Bloch oscillations prevent a current from being set up in the system (see (2.12)). It is furthermore expected that if the electron is in contact with a thermal environment, the resulting scattering mechanisms will suppress the Bloch oscillations and lead to a steady current. In the model considered here, the environment is described by a chain of two-level atoms with which the electron interacts in the RI scheme. We show that a dc current is indeed created due to the interaction of the particle with its environment. In addition to drifting in the direction of the applied field, the electron diffuses around its mean position.

We also study the energy transfer between the particle and its environment, as well as its entropy production (we are here in a non-equilibrium situation because of the constant electric field). The results concerning this model come from [BDP].

2.1 The "ideal" one-atom maser or thermal relaxation in a QED cavity

2.1.1 Description of the model and the RDM

We consider the situation where the atoms of the beam are prepared in a stationary mixture of two states with energies $\tilde{E} < E_0$, and without loss of generality we set $\tilde{E} = 0$. We assume the cavity to be nearly resonant with the transitions between these two states. Neglecting the non-resonant modes of the cavity, we can describe its quantized electromagnetic field by a single harmonic oscillator of frequency $E \simeq E_0$.

The Hilbert space for a single atom is $\mathfrak{h}_{\mathcal{E}} = \mathbb{C}^2 \simeq \Gamma_-(\mathbb{C})$, the Fermionic Fock space over $\mathbb{C}$. The Hamiltonian of a single atom is thus

$$h_{\mathcal{E}} = E_0 b^* b,$$

where b^*, b denote the creation/annihilation operators on $\mathfrak{h}_{\mathcal{E}}$, see (1.21) in Section 1.6. The Hilbert space of the cavity field is $\mathfrak{h}_{\mathcal{S}} := \ell^2(\mathbb{N}) = \Gamma_+(\mathbb{C})$, the Bosonic Fock space over $\mathbb{C}$. Its Hamiltonian is

$$h_{\mathcal{S}} = E a^* a \equiv EN,$$

where a^*, a are the creation/annihilation operators on $\mathfrak{h}_{\mathcal{S}}$ satisfying the commutation relation $[a, a^*] = \mathbb{1}$ and N is the number operator on $\Gamma_+(\mathbb{C})$.

In the dipole approximation, an atom interacts with the cavity field through its electric dipole moment. The full dipole coupling is given by $\frac{\lambda}{2}(a + a^*) \otimes (b + b^*)$, acting on $\mathfrak{h}_{\mathcal{S}} \otimes \mathfrak{h}_{\mathcal{E}}$, where $\lambda \in \mathbb{R}$ is a coupling constant. Neglecting the counter rotating term $a \otimes b + a^* \otimes b^*$ in this coupling (this is the so called *rotating wave approximation*) leads to the well known Jaynes-Cummings Hamiltonian

$$h = h_{\mathcal{S}} \otimes \mathbb{1}_{\mathcal{E}} + \mathbb{1}_{\mathcal{S}} \otimes h_{\mathcal{E}} + \lambda v, \qquad v = \frac{1}{2}(a^* \otimes b + a \otimes b^*), \tag{2.1}$$

for the coupled system $\mathcal{S} + \mathcal{E}$ (see e.g. [Ba, CDG, Du]). (The toy model studied in Section 1.6 is very similar. One simply replaces the bosonic Fock space $h_{\mathcal{S}} = \Gamma_+(\mathbb{C})$ by the fermionic one $\Gamma_-(\mathbb{C}) = \mathbb{C}^2$.) The rotating wave approximation, and thus the dynamics generated by the Jaynes-Cummings Hamiltonian, is known to be in good agreement with experimental data as long as the detuning parameter $\Delta \equiv E - E_0$ satisfies $|\Delta| \ll \min(E_0, E)$ and the coupling is small $|\lambda| \ll E_0$. To our knowledge, there is however no mathematically precise statement about this approximation.

Finally, the initial state of the atoms will be the equilibrium state at inverse temperature β, i.e. $\rho_{\mathcal{E},\beta} \equiv \mathrm{e}^{-\beta h_{\mathcal{E}}} / \mathrm{Tr}\, \mathrm{e}^{-\beta h_{\mathcal{E}}}$.

As for the toy model of Section 1.6, using the fact that h commutes with the total number operator $N^{\rm tot} = a^*a \otimes \mathbb{1}_{\mathcal{E}} + \mathbb{1}_{\mathcal{S}} \otimes b^*b$, we can calculate explicitly $\mathrm{e}^{-\mathrm{i}\tau h}$ and hence the RDM $\mathcal{L}_\beta$ associated to this RI system. One gets

$$\mathcal{L}_\beta(\rho) = \sum_{\sigma,\sigma'=0,1} V_{\sigma'\sigma}\rho V_{\sigma'\sigma}^*, \tag{2.2}$$

where the operators $V_{\sigma'\sigma}$ are given by

$$\begin{aligned} V_{00} &= \frac{1}{\sqrt{Z_{\mathcal{E},\beta}}}\,\mathrm{e}^{-\mathrm{i}\tau EN}\,C(N), \qquad V_{10} = \frac{1}{\sqrt{Z_{\mathcal{E},\beta}}}\,\mathrm{e}^{-\mathrm{i}\tau EN}\,S(N+1)\,a, \\ V_{01} &= \frac{\mathrm{e}^{-\beta E_0/2}}{\sqrt{Z_{\mathcal{E},\beta}}}\,\mathrm{e}^{-\mathrm{i}\tau EN}\,S(N)\,a^*, \quad V_{11} = \frac{\mathrm{e}^{-\beta E_0/2}}{\sqrt{Z_{\mathcal{E},\beta}}}\,\mathrm{e}^{-\mathrm{i}\tau EN}\,C(N+1)^*, \end{aligned} \tag{2.3}$$

with

$$C(N) \equiv \cos(\pi\sqrt{\xi N+\eta}) + \mathrm{i}\eta^{1/2}\,\frac{\sin(\pi\sqrt{\xi N+\eta})}{\sqrt{\xi N+\eta}}, \qquad S(N) \equiv \xi^{1/2}\,\frac{\sin(\pi\sqrt{\xi N+\eta})}{\sqrt{\xi N+\eta}},$$

and where

$$\eta \equiv \left(\frac{\Delta\tau}{2\pi}\right)^2, \qquad \xi \equiv \left(\frac{\lambda\tau}{2\pi}\right)^2, \tag{2.4}$$

are the dimensionless detuning parameter and coupling constant.

2.1.2 Spectral analysis of the RDM

We know from the general results on RI systems that $\mathcal{L}_\beta$ is a contraction on $\mathcal{B}^1(\mathfrak{h}_{\mathcal{S}})$ (Proposition 2 in the Introduction), and that the state $\rho(n)$ of $\mathcal{S}$ evolves according to the discrete semigroup $\mathcal{L}_\beta^n$, i.e. $\rho(n) = \mathcal{L}_\beta^n(\rho)$. To understand the asymptotic behavior of $\rho(n)$, we shall thus study the spectral properties of $\mathcal{L}_\beta$. In particular, we will be interested in its peripheral eigenvalues $\mathrm{e}^{\mathrm{i}\theta}$, for $\theta \in \mathbb{R}$, and especially in the eigenvalue 1 (the corresponding eigenstate(s) will give the candidates for the asymptotic state(s)).

To understand the difficulty in the spectral analysis, assume that the atom-field coupling is turned off. The reduced dynamics is then nothing but the free evolution of $\mathcal{S}$, i.e. $\mathcal{L}_\beta(\rho) = \mathrm{e}^{-i\tau h_{\mathcal{S}}}\rho\,\mathrm{e}^{i\tau h_{\mathcal{S}}}$. It is easy to see that the spectrum of $\mathcal{L}_\beta$ is then pure point:

$$\mathrm{sp}(\mathcal{L}_\beta) = \mathrm{sp}_{\rm pp}(\mathcal{L}_\beta) = \{\mathrm{e}^{\mathrm{i}\tau Ed} \,|\, d \in \mathbb{Z}\}.$$

This spectrum is finite if $\tau E \in 2\pi\mathbb{Q}$ and densely fills the unit circle in the opposite case. In any case, all the eigenvalues (and in particular 1) are infinitely degenerate. This explains why perturbation theory in λ fails for this model. Note also that since $\mathfrak{h}_{\mathcal{S}}$ has infinite dimension, it is not automatic that $\mathcal{L}_\beta$ has 1 as an eigenvalue (we only know that it is in the spectrum since it is in the one of $\mathcal{L}_\beta^*$).

To describe the spectral results, we need to introduce a notion of resonance. Recall that such a resonance phenomenon already occured in the toy model ($\nu\tau \notin 2\pi\mathbb{N}$). An essential feature

of the dynamics generated by the Jaynes-Cummings Hamiltonian h are Rabi oscillations. In the presence of n photons, the probability for the atom to make a transition from its ground state to its excited state is a periodic function of time. The circular frequency of this oscillation is given by $\sqrt{\lambda^2 n + \Delta^2}$. (In our units, λ is thus the one photon Rabi-frequency of the atom in a perfectly tuned cavity.) If the interaction time τ is a multiple of the Rabi-oscillation period for n photons, then no transition will be possible from the n-photon state to the previous one. Such a *resonance* occurs when, for some integer k,

$$\tau = k\frac{2\pi}{\sqrt{\lambda^2 n + \Delta^2}} \quad \Longleftrightarrow \quad \xi n + \eta = k^2,$$

where η and ξ are defined in (2.4). We therefore introduce the following set

Definition 2.1. $R(\eta, \xi) := \{n \in \mathbb{N} \,|\, \xi n + \eta = k^2$ *for some* $k \in \mathbb{N}\}$. *An element* $n \in R(\xi, \eta)$ *is called a Rabi resonance.*

The Hilbert space $\mathfrak{h}_{\mathcal{S}}$ thus has a decomposition $\mathfrak{h}_S = \bigoplus_{k=1}^{r} \mathfrak{h}_{\mathcal{S}}^{(k)}$, where $r-1$ is the number of Rabi resonances, $\mathfrak{h}_{\mathcal{S}}^{(k)} \equiv \ell^2(I_k)$ and $\{I_k \,|\, k = 1, \ldots, r\}$ is the partition of $\mathbb{N}$ induced by the resonances. We call $\mathfrak{h}_{\mathcal{S}}^{(k)}$ the k-th Rabi sector and denote by P_k the corresponding orthogonal projection.

It is easy to show that, according to the arithmetic properties of ξ and η (rational or not), the set $R(\eta, \xi)$ possesses either no, one ore infinitely many elements ([BP], Lemma 3.2). We shall say accordingly that the system is non-resonant, simply resonant or fully resonant. A fully resonant system will be called degenerate if there exist $n \in \{0\} \cup R(\eta, \xi)$ and $m \in R(\eta, \xi)$ such that $n < m$ and $n+1, m+1 \in R(\eta, \xi)$, i.e. there are two pairs of consecutive Rabi resonances. (Such degenerate systems exist, if e.g. $\xi = 840$ and $\eta = 1$ then $(1, 2)$ and $(52, 53)$ are pairs of consecutive resonances. We refer to [BP] for more details on degenerate systems.)

The main ingredients for the spectral analysis of $\mathcal{L}_\beta$ are:

1) *The gauge symmetry.*

For any $\theta \in \mathbb{R}$, $\mathcal{L}(\mathrm{e}^{-i\theta N}\rho\,\mathrm{e}^{i\theta N}) = \mathrm{e}^{-i\theta N}\mathcal{L}(\rho)\,\mathrm{e}^{i\theta N}$, which follows from $[h, N^{\mathrm{tot}}] = [h_{\mathcal{E}}, \rho_{\mathcal{E},\beta}] = 0$. As a consequence, $\mathcal{L}_\beta$ leaves invariant the subspaces

$$\mathcal{B}^{1,(d)}(\mathfrak{h}_{\mathcal{S}}) \equiv \{X \in \mathcal{B}^1(\mathfrak{h}_{\mathcal{S}}) \,|\, \mathrm{e}^{-\mathrm{i}\theta N} X \mathrm{e}^{\mathrm{i}\theta N} = \mathrm{e}^{\mathrm{i}\theta d} X \text{ for all } \theta \in \mathbb{R}\},$$

and hence admits a decomposition

$$\mathcal{L}_\beta = \bigoplus_{d \in \mathbb{Z}} \mathcal{L}_\beta^{(d)},$$

so that one can analyze separately the $\mathcal{L}_\beta^{(d)}$.

2) *How* $\mathcal{L}_\beta$ *acts on diagonal states, i.e. on* $\mathcal{B}^{1,(0)}(\mathfrak{h}_{\mathcal{S}})$.

Because of the gauge symmetry, if ρ is an invariant state so is its "diagonal part" $\rho_0 = \sum_n \langle n|\rho n\rangle |n\rangle\langle n| \in \mathcal{B}^{1,(0)}$. It is thus important to understand the diagonal invariant states.

If we denote by x_n the diagonal elements of $X \in \mathcal{B}^{1,(0)}(\mathfrak{h}_{\mathcal{S}})$, we can identify $\mathcal{B}^{1,(0)}(\mathfrak{h}_{\mathcal{S}})$ with $\ell^1(\mathbb{N})$. Introducing the number operator $(Nx)_n \equiv n x_n$ and the finite difference operators

$$(\nabla x)_n \equiv \begin{cases} x_0 & \text{for } n = 0, \\ x_n - x_{n-1} & \text{for } n \geq 1, \end{cases} \qquad (\nabla^* x)_n \equiv x_n - x_{n+1} \text{ for } n \geq 0,$$

a simple algebra from (2.2)-(2.3) leads to

$$\mathcal{L}_\beta^{(0)} = \mathbb{1} - \nabla^* D(N) \mathrm{e}^{-\beta E_0 N} \nabla \mathrm{e}^{\beta E_0 N}, \tag{2.5}$$

where

$$D(N) := \frac{1}{1+\mathrm{e}^{-\beta E_0}} \sin^2(\pi\sqrt{\xi N + \eta}) \frac{\xi N}{\xi N + \eta}. \tag{2.6}$$

In particular, the diagonal invariant states ρ are solutions of $D(N)\mathrm{e}^{-\beta E_0 N}\nabla \mathrm{e}^{\beta E_0 N}\rho = 0$. Hence they satisfy $(\mathrm{e}^{-\beta E_0 N}\nabla \mathrm{e}^{\beta E_0 N}\rho)_n = 0 \Leftrightarrow \rho_n = \mathrm{e}^{-\beta E_0}\rho_{n-1}$ unless $D(n) = 0$, i.e. n is a Rabi resonance. We therefore have three situations:

- If the system is non-resonant, it follows from (2.6) that $D(n) = 0$ if and only if $n = 0$ and hence there is a unique diagonal invariant state $\frac{\mathrm{e}^{-\beta E_0 N}}{\mathrm{Tr}\ \mathrm{e}^{-\beta E_0 N}} = \rho_{\mathcal{S},\beta^*}$ where $\beta^* = \beta\frac{E_0}{E}$ if $\beta > 0$ (this is the same renormalization as for the toy model of Section 1.6) and none if $\beta \leq 0$.

- If the system is simply resonant there exists $n_1 \in \mathbb{N}^*$ such that $D(n) = 0$ if and only if $n = 0$ or $n = n_1$. The eigenvalue equation then splits into two decoupled systems

$$\rho_n = \mathrm{e}^{-\beta E_0}\rho_{n-1}, \ \ n \in I_1 \equiv \{1, \ldots, n_1 - 1\},$$
$$\rho_n = \mathrm{e}^{-\beta E_0}\rho_{n-1}, \ \ n \in I_2 \equiv \{n_1 + 1, \ldots\}.$$

  The first one yields the invariant state $\frac{\mathrm{e}^{-\beta E_0 N}P_1}{\mathrm{Tr}\ (\mathrm{e}^{-\beta E_0 N}P_1)} = \rho^{(1)}_{\mathcal{S},\beta^*}$, for any $\beta \in \mathbb{R}$. The second system gives another invariant state $\frac{\mathrm{e}^{-\beta E_0 N}P_2}{\mathrm{Tr}\ (\mathrm{e}^{-\beta E_0 N}P_2)} = \rho^{(2)}_{\mathcal{S},\beta^*}$, provided $\beta > 0$.

- If the system is fully resonant, $D(n)$ has an infinite sequence $n_0 = 0 < n_1 < n_2 < \cdots$ of zeros. The eigenvalue equation now splits into an infinite number of finite dimensional systems

$$\rho_n = \mathrm{e}^{-\beta E_0}\rho_{n-1}, \ \ n \in I_k \equiv \{n_{k-1} + 1, \ldots, n_k - 1\},$$

  where $k = 1, 2, \ldots$ For any $\beta \in \mathbb{R}$, we thus have an infinite number of invariant states $\frac{\mathrm{e}^{-\beta E_0 N}P_k}{\mathrm{Tr}\ (\mathrm{e}^{-\beta E_0 N}P_k)} = \rho^{(k)}_{\mathcal{S},\beta^*}$, one for each Rabi sector.

3) *The following Perron-Frobenius type Theorem due to Schrader ([Sch], Theorem 4.1).*

Theorem 2.2. *Let ϕ be a 2-positive map on $\mathcal{B}^1(\mathcal{H})$ such that its spectral radius $r(\phi) = \|\phi\|$. If λ is a peripheral eigenvalue of ϕ with eigenvector X, i.e. $\phi(X) = \lambda X$, $X \neq 0$, $|\lambda| = r(\phi)$, then $|X| = \sqrt{X^*X}$ is an eigenvector of ϕ to the eigenvalue $r(\phi)$: $\phi(|X|) = r(\phi)|X|$.*

Since the RDM $\mathcal{L}_\beta$ is completely positive trace preserving map we can apply Theorem 2.2 to it. Hence, if $\mathrm{e}^{\mathrm{i}\theta}$ is a peripheral eigenvalue of $\mathcal{L}_\beta^{(d)}$ for some d, with eigenvector X, then $|X| \in \mathcal{B}^{1,(0)}(\mathfrak{h}_\mathcal{S})$ is an invariant state of $\mathcal{L}_\beta^{(0)}$, which we all know.

Putting all these ingredients together we have a full description of the peripheral eigenvalues of $\mathcal{L}_\beta$.

Lemma 2.3. *[BP] 1. The only peripheral eigenvalue of $\mathcal{L}_\beta^{(0)}$ is 1.*

2. If the system is not degenerate, then the only peripheral eigenvalue of $\mathcal{L}_\beta$ is 1 and the corresponding eigenvectors are diagonal.

3. If the system is degenerate we note $N(\eta,\xi) := \{n \in \{0\} \cup R(\eta,\xi) \,|\, n+1 \in R(\eta,\xi)\}$ and $\mathcal{D}(\eta,\xi) := \{d = n-m \,|\, n,m \in N(\eta,\xi), n \neq m\}$. In this case the set of peripheral eigenvalues of $\mathcal{L}_\beta$ is given by

$$\{1\} \cup \{\mathrm{e}^{\mathrm{i}(\tau\omega+\xi\pi)d} \,|\, d \in \mathcal{D}(\eta,\xi)\}.$$

2.1.3 Convergence results

Thermal relaxation is an ergodic property of the map $\mathcal{L}_\beta$ and of its invariant states. For any density matrix ρ, we denote the orthogonal projection on the closure of $\mathrm{Ran}\,\rho$ by $s(\rho)$, the support of ρ. We also write $\mu \ll \rho$ whenever $s(\mu) \leq s(\rho)$.

A state ρ is ergodic, respectively mixing, for the semigroup generated by $\mathcal{L}_\beta$ whenever

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \left(\mathcal{L}_\beta^n(\mu)\right)(A) = \rho(A), \tag{2.7}$$

respectively

$$\lim_{n\to\infty} \left(\mathcal{L}_\beta^n(\mu)\right)(A) = \rho(A), \tag{2.8}$$

holds for all states $\mu \ll \rho$ and all $A \in \mathcal{B}(\mathfrak{h}_\mathcal{S})$. ρ is exponentially mixing if the convergence in (2.8) is exponential, i.e. if

$$\left|\left(\mathcal{L}_\beta^n(\mu)\right)(A) - \rho(A)\right| \leq C_{A,\mu}\,\mathrm{e}^{-\alpha n},$$

for some constant $C_{A,\mu}$ which may depend on A and μ and some $\alpha > 0$ independent of A and μ. A mixing state is ergodic and an ergodic state is clearly invariant.

A state ρ is faithful iff $\rho > 0$, that is $s(\rho) = Id$. Thus, if ρ is a faithful ergodic (resp. mixing) state the convergence (2.7) (resp. (2.8)) holds for every state μ and one has global relaxation. In this case, ρ is easily seen to be the only ergodic state of $\mathcal{L}_\beta$. Conversely, one can show that if $\mathcal{L}_\beta$ has a unique faithful invariant state, this state is ergodic:

Theorem 2.4. *[BP] Let ϕ be a completely positive trace preserving map on $\mathcal{B}^1(\mathcal{H})$. If ϕ has a faithful invariant state ρ_{inv} and 1 is a simple eigenvalue of ϕ then ρ_{inv} is ergodic.*

Using Lemma 2.3, we have the following theorem which is the main result of this section.

Theorem 2.5. *[BP] 1. If the system is non-resonant then $\mathcal{L}_\beta$ has no invariant state for $\beta \leq 0$ and the unique ergodic state*

$$\rho_{\mathcal{S},\beta^*} = \frac{\mathrm{e}^{-\beta^* h_\mathcal{S}}}{\mathrm{Tr}\,\mathrm{e}^{-\beta^* h_\mathcal{S}}}$$

for $\beta > 0$. In the latter case any initial state relaxes in the mean to the thermal equilibrium state at inverse temperature $\beta^* = \beta\frac{E_0}{E}$.

2. If the system is simply resonant then $\mathcal{L}_\beta$ has the unique ergodic state $\rho^{(1)}_{\mathcal{S},\beta^*}$ if $\beta \leq 0$ and two ergodic states $\rho^{(1)}_{\mathcal{S},\beta^*}$, $\rho^{(2)}_{\mathcal{S},\beta^*}$ if $\beta > 0$. In the latter case, for any initial state ρ, one has

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \left(\mathcal{L}^n_\beta(\rho)\right)(A) = \rho(P_1)\, \rho^{(1)}_{\mathcal{S},\beta^*}(A) + \rho(P_2)\, \rho^{(2)}_{\mathcal{S},\beta^*}(A),$$

for all $A \in \mathcal{B}(\mathfrak{h}_\mathcal{S})$ (recall P_k is the projection onto the k-th Rabi sector).

3. If the system is fully resonant then for any $\beta \in \mathbb{R}$, $\mathcal{L}_\beta$ has infinitely many ergodic states $\rho^{(k)}_{\mathcal{S},\beta^*}$, $k = 1, 2, \ldots$ Moreover, if the system is non-degenerate,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \left(\mathcal{L}^n_\beta(\rho)\right)(A) = \sum_{k=1}^{\infty} \rho(P_k)\, \rho^{(k)}_{\mathcal{S},\beta^*}(A), \tag{2.9}$$

holds for any initial state ρ and all $A \in \mathcal{B}(\mathfrak{h}_\mathcal{S})$.

4. If the system is non-degenerate, any invariant state is diagonal and can be represented as a convex linear combination of ergodic states.

5. Whenever the state $\rho^{(k)}_{\mathcal{S},\beta^*}$ is ergodic, it is also exponentially mixing if the corresponding Rabi sector $\mathfrak{h}^{(k)}_\mathcal{S}$ is finite dimensional.

Remarks. i) In the non-degenerate cases, this result implies some weak form of decoherence in the energy eigenbasis of the cavity field: the time averaged off-diagonal part of the state $\mathcal{L}^n_\beta(\rho)$ decays with time.

ii) If the system is degenerate, (2.9) and the conclusions of Assertion 4. still hold provided a further non-resonance condition is satisfied. Namely, if $\mathrm{e}^{\mathrm{i}(\tau E + \xi\pi)d} \neq 1$ for all $d \in \mathcal{D}$ (see Lemma 2.3), then all eigenvectors of $\mathcal{L}_\beta$ to the eigenvalue 1 are diagonal.

iii) Numerical experiments support the conjecture that all the ergodic states are mixing. However, our analysis does not provide a proof of this conjecture if $\mathfrak{h}^{(k)}_\mathcal{S}$ is infinite dimensional. In fact, we will see in the next section that $\mathcal{L}_\beta$ has an infinite number of metastable states in the non-resonant and simply resonant cases. As a result, we expect slow, i.e. non-exponential, relaxation.

2.1.4 Metastable states

If the system is non-resonant we say that $m \in \mathbb{N}^*$ is a Rabi quasi-resonance if it satisfies $D(m) < D(m \pm 1)$: these are the values of m for which, although it is non-zero, the transition rate from $|m-1\rangle$ to $|m\rangle$ is small (and hence more difficult to occur), see (2.5)-(2.6). Let $(m_k)_{k\in\mathbb{N}^*}$ be the strictly increasing sequence of quasi-resonances. It is straightforward to show that $D(m_k) = O(k^{-2})$ as $k \to \infty$. This implies that for large k the "quasi Rabi sectors" $\ell^2(\{m_k, \ldots, m_{k+1}-1\})$ are very weakly coupled and explains why one expects slow relaxation. To make this statement more precise let

$$D_0(n) := \begin{cases} 0 & \text{if } n \in \{m_1, m_2, \ldots\}, \\ D(n) & \text{otherwise,} \end{cases}$$

and $\mathcal{L}^{(0)}_{\beta,0} := \mathbb{1} - \nabla^* D_0(N) \mathrm{e}^{-\beta E_0 N} \nabla \mathrm{e}^{\beta E_0 N}$. One immediately concludes that

$$\mathcal{L}^{(0)}_{\beta} = \mathcal{L}^{(0)}_{\beta,0} + \mathcal{T}, \tag{2.10}$$

where $\mathcal{T}$ is a trace class operator. The analysis of the fully resonant case shows that 1 is an infinitely degenerate eigenvalue of $\mathcal{L}^{(0)}_{\beta,0}$. The corresponding positive eigenvectors

$$\tilde{\rho}^{(k)}_{\mathcal{S},\beta^*} = \frac{\mathrm{e}^{-\beta^* h_{\mathcal{S}}} \widetilde{P}_k}{\mathrm{Tr}\left(\mathrm{e}^{-\beta^* h_{\mathcal{S}}} \widetilde{P}_k\right)}$$

where $\widetilde{P}_k$ denotes the orthogonal projection onto $\ell^2(\{0, \ldots, m_k - 1\})$, are metastable states of the system. Because of these almost invariant states, the global relaxation process is extremely slow in the non-resonant and simply resonant cases. In spectral terms, (2.10) shows that 1 is always in the essential spectrum of $\mathcal{L}_\beta$. It follows that relaxation can not be exponential in infinite dimensional Rabi sectors.

One way to see metastable states in action consists in cooling the cavity with cold atoms. Figure 2.1 shows the result of such a calculation. The solid line is the initial state of the cavity which we chose to be thermal equilibrium with an average photon number of 22. The dashed line is the stationary state $\rho_{\mathcal{S},\beta^*}$, thermal equilibrium with an average of 7 photons. The broken line is the state of the cavity after 5000 interactions. The vertical dashed lines mark the positions of the Rabi quasi-resonances m_k. The picture shows clearly that local equilibrium is achieved in each interval $[m_k, m_{k+1}[$: the slope of the broken line agrees with that of the invariant state on these intervals. However only the first three intervals have reached a common equilibrium. The average photon number at this stage is still slightly larger than 17. It requires 50000 interactions for this number to drop under 10. Figure 2.2 shows the corresponding state of the cavity.

Another way to see these metastable states is to compute the evolution of the first metastable state $\tilde{\rho}^{(1)}_{\mathcal{S},\beta^*}$ and the relative entropies

$$\mathfrak{D}_k(n) := \mathrm{Ent}\left(\mathcal{L}^n_\beta\left(\tilde{\rho}^{(1)\,\beta^*}_{\mathcal{S}}\right) \,\middle|\, \tilde{\rho}^{(k)\,\beta^*}_{\mathcal{S}}\right),$$

see [BP], where the relative entropy of two states is $\mathrm{Ent}(\mu \,|\, \nu) = -\mathrm{Tr}\,\mu(\log\mu - \log\nu)$ and is a measure of the "distance" between μ and ν.

2.2 Diffusion in a tight binding band

2.2.1 Description of the model

The small system.

The system $\mathcal{S}$ consists in a spinless particle on the one-dimensional lattice $\mathbb{Z}$ and submitted to a constant external force $F \geq 0$. The quantum Hilbert space and Hamiltonian of the particle are

$$\mathfrak{h}_{\mathcal{S}} = \ell^2(\mathbb{Z}), \qquad h_{\mathcal{S}} = -\Delta - FX,$$

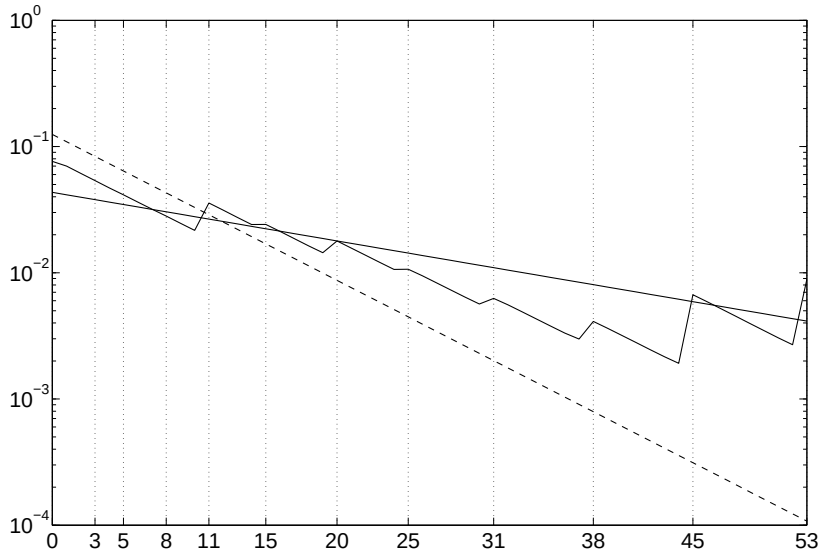

Figure 2.1: Cooling the cavity: 5000 interactions.

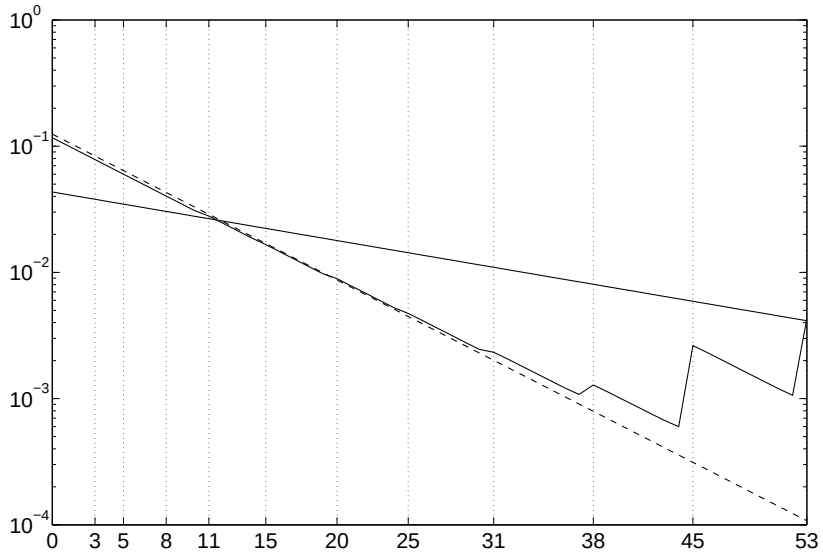

Figure 2.2: Cooling the cavity: 50000 interactions.

where Δ is the discrete nearest neighbor Laplacian and X the lattice position operator

$$-\Delta = \sum_{x\in\mathbb{Z}} \big(2\,|x\rangle\langle x| - |x{+}1\rangle\langle x| - |x\rangle\langle x{+}1|\big), \quad X = \sum_{x\in\mathbb{Z}} x\,|x\rangle\langle x|.$$

We shall also identify $\mathfrak{h}_{\mathcal{S}}$ with $L^2(\mathbb{T}^1, \mathrm{d}\xi)$ via the discrete Fourier transform, so that

$$-\Delta = 2(1-\cos\xi), \qquad X = \mathrm{i}\partial_\xi.$$

Here $\mathbb{T}^1 \simeq [0, 2\pi[$ is the first Brillouin zone and ξ the crystal momentum. If $T = \sum_{x\in\mathbb{Z}} |x{+}1\rangle\langle x| = \mathrm{e}^{-\mathrm{i}\xi}$ denotes the translation operator, one easily shows that:

1. When $F = 0$, $h_{\mathcal{S}}$ has a single band of absolutely continuous spectrum, $\mathrm{sp}(h_{\mathcal{S}}) = [0, 4]$, and the motion of the particle is described by

$$T(t) = \mathrm{e}^{\mathrm{i}th_{\mathcal{S}}} T \mathrm{e}^{-\mathrm{i}th_{\mathcal{S}}} = T, \quad X(t) = \mathrm{e}^{\mathrm{i}th_{\mathcal{S}}} X \mathrm{e}^{-\mathrm{i}th_{\mathcal{S}}} = X + \mathrm{i}(T - T^*)t,$$

   showing its ballistic nature.

2. When $F \neq 0$, $h_{\mathcal{S}}$ has discrete spectrum, $\mathrm{sp}(h_{\mathcal{S}}) = 2 - F\mathbb{Z}$. This is the well-known Wannier-Stark ladder. In the position representation, the normalized eigenvector ψ_k to the eigenvalue $E_k = 2 - Fk$ is given by

$$\psi_k(x) = J_{k-x}\left(\frac{2}{F}\right), \tag{2.11}$$

   where the J_ν are Bessel functions. From their asymptotic behavior for large ν (see e.g. Formula (10.19.1) in [OLBC]) we infer that

$$\psi_k(x) \sim \frac{1}{\sqrt{2\pi|k-x|}} \left(\frac{\mathrm{e}}{F|k-x|}\right)^{|k-x|} \quad \text{for } |k-x| \to \infty,$$

   which shows that $\psi_k(x)$ is sharply localized around $x = k$. The motion of the particle, described by

$$\left.\begin{aligned} T(t) &= \mathrm{e}^{\mathrm{i}th_{\mathcal{S}}} T \mathrm{e}^{-\mathrm{i}th_{\mathcal{S}}} = \mathrm{e}^{-\mathrm{i}tF} T, \\ X(t) &= \mathrm{e}^{\mathrm{i}th_{\mathcal{S}}} X \mathrm{e}^{-\mathrm{i}th_{\mathcal{S}}} = X + \frac{4}{F}\sin\left(\frac{Ft}{2}\right)\sin\left(\xi + \frac{Ft}{2}\right), \end{aligned}\right\} \tag{2.12}$$

   is now confined by Bloch oscillations.

The environment.

As in Section 2.1, it consists of 2-level atoms, each of which has a quantum Hilbert space $\mathfrak{h}_{\mathcal{E}} = \mathbb{C}^2$ which we identify with $\Gamma_-(\mathbb{C})$, the fermionic Fock space over $\mathbb{C}$, and a Hamiltonian given by

$$h_{\mathcal{E}} = Eb^*b,$$

where $E \geq 0$ is the Bohr frequency of the atom and b^*, b are the usual Fermi creation and annihilation operators.

The initial state of the two-level atoms will be their equilibrium state at inverse temperature β described by the density matrix

$$\rho_\beta = Z_\beta^{-1} \mathrm{e}^{-\beta h_\mathcal{E}}, \quad Z_\beta = \mathrm{Tr}(\mathrm{e}^{-\beta h_\mathcal{E}}) = 1 + \mathrm{e}^{-\beta E}.$$

The interaction.

The interaction between the particle and the two-level atom is chosen so that its effect is to give a right or left kick to the particle, depending on whether the atom is in its ground state or in its excited state. More precisely, we set

$$v = \sum_{x\in\mathbb{Z}} \big(|x+1\rangle\langle x| \otimes b^* + |x\rangle\langle x+1| \otimes b \big) = T \otimes b^* + T^* \otimes b.$$

To understand this interaction, note that when $F > 0$, the translation operator T can also be thought of as a lowering operator for the particle. Indeed, from (2.11) one finds

$$T\psi_k = \psi_{k+1}. \tag{2.13}$$

Similarly, T^* acts as a raising operator. As a result, v describes an exchange of energy between the two-level system and the particle. The model considered here is thus very similar to the one studied in Section 2.1 except that the spectrum of $h_\mathcal{S}$, contrary to the spectrum of the mode of the electromagnetic field, is not bounded from below. As a result, the system we treat here has no invariant state (see the end of Section 2.2.2).

2.2.2 Interaction with a single atom. The RDM $\mathcal{L}_\beta$

As for the Jaynes-Cummings Hamiltonian, $h = h_\mathcal{S} + h_\mathcal{E} + \lambda v$ can easily be diagonalized by exploiting the fact that it commutes with the "number operator"

$$N = \frac{h_\mathcal{S} - 2}{F} + \frac{h_\mathcal{E}}{E}. \tag{2.14}$$

Introducing the unitary operator

$$U = (Tb^*b + bb^*)\cos\theta - (Tb^* - b)\sin\theta,$$

where θ is chosen such that

$$\cos(2\theta) = \frac{E-F}{\omega_0}, \qquad \sin(2\theta) = \frac{2\lambda}{\omega_0},$$

and

$$\omega_0 = \sqrt{(E-F)^2 + 4\lambda^2},$$

one gets the following explicit formula for the propagator,

$$\mathrm{e}^{\mathrm{i}th} = U \mathrm{e}^{\mathrm{i}t(E-F)/2} \mathrm{e}^{\mathrm{i}t\omega_0(b^*b - 1/2)} \mathrm{e}^{\mathrm{i}th_\mathcal{S}} U^*. \tag{2.15}$$

It follows then that

$$\begin{aligned}\mathrm{e}^{\mathrm{i}th}X\,\mathrm{e}^{-\mathrm{i}th} &= \mathrm{e}^{\mathrm{i}th_{\mathcal{S}}}X\mathrm{e}^{-\mathrm{i}th_{\mathcal{S}}}\\ &\quad+\left(\frac{4\lambda^2}{\omega_0^2}(bb^*-b^*b)+\frac{2\lambda(E-F)}{\omega_0^2}(Tb^*+T^*b)\right)\sin^2\left(\frac{\omega_0 t}{2}\right)\\ &\quad-\mathrm{i}\frac{\lambda}{\omega_0}(Tb^*-T^*b)\sin(\omega_0 t).\end{aligned}$$

We conclude that the coupling to a single atom does not substantially alter the long term behavior of the particle: it turns the periodic Bloch oscillations (2.12) of frequency $\omega_{\mathrm{Bloch}}=F$ into quasi-periodic motion with the two frequencies ω_{Bloch} and ω_0. In particular, when $F\neq 0$, the motion remains bounded. As we will see, the situation is very different for repeated interactions with a sequence of atoms.

The following result describes the RDM $\mathcal{L}_\beta$ of this system. It follows directly from (2.15).

Lemma 2.6. *For any $\rho\in\mathcal{B}^1(\mathfrak{h}_{\mathcal{S}})$, one has $\mathcal{L}_\beta(\rho)=\mathcal{U}\circ\widetilde{\mathcal{L}}_\beta(\rho)=\widetilde{\mathcal{L}}_\beta\circ\mathcal{U}(\rho)$ with*

$$\mathcal{U}(\rho)=\mathrm{e}^{-\mathrm{i}\tau h_{\mathcal{S}}}\rho\,\mathrm{e}^{\mathrm{i}\tau h_{\mathcal{S}}},\qquad \widetilde{\mathcal{L}}_\beta(\rho)=p_-T^*\rho T+p_0\rho+p_+T\rho T^*, \tag{2.16}$$

where

$$p_-=\frac{\mathrm{e}^{-\beta E}}{1+\mathrm{e}^{-\beta E}}\,p,\quad p_0=1-p,\quad p_+=\frac{1}{1+\mathrm{e}^{-\beta E}}\,p,$$

with $p=\frac{4\lambda^2}{\omega_0^2}\sin^2\left(\frac{\omega_0\tau}{2}\right)$.

If ρ describes the state of the particle, then $T^*\rho T$ (respectively $T\rho T^*$) represents the same state translated by one lattice spacing to the left (respectively right). Note moreover that

$$p_-+p_0+p_+=1,$$

so that the reduced dynamics consists of a free evolution with the Hamiltonian $h_{\mathcal{S}}$, followed by a random translation by ±1 or 0, and with probabilities $p_\pm$ or p_0. Note that the dynamics is trivial if $p=0$, i.e. if $\omega_0\tau=2\pi m$ with $m\in\mathbb{Z}$. In that case there is no translation and the particle evolves according to $h_{\mathcal{S}}$. This can be seen directly on (2.15) by noticing that in such a case $Uh_{\mathcal{S}}U^*=h_{\mathcal{S}}+Fb^*b$. It follows that the propagator factorizes

$$\mathrm{e}^{\mathrm{i}\tau h}=(-1)^m\mathrm{e}^{\mathrm{i}\tau(E-F)/2}\,\mathrm{e}^{\mathrm{i}\tau h_{\mathcal{S}}}\otimes\mathrm{e}^{\mathrm{i}\tau Fb^*b},$$

and, up to an inessential phase factor and a renormalization of the atomic Bohr frequency, the particle and the two-level system evolve as if they were not coupled. This resembles the "Rabi oscillation" phenomenon which appears in the Jaynes-Cummings model (see Section 2.1.2). In the following we will avoid this resonance and assume $p\neq 0$.

We can now see that the system has no stationary state as we already mentioned, i.e. there exists no density matrix ρ on $\mathfrak{h}_{\mathcal{S}}$ such that $\mathcal{L}_\beta(\rho)=\rho$. Indeed, it follows from the gauge

invariance $\mathcal{L}_\beta(\mathrm{e}^{-\mathrm{i}th_\mathcal{S}}\rho\,\mathrm{e}^{\mathrm{i}th_\mathcal{S}}) = \mathrm{e}^{-\mathrm{i}th_\mathcal{S}}\mathcal{L}_\beta(\rho)\,\mathrm{e}^{\mathrm{i}th_\mathcal{S}}$ that the subspaces $\mathcal{B}^{1,(d)}(\mathfrak{h}_\mathcal{S})$, $d \in \mathbb{Z}$, defined by

$$\begin{aligned}\mathcal{B}^{1,(d)}(\mathfrak{h}_\mathcal{S}) &= \{\rho \in \mathcal{B}^1(\mathfrak{h}_\mathcal{S}) \,|\, \mathrm{e}^{-\mathrm{i}th_\mathcal{S}}\rho\,\mathrm{e}^{\mathrm{i}th_\mathcal{S}} = \mathrm{e}^{\mathrm{i}td}\rho \text{ for all } t \in \mathbb{R}\}\\ &= \{\rho \in \mathcal{B}^1(\mathfrak{h}_\mathcal{S}) \,|\, \rho = \sum_{k\in\mathbb{Z}} \rho_k |\psi_k\rangle\langle\psi_{k+d}|\},\end{aligned}$$

are globally invariant under $\mathcal{L}_\beta$. Hence, if a state ρ is stationary, so is its diagonal part $\rho_0 = \sum_k p_k|\psi_k\rangle\langle\psi_k|$, where $p_k = \langle\psi_k|\rho\psi_k\rangle$. From (2.16) one gets

$$p_{k-1} - Z_\beta p_k + \mathrm{e}^{-\beta E} p_{k+1} = 0,$$

which implies that $p_k = a + b\,\mathrm{e}^{\beta E k}$ for some constants $a, b \in \mathbb{R}$. But this contradicts the fact that $1 = \mathrm{Tr}\,\rho = \sum_k p_k$.

2.2.3 Asymptotic behaviour of the particle

Our first concern is the asymptotic behaviour of the particle, and more precisely with the large time behaviour of expectation values of some functions of the position X. Given an observable B on $\mathfrak{h}_\mathcal{S}$, we write $\langle B\rangle_n = \mathrm{Tr}(B\mathcal{L}_\beta^n(\rho))$, for its expectation value at time $t = n\tau$. As already announced, the following theorem shows that the repeated interactions make the motion of the particle diffusive. The motion is characterized by a drift velocity

$$v_\mathrm{d} = v_\mathrm{d}(E,F) = \frac{p}{\tau}\tanh\left(\frac{\beta E}{2}\right),$$

and a diffusion constant

$$D = D(E,F) = \frac{p}{2\tau}\left(1 - p\tanh^2\left(\frac{\beta E}{2}\right)\right).$$

More precisely, the following holds.

Theorem 2.7. *[BDP] Assume that $F > 0$, $\lambda \neq 0$ and $\omega_0\tau \notin 2\pi\mathbb{Z}$ so that $p \in]0,1]$. Let the density matrix $\rho \in \mathcal{B}^1(\mathfrak{h}_\mathcal{S})$ describe the initial state of the particle and denote by μ_n the spectral measures of the position observable X on the states $\rho(n)$,*

$$\mu_n(f) = \int f(x)\,\mathrm{d}\mu_n(x) = \langle f(X)\rangle_n.$$

1. *The Central Limit Theorem (CLT) holds: For any bounded continuous function f on $\mathbb{R}$,*

$$\lim_{n\to\infty}\int f\left(\frac{x - v_\mathrm{d} n\tau}{\sqrt{2Dn\tau}}\right)\mathrm{d}\mu_n(x) = \int f(x)\,\mathrm{e}^{-x^2/2}\,\frac{\mathrm{d}x}{\sqrt{2\pi}}.$$

2. *If $\mathrm{Tr}\left(X^2\rho\right) < +\infty$, then*

$$\lim_{n\to\infty}\frac{\langle X\rangle_n}{n\tau} = v_\mathrm{d}, \quad \lim_{n\to\infty}\frac{\langle (X - v_\mathrm{d} n\tau)^2\rangle_n}{n\tau} = 2D.$$

3. *If* $\mathrm{Tr}\left(\mathrm{e}^{\alpha|X|}\rho\right) < +\infty$ *for all* $\alpha > 0$*, then a Large Deviation Principle (LDP) holds in the sense that, for any interval* $J \subset \mathbb{R}$*,*

$$\lim_{n\to\infty} \frac{1}{n} \log \mu_n(nJ) = -\inf_{x\in J} I(x),$$

where $I(x)$ *is the Legendre-Fenchel transform of*

$$e(\alpha) = \log\left((1-p) + p\frac{\cosh(\frac{1}{2}\beta E + \alpha)}{\cosh\frac{1}{2}\beta E}\right), \tag{2.17}$$

i.e. $I(x) = \sup_{\alpha\in\mathbb{R}} \alpha x - e(\alpha)$.

Note that when $E = F$, the mobility

$$\mu = \lim_{F\to 0} \frac{v_\mathrm{d}}{F} = \frac{\beta \sin^2(\lambda\tau)}{2\tau},$$

and the diffusion constant

$$D = \mu\beta^{-1}\left(1 - \sin^2(\lambda\tau)\tanh^2\left(\frac{\beta F}{2}\right)\right),$$

satisfy the Einstein relation

$$\lim_{F\to 0} D = \mu\beta^{-1} = \mu k_B T.$$

The rate function in Part 3 is explicitly given by

$$I(x) = \begin{cases} -x\left(\dfrac{\beta E}{2} + \log\left(\dfrac{R(x)-x}{a(1-x)}\right)\right) - \log\left(\dfrac{(1-p)(R(x)+1)}{1-x^2}\right) & \text{for } x \in [-1,1], \\ +\infty & \text{otherwise,} \end{cases}$$

where

$$a = \frac{p}{(1-p)\cosh(\beta E/2)}, \qquad R(x) = \sqrt{x^2 + a^2(1-x^2)}.$$

It is strictly convex on $[-1,1]$ and satisfies $I(v_\mathrm{d}\tau) = 0$ and $I(x) > 0$ for $x \neq v_\mathrm{d}\tau$.

Note that the drift velocity and diffusion constant do not depend on the initial state of the particle. The CLT gives us the probability to find the particle at time $n\tau$ in a region of size $O(\sqrt{n})$ around the mean value $v_\mathrm{d} n\tau$, whereas the LDP gives information on this probability for a region of size $O(n)$. To put it differently, it yields information on the probability that the particle's mean speed falls asymptotically in an interval of size $O(1)$. Loosely speaking, it says that

$$\mu_n(\{n(v_\mathrm{d} + \delta v)\tau\}) \simeq \mathrm{e}^{-nI((v_\mathrm{d}+\delta v)\tau)}.$$

The peculiar symmetry $e(-\beta E - \alpha) = e(\alpha)$ immediately leads to the relation $I(x) = -\beta E x + I(-x)$ which tells us that

$$\lim_{\delta v\downarrow 0}\lim_{n\to\infty} \frac{1}{n\tau} \log \frac{\mu_n(n[-v-\delta v, -v+\delta v]\tau)}{\mu_n(n[v-\delta v, v+\delta v]\tau)} = -\beta E v,$$

i.e. that negative mean velocities are exponentially less likely than positive ones. One can recognize here a form of fluctuation theorem. Indeed, we shall see in the next section that the symmetry of the function $e(\alpha)$ is a direct consequence of a remnant of the Evans-Searles (or transient) fluctuation theorem (2.18).

2.2.4 Statistics of energy changes

One can also study the statistics of the energy changes of the particle, the environment and the whole system. Note that contrary to what happened in Chapter 1, the system $\mathcal{S}$ is infinite, its Hamiltonian is neither bounded from below nor from above, and can thus continuously gain or lose energy.

To study the change in the energy of the atomic reservoir we use the following operational procedure. The reservoir being initially in thermal equilibrium at inverse temperature β and the particle in the state ρ, we measure the total energy of the reservoir and the particle just before the first interaction and just after the n-th interaction. These successive measurements yield the four values $E_{\mathrm{p},0}, E_{\mathrm{p},n} \in \mathrm{sp}\, H_{\mathrm{p}}$ and $E_{\mathrm{env},0}, E_{\mathrm{env},n} \in \mathrm{sp}\, H_{\mathrm{env}}$. It will be convenient to express the resulting change in energy in terms of the "entropy like" quantities

$$\Delta S_{\mathrm{p},n} = \beta^*(E_{\mathrm{p},n} - E_{\mathrm{p},0}), \qquad \Delta S_{\mathrm{env},n} = -\beta(E_{\mathrm{env},n} - E_{\mathrm{env},0}),$$

where $\beta^* = \beta E/F$ (it is the same renormalization procedure as for the one-atom maser and the toy model). We denote by $\mathbb{P}^n$ the joint probability distribution of $\Delta S_{\mathrm{p},n}$, $\Delta S_{\mathrm{env},n}$ and by $\mathbb{E}^n$ the corresponding expectation.

Remark 2.1. *When applied to electric charge, or more generally to particle number, the two measurements process described above goes under the name full counting statistics (see e.g. [ABGK] and references therein). The present application is closer to the approach to current fluctuations found in [dR].*

Theorem 2.8. *[BDP]*

1. $\mathbb{P}^n[\Delta S_{\mathrm{p},n} = \Delta S_{\mathrm{env},n}] = 1$. *Hence, in the following we set*

$$\Delta S_n = \Delta S_{\mathrm{p},n} = \Delta S_{\mathrm{env},n}.$$

2. *The cumulant generating function of ΔS_n is given by*

$$\log \mathbb{E}^n \left[\mathrm{e}^{\alpha \Delta S_n}\right] = n e(-\alpha\beta E).$$

3. *Its mean value and variance are*

$$\mathbb{E}^n \left[\frac{\Delta S_n}{n}\right] = -\beta E v_{\mathrm{d}} \tau, \qquad \mathbb{E}^n \left[\frac{(\Delta S_n + \beta E v_{\mathrm{d}} n\tau)^2}{n}\right] = (\beta E)^2 2D\tau.$$

4. *The CLT holds: For any bounded continuous function f,*

$$\lim_{n\to\infty} \mathbb{E}^n \left[f\left(\frac{\Delta S_n + \beta E v_{\mathrm{d}} n\tau}{\beta E\sqrt{2Dn\tau}}\right)\right] = \int f(x)\, \mathrm{e}^{-x^2/2}\, \frac{\mathrm{d}x}{\sqrt{2\pi}}.$$

5. *The sequence $(\mathbb{P}^n)_{n\in\mathbb{N}}$ satisfies a LDP: For any intervals $J \subset \mathbb{R}$,*

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}^n \left(\frac{\Delta S_n}{n} \in J \right) = - \inf_{s\in J} \phi(s),$$

with the rate function $\phi(s) = \sup_{\alpha\in\mathbb{R}}(\alpha s - e(-\alpha\beta E))$.

6. *It satisfies the transient fluctuation theorem*

$$\frac{\mathbb{P}\left(\frac{\Delta S_n}{n} = -s\right)}{\mathbb{P}\left(\frac{\Delta S_n}{n} = s\right)} = \mathrm{e}^s. \tag{2.18}$$

Part 1. clearly reflects the fact that the "number operator" (2.14) commutes with h so that

$$\beta^* h_{\mathcal{S}} + \beta h_{\mathrm{env}} = \beta E \left(\frac{h_{\mathcal{S}}}{F} + \frac{h_{\mathrm{env}}}{E} \right),$$

is preserved by the repeated interaction dynamics.

The particle's drift velocity is v_{d}, and one sees therefore that, as expected, its energy loss per unit time equals the work done by F per unit time. Simultaneously, the environment gains energy at a rate Ev_{d}: indeed, the particle moves on average v_{d} steps to the right per unit time, which corresponds to v_{d} elements of the chain gaining an energy E. This leads to an average energy gain or loss of $(E - F)v_{\mathrm{d}}$ for the full system. In the special case $E = F$, these rates are equal, and the total system neither looses nor gains energy. This is a consequence of the fact that the interaction term in the Hamiltonian commutes with the free Hamiltonian in this case. In general, the total energy is not preserved, which is a reflection of the fact that the Hamiltonian of the total system is time-dependent.

The transient fluctuation theorem (2.18) traduces the symmetry $\theta(\alpha) = \theta(1 - \alpha)$ of the cumulant generating function $\theta(\alpha) = e(-\alpha\beta E)$ of ΔS_n, showing that the symmetry for $e(\alpha)$ observed in Theorem 2.7 is indeed a consequence of the transient fluctuation theorem. Of course this symmetry is evident from (2.17), but it is actually a consequence of time-reversal invariance as we shall briefly explain. We refer to [BDP] for more details.

Let $\mathcal{L}^\alpha_\beta(A) := \mathrm{Tr}_{\mathfrak{h}_\mathcal{E}} \left((\mathbb{1} \otimes \rho^\alpha_\beta)\mathrm{e}^{-\mathrm{i}th}(A \otimes \rho^{1-\alpha}_\beta)\mathrm{e}^{\mathrm{i}th} \right)$ (this map appears naturally in the calculation of the cumulant generating function, see [BDP]). One can then show that $\mathcal{L}^\alpha_\beta$ is also a completely positive map with spectral radius equal to $\theta(\alpha)$. If now C denotes the complex conjugation on $\ell^2(\mathbb{Z})$, i.e. $C\psi(x) = \overline{\psi(x)}$, then the anti-linear involution $\mathcal{C}(A) = CAC^*$ implements time-reversal of the particle's dynamics and one can prove that, for all $\alpha \in \mathbb{R}$, $(\mathcal{L}^\alpha_\beta)^* = \mathcal{C} \circ \mathcal{L}^{1-\alpha}_\beta \circ \mathcal{C}$, from which the symmetry $\theta(1 - \alpha) = \theta(\alpha)$ immediately follows.

Since $-\frac{h_{\mathcal{S}}}{F} \simeq X$, we expect a very similar result for the position increment $\Delta X_n = X_n - X_0$ obtained from a double measurement of X at time $t = 0$ and $t = n\tau$. Indeed, the distribution $\mathbb{Q}^n$ of ΔX_n satisfies the following.

Theorem 2.9. *[BDP]*

1. *The cumulant generating function of ΔX_n satisfies*

$$g(\alpha) = \lim_{n\to\infty} \frac{1}{n} \log \mathbb{Q}^n \left[\mathrm{e}^{\alpha \Delta X_n} \right] = e(\alpha).$$

2. *Its mean value and variance satisfy*

$$\lim_{n\to\infty} \mathbb{Q}^n \left[\frac{\Delta X_n}{n\tau} \right] = v_{\mathrm{d}}, \qquad \mathbb{Q}^n \left[\frac{(\Delta X_n - v_{\mathrm{d}} n\tau)^2}{n\tau} \right] = 2D.$$

3. *The CLT holds: For any bounded continuous function f,*

$$\lim_{n\to\infty} \mathbb{Q}^n \left[f\left(\frac{\Delta X_n - v_{\mathrm{d}} n\tau}{\sqrt{2Dn\tau}} \right) \right] = \int f(x)\, \mathrm{e}^{-x^2/2}\, \frac{\mathrm{d}x}{\sqrt{2\pi}}.$$

4. *The sequence $(\mathbb{Q}^n)_{n\in\mathbb{N}}$ satisfies a LDP: For any intervals $J \subset \mathbb{R}$,*

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{Q}^n \left[\frac{\Delta X_n}{n} \in J \right] = - \inf_{x\in J} I(x).$$

5. *It satisfies the asymptotic fluctuation theorem*

$$\beta E(v - \delta v) \leq \lim_{n\to\infty} \frac{1}{n} \log \frac{\mathbb{Q}^n \left[\frac{\Delta X_n}{n\tau} \in [-v - \delta v, -v + \delta v] \right]}{\mathbb{Q}^n \left[\frac{\Delta X_n}{n\tau} \in [v - \delta v, v + \delta v] \right]} \leq \beta E(v + \delta v),$$

for $v \in]-1, 1[$ and $\delta v > 0$, small enough.

Remark. This last fluctuation theorem is neither of transient type (it is valid only in the large time limit) nor of stationary type since the dynamics has no stationary state.

Note also the similarity with Theorem 2.7. This last theorem describes the position increment of the particle instead of the position itself. In particular, it does not require sharp localization of the particle position at time 0, contrary to Theorem 2.7.

Chapter 3

Hamiltonians on the bosonic Fock space

In this chapter we study two classes of Hamiltonians which are used as simplified versions of Hamiltonians arising in various contexts of quantum physics. The common feature of these Hamiltonians is that they will act, at least partially, on the bosonic (or symmetric) Fock space $\Gamma_+(\mathfrak{h})$ over the one-particle space $\mathfrak{h}$. The results presented here come from the two articles [BD1, BD2].

Throughout the chapter we will denote by $\Gamma_+(\mathfrak{h})$ the bosonic Fock space, i.e. $\Gamma_+(\mathfrak{h}) = \oplus_{n=0}^{\infty} \otimes_{\mathrm{s}}^n \mathfrak{h}$, by $\Gamma_+^n(\mathfrak{h}) := \otimes_{\mathrm{s}}^n \mathfrak{h}$ the n-particle sector and by $\Gamma_+^{\mathrm{fin}}(\mathfrak{h})$ the finite particle subspace, i.e. $\Gamma_+^{\mathrm{fin}}(\mathfrak{h}) = \{\Psi = (\Psi^{(n)})_n \in \Gamma_+(\mathfrak{h}) \mid \Psi^{(n)} = 0 \text{ for all but finitely many } n'\mathrm{s}\}$.

3.1 Generalized spin boson Hamiltonian

In this section, we consider a class of Hamiltonians which arise in quantum physics as simplified Hamiltonians describing a small quantum system described by the Hilbert space $\mathcal{H}_{\mathcal{S}}$ interacting with a bosonic field. For instance, the dipole approximation to nonrelativistic QED is of this form. More precisely, we are interested in self-adjoint operators of the form

$$H \;=\; H_{\mathcal{S}} \otimes \mathbb{1} + \mathbb{1} \otimes \int h(k) a^*(k) a(k) \mathrm{d}k + \int v(k) \otimes a^*(k) \mathrm{d}k + \int v(k)^* \otimes a(k) \mathrm{d}k, \quad (3.1)$$

acting on the Hilbert space $\mathcal{H} = \mathcal{H}_{\mathcal{S}} \otimes \Gamma_+(\mathfrak{h})$, where $\mathfrak{h} = L^2(\mathbb{R}^d; \mathbb{C}^n)$. The operator $H_{\mathcal{S}}$ is assumed to be self-adjoint on $\mathcal{H}_{\mathcal{S}}$, $h(k)$ describes the dispersion relation of the bosons and is a multiplication operator by a non-negative self-adjoint matrix on $\mathbb{C}^n$, and $v(k)$ is a function with values in operators on $\mathcal{H}_{\mathcal{S}}$ which is responsible for the interaction between the small system and the bosons.

There is no universally accepted name for operators of the form (3.1). In [DG1, DJ1, GGM, Ge] they are called Pauli-Fierz operators or Pauli-Fierz Hamiltonians. However, sometimes the same expression "Pauli-Fierz Hamiltonian" is used to denote slightly different objects [HVV, HS]. In order to avoid confusion, we will call them "generalized spin boson" Hamiltonians (the

usual spin-boson Hamiltonian fits into this framework with $H_{\mathcal{S}} = \sigma_z$ acting on $\mathcal{H}_{\mathcal{S}} = \mathbb{C}^2$, $h(k)$ is a multiplication operator on $\mathfrak{h} = L^2(\mathbb{R}^d)$ and represents the energy of a boson of momentum k, e.g. $h(k) = \sqrt{m^2 + k^2}$ where $m \geq 0$ is the mass of the bosons, and $v(k) = \sigma_x \times f(k)$ with $f(k)$ a so-called form factor describing the interaction).

One of the results about generalized spin-boson Hamiltonians that can be found in the literature says that the essential spectrum of H is shifted to the right from the bottom of the spectrum of H by the "mass" $\inf \mathrm{sp}(h)$. This theorem was first proven in [DG1]. It resembles the HVZ Theorem about many-body Schrödinger operators [RS4], and we shall call it a HVZ-type theorem. It is obvious that if the "mass" is positive, the HVZ-type theorem then implies the existence of a ground state. It turns out that even in the massless case, under some additional assumptions, one can show that there exists a ground state of H that "sits" at the tip of the continuous spectrum. This result was first proven in [AH, BFS] for a small coupling constant. In [Sp] this result was extended to an arbitrary coupling constant for a Hamiltonian satisfying an appropriate condition that allows to use the Perron-Frobenius method. In the work of Gérard [Ge] the existence of a ground state was proven for a large class of generalized spin-boson Hamiltonians without using the Perron-Frobenius method. See also later work [A, GLL].

In [BD1] we extend the HVZ-type theorem of [DG1] and the theorem about the existence of a ground state from [Ge] to a larger class of Hamiltonians. In Equation (3.1) we used the formalism of "operator valued distributions" $a^*(k)$, $a(k)$, which is a common approach to creation/annihilation operators. Here, we will not use this formalism and in particular consider interactions v which will not be "fibered" with respect to the variable k and also defined only as quadratic forms (from $\mathcal{H}_{\mathcal{S}}$ to $\mathcal{H}_{\mathcal{S}} \otimes \mathfrak{h}$). Operators of the form (3.1) appear in various contexts in quantum physics, and it is therefore natural to consider as general assumptions as possible. For example, the use of quadratic forms turns out to be of some importance if one wants to allow some weak ultraviolet singularities in the study of the Nelson Hamiltonian [Mö].

The use of more general/abstract Hamiltonians leads to somehow generalize the techniques commonly used in this context. One of the techniques that proved powerful in the study of 2nd quantized Hamiltonians is the so-called pullthrough formula. It was used in the early works of Glimm, Jaffe and Rosen on constructive quantum field theory, for instance in the work of Rosen on higher order estimates [Ro]. In [AH, BFS, Ge] it is used as an important step in the proof of the existence of a ground state for Pauli-Fierz operators. The version of the pullthrough formula usually employed has the form

$$\mathbb{1} \otimes a(k)\, H = (H + h(k))\, \mathbb{1} \otimes a(k) + v(k) \otimes \mathbb{1}. \tag{3.2}$$

This formula depends explicitly on the identification of the one-particle space with the space $L^2(\mathrm{d}k)$, which should not play a role in the arguments. (Besides, the annihilation operator $a(k)$ is not even closable. It can be interpreted as an operator valued distribution, which is a little awkward mathematically.) In [BD1], a more abstract version of (3.2) has been proposed. It takes the form

$$A\, H = (H \otimes \mathbb{1}_{\mathfrak{h}} + \mathbb{1}_{\mathcal{H}} \otimes h)\, A + v,$$

where $A : \Gamma_+(\mathfrak{h}) \to \Gamma_+(\mathfrak{h}) \otimes \mathfrak{h}$ is a so-called Pullthrough Annihilation Operator (the adjoint of A is simply $A^* : \Gamma_+(\mathfrak{h}) \otimes \mathfrak{h} \ni \Psi \otimes f \mapsto a^*(f)\Psi \in \Gamma_+(\mathfrak{h})$, see Section 4 of [BD1] for more

details). This abstract version of the pullthrough formula has recently been used in [GeHPS] to study the existence of a ground state for the Nelson model on static space-time.

3.1.1 Description of the model and Assumptions

We assume that $H_{\mathcal{S}}$ is a positive self-adjoint operator on $\mathcal{H}_{\mathcal{S}}$ (actually, bounded from below is sufficient but we take it positive for simplicity), and h is a positive self-adjoint operator on $\mathfrak{h} = L^2(\mathbb{R}^d;\mathbb{C}^n)$. We call $m := \inf \mathrm{sp}(h)$ the mass of the bosons. The free Hamiltonian is defined as

$$H_0 = H_{\mathcal{S}} \otimes \mathbb{1} + \mathbb{1} \otimes \mathrm{d}\Gamma(h).$$

One of the generalization is that we allow for coupling operators v which are only defined as quadratic forms from $\mathcal{H}_{\mathcal{S}}$ to $\mathcal{H}_{\mathcal{S}} \otimes \mathfrak{h}$. Contrary to (3.1) where $v\psi := v(k)\psi \in L^2(\mathbb{R}^d;\mathcal{H}_{\mathcal{S}}) \simeq \mathcal{H}_{\mathcal{S}} \otimes L^2(\mathbb{R}^d)$), we do not necessarily assume that v is "fibered" with respect to k and hence write $a(v)/a^*(v)$ for $\int v(k)^* \otimes a(k)\mathrm{d}k$ / $\int v(k) \otimes a^*(k)\mathrm{d}k$. Of course, $a(v)$ and $a^*(v)$ will be defined only as quadratic forms as well.

Let us briefly recall the basic notation on unbounded forms that we will use. If $\mathcal{H}_1$ and $\mathcal{H}_2$ are Hilbert spaces, a form v from $\mathcal{H}_1$ to $\mathcal{H}_2$ is a map

$$\mathrm{Dom}_{\mathrm{l}} \times \mathrm{Dom}_{\mathrm{r}} \ni (\Phi,\Psi) \mapsto \langle \Phi, v\Psi\rangle \in \mathbb{C},$$

where $\mathrm{Dom}_{\mathrm{r/l}}$ are subspaces of $\mathcal{H}_{1/2}$ called the right/left domain of v. If v can be extended to a bounded sesquilinear map, then we denote by $\|v\|$ its norm. If moreover h_1, resp. h_2, is a closed densely defined operator on $\mathcal{H}_1$, resp. $\mathcal{H}_2$, then $h_2 v h_1$ denotes the form $(\Phi,\Psi) \mapsto \langle h_2^*\Phi | v h_1 \Psi\rangle$, with right domain $\mathrm{Dom}_{\mathrm{r}}(h_2 v h_1) := \{\Psi \in \mathrm{Dom}(h_1) \,|\, h_1\Psi \in \mathrm{Dom}_{\mathrm{r}} v\}$ and left domain $\mathrm{Dom}_{\mathrm{l}}(h_2 v h_1) := \{\Phi \in \mathrm{Dom}(h_2^*) \,|\, h_2^*\Phi \in \mathrm{Dom}_{\mathrm{l}} v\}$.

We can now describe the interaction operator V. Let $\mathcal{D}_0$ be a dense subspace of $\mathcal{H}_{\mathcal{S}}$ contained in $\mathrm{Dom}\, H_{\mathcal{S}}^{1/2}$, $\mathfrak{h}_0$ a dense subspace of $\mathfrak{h}$ contained in $\mathrm{Dom}\, h^{1/2}$, and v a quadratic form from $\mathcal{H}_{\mathcal{S}}$ to $\mathcal{H}_{\mathcal{S}} \otimes \mathfrak{h}$ with right domain $\mathcal{D}_0$ and left domain $\mathcal{D}_0 \otimes_{\mathrm{alg}} \mathfrak{h}_0$, i.e. $v : \mathcal{D}_0 \otimes_{\mathrm{alg}} \mathfrak{h}_0 \times \mathcal{D}_0 \ni (\phi,\psi) \mapsto \langle \phi, v\psi\rangle \in \mathbb{C}$, and where $\otimes_{\mathrm{alg}}$ denotes the algebraic tensor product. We define the annihilation form $a(v)$ as a form with left and right domain $\mathcal{D}_0 \otimes_{\mathrm{alg}} \Gamma_{\mathrm{alg}}(\mathfrak{h}_0)$ (the algebraic symmetric Fock space over $\mathfrak{h}_0$). It is defined for $\Phi \in \mathcal{D}_0 \otimes_{\mathrm{alg}} \Gamma^m_{\mathrm{alg}}(\mathfrak{h}_0)$ and $\Psi \in \mathcal{D}_0 \otimes_{\mathrm{alg}} \Gamma^n_{\mathrm{alg}}(\mathfrak{h}_0)$ as

$$\langle \Phi, a(v)\Psi\rangle := \begin{cases} 0, & m \neq n-1, \\ \sqrt{n}(\Phi | v^* \otimes \mathbb{1}^{\otimes(n-1)}\, \Psi), & m = n-1. \end{cases}$$

The creation form $a^*(v)$ is defined as $a^*(v) := (a(v))^*$. (Here the * denotes the adjoint form.) The interaction V is then defined as the quadratic form

$$V = a(v) + a^*(v).$$

For the convenience of the reader, we finally gather here the various assumptions we will use throughout this section.

The first assumption **(SB1)** is the minimal assumption which serves to define the operator $H = H_0 + V$.

(SB1) $\lim_{t\to+\infty} \|(\mathbb{1}_{\mathcal{H}_S} \otimes h^{-1/2})v(t + H_S)^{-1/2}\| < 1.$

The second assumption describes the confinment of the small system (one may think of a particle submitted to a confining potential, or even a finite dimensional system).

(SB2) $(1 + H_S)^{-1}$ is compact.

Our next assumption concerns the dispersion relation of the bosons.

(SB3) h is the multiplication operator by a continuous function: $\mathbb{R}^d \ni k \mapsto h(k) \in \mathcal{B}(\mathbb{C}^n)$ such that $h(k)$ is selfadjoint positive for all k, $\nabla h \in L^\infty(\mathbb{R}^d, \mathcal{B}(\mathbb{C}^n))$ and $\lim_{|k|\to\infty} (\inf h(k)) = +\infty$.

If we denote by x the operator on $\mathfrak{h}$ equal to $x = -i\nabla_k$, then, using Assumption **(SB3)**, one easily sees that for any $r > 0$, $q > 0$, the operator $1_{[-r,r]}(|x|)1_{[0,q]}(h)$ is compact. Moreover, let f, g be bounded measurable functions on $\mathbb{R}$ such that $\lim_{|t|\to\infty} f(t) = 0$, $\lim_{t\to+\infty} g(t) = 0$, then $f(|x|)g(h)$ is compact. This assumption is used, e.g. in the HVZ-type theorem 3.2, to perform localization in configuration space.

(SB4) $(\mathbb{1}_{\mathcal{H}_S} \otimes h^{-1/2})v(1 + H_S)^{-1/2}$ is compact.

(SB5) $(\mathbb{1}_{\mathcal{H}_S} \otimes h^{-1})v(1 + H_S)^{-1/2}$ is compact.

Note that Assumption **(SB4)** implies **(SB1)**. In fact, **(SB4)** implies $\lim_{t\to\infty} \|(\mathbb{1}_{\mathcal{H}_S} \otimes h^{-1/2})v(t + H_S)^{-1/2}\| = 0$. Note also that **(SB4)** implies **(SB5)** when $\inf \mathrm{sp}(h) > 0$.

(SB6) v can be split as $v = \mathbb{1}_{\mathcal{H}_S} \otimes |z\rangle + v_{\mathrm{ren}}$, where $z \in \mathrm{Dom}(h^{-1/2})$ and $(\mathbb{1}_{\mathcal{H}_S} \otimes h^{-1})v_{\mathrm{ren}}(1 + H_S)^{-1/2}$ is bounded.

3.1.2 Main results

Of course the first issue is to show that the generalized spin-boson Hamiltonian is well defined. This is the purpose of the following:

Proposition 3.1. *[BD1] Suppose that* $\alpha := \lim_{t\to\infty} \|(\mathbb{1}_{\mathcal{H}_S} \otimes h^{-1/2})v(t + H_S)^{-1/2}\| < \infty$. *Then the form* V *is form bounded with respect to* H_0 *with the* H_0*-form bound* $\leq \alpha$. *As a consequence, if Assumption* **(SB1)** *holds, then the operator* $H := H_0 + V$ *is well defined as the form sum. The form domains of* H_0 *and* H *coincide, that is* $\mathrm{Dom}|H|^{\frac{1}{2}} = \mathrm{Dom}|H_0|^{\frac{1}{2}}$.

The first main result is the HVZ type theorem which says that the essential spectrum of H is shifted to the right from the bottom of the spectrum of H by the "mass" $\inf \mathrm{sp}(h)$. Our result is similar to the one proven in [DG1] but with weaker assumptions.

Theorem 3.2. *[BD1] Suppose Assumptions* **(SB2)**, **(SB3)**, **(SB4)** *are true. Then*

$$\mathrm{sp}_{\mathrm{ess}} H = [\inf \mathrm{sp}(H) + \inf \mathrm{sp}(h), +\infty[.$$

Our next result concerns the existence of a ground state and is a generalization of a result of [Ge]. Theorem 3.2 clearly implies the existence of a ground state if $\inf \mathrm{sp}(h) > 0$. In the case $\inf \mathrm{sp}(h) = 0$, we need the stronger assumption **(SB5)** (recall that **(SB4)** implies **(SB5)** when $\inf \mathrm{sp}(h) > 0$). It is well-known that the main issue to prove the existence of a ground state is related to the so-called *infrared catastrophy*: if the mass of the bosons is zero, i.e $\inf \mathrm{sp}(h) = 0$, one can a priori have infinitely many "soft" bosons (bosons of low energy) in the ground state which makes it leave the initial Fock space (one then has to turn to another representation, see e.g. [Ar]). The role of assumption **(SB5)** is precisely to control the number of these soft bosons.

Theorem 3.3. *[BD1] Suppose that Assumptions* **(SB2)**, **(SB3)**, **(SB4)**, **(SB5)** *are satisfied. Then* $\inf \mathrm{sp}(H) \in \mathrm{sp}_{\mathrm{pp}}(H)$. *In other words,* H *has a ground state.*

In the particular case where v can be splitted as in Assumption **(SB6)**, our last result gives a necessary condition for the existence of a ground state. This is a generalization of a result of [DG2].

Theorem 3.4. *[BD1] Assume* v *satisfies Assumption* **(SB1)** *and* **(SB6)**. *If* H *has a ground state, then* $z \in \mathrm{Dom}(h^{-1})$.

One can see this result as a sort of reciprocal of Theorem 3.3. Indeed, we have the following corollary.

Corollary 3.5. *[BD1] Suppose Assumptions* **(SB2)** *and* **(SB3)** *are satisfied. If* v *satisfies* **(SB6)** *with* v_{ren} *satisfying also Assumptions* **(SB4)** *and* **(SB5)**. *Then,* H *has a ground state if and only if* $z \in \mathrm{Dom}(h^{-1})$.

3.2 Quadratic Hamiltonian on Symmetric Fock space

In this section we consider purely quadratic Hamiltonians (also called Bogoliubov Hamiltonians for reasons which will become more transparent later) on the symmetric Fock space, that is, of self-adjoint operators on a bosonic Fock space formally given by an expression of the form

$$\begin{aligned} H &= \int h(k) a^*(k) a(k) \mathrm{d}k + \frac{1}{2} \int v(k,k') a^*(k) a^*(k') \mathrm{d}k \mathrm{d}k' \\ &\quad + \frac{1}{2} \int \bar{v}(k,k') a(k) a(k') \mathrm{d}k \mathrm{d}k' + c, \end{aligned}$$

where $h(k)$ is a real function on $\mathfrak{h} = L^2(K, \mathrm{d}k)$ for some measure space $(K, \mathrm{d}k) =$ (i.e. the "free" one-particle Hamiltonian h is a multiplication operator), $v(k,k')$ is a complex symmetric

function on $K \times K$ and c is a constant which may be infinite. Actually, as in Section 3.1, we will not use such a "fibration" with respect to k and write (still formally)

$$H = \mathrm{d}\Gamma(h) + \frac{1}{2}a_2^*(v) + \frac{1}{2}a_2(v) + c. \tag{3.3}$$

When $\mathfrak{h}$ is finite dimensional this expression is not just formal, h is then interpreted as a self-adjoint operator on $\mathfrak{h}$, and v can be usually interpreted in two ways: either as a vector in $\otimes_{\mathrm{s}}^2\mathfrak{h}$ – a symmetric 2-particle vector, or as an antilinear operator on $\mathfrak{h}$ satisfying $\bar{v} = v^*$ where $f \mapsto \bar{f}$ is a fixed complex conjugation on $\mathfrak{h}$[a]. The operators $a_2(v)/a_2^*(v)$ denote here the quadratic annihilation/creation operator associated to the 2-particle vector v and are defined by

$$a_2^*(v)\Psi := \sqrt{n+2}\sqrt{n+1}\, v \otimes_{\mathrm{s}} \Psi, \qquad \Psi \in \Gamma_+^n(\mathfrak{h}),$$

$$a_2(v)\Psi := \sqrt{n+2}\sqrt{n+1}\, \langle v| \otimes \mathbb{1}^{\otimes n}\, \Psi, \qquad \Psi \in \Gamma_+^{n+2}(\mathfrak{h}),$$

where $\langle v| \otimes \mathbb{1}^{\otimes n} : \Gamma_+^{n+2}(\mathfrak{h}) \ni f_1 \otimes \cdots \otimes f_{n+2} \mapsto \langle v|f_1 \otimes f_2\rangle f_3 \otimes \cdots \otimes f_{n+2} \in \Gamma_+^n(\mathfrak{h})$. (These operators are well defined on $\Gamma_+^{\mathrm{fin}}(\mathfrak{h})$ and can be extended to $\mathrm{Dom}(N)$ where N is the number operator.)

There exist explicit formulas for various quantities related to Bogoliubov Hamiltonians. Therefore, they are often used in physics literature as useful exactly solvable models. There also exists an extensive rigorous literature devoted to Bogoliubov Hamiltonians, starting with the work of Friedrichs [F]. Later many authors, often independently of one another, studied this problem, among them one should mention Berezin [Be], Ruijsenaars [Ru1, Ru2], Araki and his collaborators [A, AY], Matsui and Shimada [MS].

Bogoliubov Hamiltonians are very well understood in the case of a finite number of degrees of freedom. Their theory becomes more difficult when the number of degrees of freedom is infinite (in other words, when the one-particle Hilbert space $\mathfrak{h}$ is infinite dimensional), which is the main topic of [BD2]. (Note, however, that even in the case of a finite number of degrees of freedom properties of Bogoliubov Hamiltonians are interesting.) In the case of an infinite number of degrees of freedom we will not use the formula (3.3) to define Bogoliubov Hamiltonians. This is due to several problems, e.g.:

1. v may actually be not an element of $\otimes_{\mathrm{s}}^2\mathfrak{h}$, but only an unbounded linear functional on this space, which means that $a_2^*(v)$ is not an operator but a quadratic form. (If we consider v as an operator on $\mathfrak{h}$, $v \in \otimes_{\mathrm{s}}^2\mathfrak{h}$ corresponds to a Hilbert-Schmidt operator on $\mathfrak{h}$ while we will consider operators which are just bounded).

2. c can be infinite, which means that the definition of H involves an infinite renormalization,

In order to define Bogoliubov Hamiltonians, we will have to come back to the "classical system" associated to the Fock space $\Gamma_+(\mathfrak{h})$. Bogoliubov Hamiltonians will then appear as generators of strongly continuous unitary groups $U(t)$ which implement on $\Gamma_+(\mathfrak{h})$, if possible, the

[a]The map $T : \mathfrak{h} \otimes \mathfrak{h} \ni \phi \otimes \psi \mapsto |\phi\rangle\langle\bar{\psi}| \in \mathcal{B}^2(\mathfrak{h})$, the space of Hilbert-Schmidt operators, extends by linearity and defines an isometry from $\mathfrak{h} \otimes \mathfrak{h}$ to the set $\mathcal{B}^2(\mathfrak{h})$ of all Hilbert-Schmidt operators on $\mathfrak{h}$. If moreover $v \in \mathfrak{h} \otimes_{\mathrm{s}} \mathfrak{h}$, then $\overline{T(v)} = T(v)^*$

group of Bogoliubov automorphisms associated to a given one-parameter group $r(t)$ of symplectic transformations on the classical space (hence the name Bogoliubov Hamiltonians), see Section 3.2.3. Of course, this definition will make Bogoliubov Hamiltonians defined only up to an additive constant, and one can ask whether there exists a natural choice that allows to fix it. We will introduce two kinds of Bogoliubov Hamiltonians: type I, characterized by the vanishing of the expectation value at the vacuum (i.e. $c = 0$), and type II, characterized by the fact that their infimum equals zero. We give sufficient conditions so that they are well defined. We also show, via a simple concrete example one can fully analyze, that there exist cases when only type I Hamiltonians are well defined, even though the classical Hamiltonian is positive (which may be interpreted as a kind of an infrared catastrophe), and cases when only type II are well defined, which means that one needs to introduce an infinite counterterm in the formula for the Hamiltonian (i.e. c in (3.3) undergoes an infinite renormalization).

3.2.1 Classical system associated to $\Gamma_+(\mathfrak{h})$

Classical system

To define the classical system associated to the Fock space $\Gamma_+(\mathfrak{h})$, we first fix a conjugation $f \mapsto \bar{f}$ on $\mathfrak{h}$. If A is an operator on $\mathfrak{h}$, $\bar{A}$ will denote the operator defined by $\bar{A}f := \overline{A\bar{f}}$. We then introduce two real vector spaces:

$$\mathcal{Y} := \{(f, \bar{f}) \ : \ f \in \mathfrak{h}\} \quad \text{and} \quad \bar{\mathcal{Y}} := \{(\bar{z}, z) \ : \ z \in \mathfrak{h}\}.$$

The space $\bar{\mathcal{Y}}$ has the meaning of the *classical phase space* of our system and $\mathcal{Y}$ serves as its dual. The duality is given by $2\mathrm{Re}\langle z|f\rangle = \langle z|f\rangle + \langle f|z\rangle$.

The form on $\mathcal{Y}$, which for $y_1 = (f_1, \bar{f}_1)$, $y_2 = (f_2, \bar{f}_2)$ is given by

$$\sigma(y_1, y_2) = 2\mathrm{Im}\langle f_1|f_2\rangle,$$

makes $\mathcal{Y}$ a symplectic space.

Elements of $\mathcal{Y}$ naturally parametrize the so-called field operators and Weyl operators on $\Gamma_+(\mathfrak{h})$. For $y = (f, \bar{f}) \in \mathcal{Y}$ they are given by

$$\phi(y) := a^*(f) + a(f), \quad W(y) := \mathrm{e}^{\mathrm{i}a^*(f)+\mathrm{i}a(f)}.$$

They satisfy the usual canonical commutation relation (CCR): $W(y_1)W(y_2) = \mathrm{e}^{-\frac{\mathrm{i}}{2}\sigma(y_1,y_2)}$.

Quantization of a classical observable

If $\beta : \bar{\mathcal{Y}} \to \mathbb{C}$ is a function that belongs to an appropriate class, i.e. a classical observable, then its Weyl quantization will be denoted $\mathrm{Op}(\beta)$. Let us list a couple of Weyl quantizations which will be relevant in this section, and where h is an operator on $\mathfrak{h}$ and $v \in \otimes^2_{\mathrm{s}}\mathfrak{h} \simeq \mathcal{B}^2_{\mathrm{s}}(\mathfrak{h})$ (the set of symmetric Hilbert-Schmidt operators):

1. $\bar{\mathcal{Y}} \ni (\bar{z}, z) \mapsto \beta(\bar{z}, z) = \langle z|hz\rangle, \ \Rightarrow \ \mathrm{Op}(\beta) = \mathrm{d}\Gamma(h) + \frac{1}{2}\mathrm{Tr}\,(h)$;
2. $\bar{\mathcal{Y}} \ni (\bar{z}, z) \mapsto \beta(\bar{z}, z) = \langle z \otimes z|v\rangle_{\mathfrak{h}\otimes\mathfrak{h}} = \langle z, v\bar{z}\rangle_{\mathfrak{h}}, \ \Rightarrow \ \mathrm{Op}(\beta) = a^*_2(v)$;
3. $\bar{\mathcal{Y}} \ni (\bar{z}, z) \mapsto \beta(\bar{z}, z) = \langle v|z \otimes z\rangle_{\mathfrak{h}\otimes\mathfrak{h}} = \langle \bar{z}, \bar{v}z\rangle_{\mathfrak{h}}, \ \Rightarrow \ \mathrm{Op}(\beta) = a_2(v)$.

As an illustration, if $\mathfrak{h} = L^2(K, \mathrm{d}k)$ then the three above examples become respectively

1. For $\beta(\bar{z}, z) = \int_K h(k)\bar{z}(k)z(k)\mathrm{d}k$, then

$$\begin{aligned} \mathrm{Op}(\beta) &= \frac{1}{2}\int_K h(k)a^*(k)a(k)\mathrm{d}k + \frac{1}{2}\int_K h(k)a(k)a^*(k)\mathrm{d}k \\ &= \int_K h(k)a^*(k)a(k)\mathrm{d}k + \frac{1}{2}\mathrm{Tr}(h). \end{aligned}$$

2. For $\beta(\bar{z}, z) = \int_{K\times K} v(k,k')\bar{z}(k)\bar{z}(k')\mathrm{d}k\mathrm{d}k'$, then

$$\mathrm{Op}(\beta) = \int_{K\times K} v(k,k')a^*(k)a^*(k')\mathrm{d}k\mathrm{d}k'.$$

3. For $\beta(\bar{z}, z) = \int_{K\times K} \bar{v}(k,k')z(k)z(k')\mathrm{d}k\mathrm{d}k'$, then

$$\mathrm{Op}(\beta) = \int_{K\times K} \bar{v}(k,k')a(k)a(k')\mathrm{d}k\mathrm{d}k'.$$

Symbol of an operator

If $z \in \mathfrak{h}$, then

$$\Omega_z := \mathrm{e}^{-\frac{|z|^2}{2}}\mathrm{e}^{a^*(z)}\Omega = W((-\mathrm{i}z, \mathrm{i}\bar{z}))\Omega$$

is the coherent vector localized arount the point in phase space $(\bar{z}, z) \in \bar{\mathcal{Y}}$. This family of vectors can be used to define the Wick symbol of an operator B acting on $\Gamma_+(\mathfrak{h})$. It is equal to the function on phase space given by

$$\mathrm{s}_B(\bar{z}, z) := \langle \Omega_z | B\Omega_z \rangle.$$

Let us give examples of Wick symbols relevant here:

1. $B = \mathrm{d}\Gamma(h) \quad \Rightarrow \quad \mathcal{Y} \ni (\bar{z}, z) \mapsto \mathrm{s}_B(\bar{z}, z) = \langle z | hz \rangle$,
2. $B = a_2^*(v) \quad \Rightarrow \quad \mathcal{Y} \ni (\bar{z}, z) \mapsto \mathrm{s}_B(\bar{z}, z) = \langle z \otimes z | v \rangle$,
3. $B = a_2(v) \quad \Rightarrow \quad \mathcal{Y} \ni (\bar{z}, z) \mapsto \mathrm{s}_B(\bar{z}, z) = \langle v | z \otimes z \rangle$.

In particular, the Wick symbol of the quadratic Hamiltonian (3.3) is

$$\mathrm{s}_B(\bar{z}, z) = \langle z | hz \rangle + \frac{1}{2}\langle z \otimes z | v \rangle + \frac{1}{2}\langle v | z \otimes z \rangle.$$

Classical Hamiltonian

Let r be a bounded linear map on $\mathcal{Y}$. It can be extended uniquely to a bounded complex linear map on $\mathfrak{h} \oplus \mathfrak{h}$, and thus can be represented as

$$r = \begin{pmatrix} p & q \\ \bar{q} & \bar{p} \end{pmatrix}, \tag{3.4}$$

where p is a linear and q an antilinear map on $\mathfrak{h}$. We say that a map r on $\mathcal{Y}$ is symplectic iff it is invertible and preserves the symplectic form σ. It is easy to see that a map r is symplectic if and only

$$\begin{cases} p^*p - \bar{q}^*\bar{q} &= 1, \quad pp^* - qq^* &= 1, \\ p^*\bar{q} - q^*p &= 0, \quad \bar{q}p^* - \bar{p}q^* &= 0. \end{cases} \tag{3.5}$$

In particular, if r is symplectic, then $p^*p \geq 1$ and therefore p is invertible.

One of the central objects in this section will be strongly continuous groups of symplectic transformations. Let $t \mapsto r(t)$ be such a transformation (with $p(t)$, $q(t)$ as in (3.5)) and a its generator, that is, $r(t) = \mathrm{e}^{ta}$. The *classical Hamiltonian of* $t \mapsto r(t)$ is defined as the function on the phase space given by

$$\bar{\mathcal{Y}} \supset \mathrm{Dom}(\bar{a}) \ni \bar{y} \mapsto \chi(\bar{y}) := \frac{1}{2}\sigma(y, ay).$$

(Note that in the case of the classical Hamiltonian we always normalize it so that $\chi(0) = 0$. An analogous normalization will not be always possible in the quantum case).

Proposition 3.6. *Suppose that a can be written as*

$$a = \mathrm{i}\begin{pmatrix} h & -v \\ \bar{v} & -\bar{h} \end{pmatrix}, \tag{3.6}$$

where h is a selfadjoint operator with domain $\mathrm{Dom}(h)$, v is a bounded operator such that $v^* = \bar{v}$, and $\mathrm{Dom}(a) = \mathrm{Dom}(h) \oplus \mathrm{Dom}(\bar{h})$. Then a generates a strongly continuous one-parameter group $(r(t))_{t\in\mathbb{R}}$ of symplectic maps and its classical Hamiltonian is

$$\bar{\mathcal{Y}} \cap \mathrm{Dom}(\bar{h} \oplus h) \ni (\bar{z}, z) \mapsto \chi(\bar{z}, z) := \langle z|hz\rangle + \frac{1}{2}\langle z \otimes z|v\rangle + \frac{1}{2}\langle v|z \otimes z\rangle. \tag{3.7}$$

A first natural definition of Bogoliubov Hamiltonians would be to consider the Weyl quantization of the classical Hamiltonian which is indeed of the form (3.3), with $c = \frac{1}{2}\mathrm{Tr}(h)$. This imposes severe restriction on h (to be trace class) and is therefore quite restrictive. We shall therefore define them in different way, and to understand how we first consider the simple case where $\mathfrak{h}$ has a finite dimension.

3.2.2 Quadratic Hamiltonians in the case of a finite number of degrees of freedom

In this section we assume that $\mathfrak{h}$ has finite dimension. The theory of Bogoliubov Hamiltonians is then fully understood. Let $r(t) = \mathrm{e}^{ta}$ be a continuous symplectic group on $\mathcal{Y}$. Then a is always of the form (3.6), so that the corresponding classical Hamiltonian is given by (3.7). In the case of a finite number of degrees of freedom we define Bogoliubov Hamiltonians associated to $t \mapsto r(t)$ by the formula

$$H = \mathrm{d}\Gamma(h) + \frac{1}{2}a_2^*(v) + \frac{1}{2}a_2(v) + c, \tag{3.8}$$

where c is an arbitrary real number. Note that when $c = 0$ the Wick symbol of H coincides with its classical Hamiltonian. The operator H is essentially self-adjoint on $\mathrm{Dom}(N)$.

A special choice of a Bogoliubov Hamiltonian is given by setting $c = \frac{1}{2}\mathrm{Tr}(h)$. As already mentioned, this leads to the Weyl quantization $\mathrm{Op}(\chi)$ of the classical Hamiltonian χ. The following theorem is well known:

Theorem 3.7. *Set*

$$k(t) := q(t)\bar{p}(t)^{-1}, \qquad l(t) := -p(t)^{-1}q(t). \tag{3.9}$$

Then

$$\begin{aligned} \mathrm{e}^{\mathrm{i}t\mathrm{Op}(\chi)} &= \det(\bar{p}(t))^{-1/2}\mathrm{e}^{-\frac{1}{2}a_2^*(l(t))}\Gamma((p(t)^{-1})^*)\mathrm{e}^{-\frac{1}{2}a_2(l(t))}, \\ \mathrm{e}^{\mathrm{i}t\mathrm{Op}(\chi)}W(y)\mathrm{e}^{-\mathrm{i}t\mathrm{Op}(\chi)} &= W(r(t)y), \;\; y \in \mathcal{Y}. \end{aligned} \tag{3.10}$$

One sees from (3.10) that the Bogoliubov Hamiltonian $H = \mathrm{Op}(\chi)$ implements the Bogoliubov dynamics defined by the continuous symplectic group $r(t)$. This is this property that we shall use in the general case to define Bogoliubov Hamiltonians. Of course, the generator H is then defined only up to a constant and we shall consider some of its natural choices.

The first choice, called type I, is when $c = 0$ and is always defined for a finite number of degrees of freedom. The other choice (so-called type II) will be characterized by the fact that its infimum is zero. Such a choice is possible only if the Bogoliubov Hamiltonian is bounded from below. In the case where $\mathfrak{h}$ is finite dimensional we have the following

Theorem 3.8. *[BD2] If dim$(\mathfrak{h})$ is finite, a Bogoliubov Hamiltonian is bounded from below iff the corresponding classical Hamiltonian is positive. Then we have*

$$\inf \mathrm{sp}\left(\mathrm{d}\Gamma(h) + \frac{1}{2}a_2^*(v) + \frac{1}{2}a_2(v)\right) = \frac{1}{4}\mathrm{Tr}\left[\begin{pmatrix} \bar{h}^2 - \bar{v}v & \bar{h}\bar{v} - \bar{v}h \\ hv - v\bar{h} & h^2 - v\bar{v} \end{pmatrix}^{1/2} - \begin{pmatrix} \bar{h} & 0 \\ 0 & h \end{pmatrix}\right].$$

In this case, the Bogoliubov Hamiltonian of type II would thus correspond to

$$c = -\frac{1}{4}\mathrm{Tr}\left[\begin{pmatrix} \bar{h}^2 - \bar{v}v & \bar{h}\bar{v} - \bar{v}h \\ hv - v\bar{h} & h^2 - v\bar{v} \end{pmatrix}^{1/2} - \begin{pmatrix} \bar{h} & 0 \\ 0 & h \end{pmatrix}\right].$$

3.2.3 Implementability of Bogoliubov automorphisms. Type I and type II Bogoliubov Hamiltonians

If r is a symplectic map on $\mathcal{Y}$, then the operators $W_r(y) := W(ry)$ also satisfy the canonical commutation relations. (One says that they form another representation of the CCR.) The map r is called *implementable* if and only if there exists a unitary operator U on $\Gamma_+(\mathfrak{h})$ such that $UW(y)U^{-1} = W(ry), \forall y \in \mathcal{Y}$. (In other words, the two representation W and W_r are unitarily equivalent.) If it exists, U is called a Bogoliubov implementer of r.

We write $r = \begin{pmatrix} p & q \\ \bar{q} & \bar{p} \end{pmatrix}$ as in (3.4) and define k and l as in (3.9). The following result is well known (see [Be] Section II.4, [Ru1, Ru2, Sh]).

Theorem 3.9. *Let r be a symplectic map. The following are equivalent:*

1. *r is implementable.*

2. $q \in \mathcal{B}^2(\mathfrak{h})$, *or equivalently,* $[r, j] \in \mathcal{B}^2(\mathfrak{h} \oplus \mathfrak{h})$ *where* $j = \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix}$.

If the above conditions hold, then

(i) the operators k and l belong to $\mathcal{B}^2_{\mathrm{s}}(\mathfrak{h})$ and $\|k\|, \|l\| < 1$;

(ii) the operator

$$U_{\mathrm{nat}} := \det(1 - k^*k)^{1/4} \mathrm{e}^{-\frac{1}{2}a_2^*(k)} \Gamma((p^{-1})^*) \mathrm{e}^{-\frac{1}{2}a_2(l)}$$

is well defined on $\Gamma_+^{\mathrm{fin}}(\mathfrak{h})$, extends to a unitary operator on $\Gamma_+(\mathfrak{h})$, and implements r. We call U_{nat} the natural Bogoliubov implementer of r*;*

(iii) all Bogoliubov implementers of r are proportional to U_{nat};

(iv) U_{nat} is the only Bogoliubov implementer whose expectation value on the vacuum is positive: $\langle \Omega | U_{\mathrm{nat}} \Omega \rangle = \det(1 - k^*k)^{1/4} > 0$.

The equivalence between 1. and 2. is known as Shale's Theorem.
We now turn to strongly continuous one-parameter groups $r(t)$ of symplectic maps.

Definition 3.10. *A one parameter symplectic group $r(t)$ is called implementable if and only if there exists a strongly continuous unitary group $U(t)$ such that, for all t, $U(t)$ is a Bogoliubov implementer of $r(t)$. If $r(t)$ is implementable, the unitary group $U(t)$ implementing $r(t)$ is called a unitary Bogoliubov dynamics and its self-adjoint generator H is called a Bogoliubov Hamiltonian (associated to $r(t)$).*

One can actually prove that $r(t)$ is unitarily implementable under very weak assumptions.

Theorem 3.11. *[BD2] Suppose $r(t)$ is a strongly continuous one-parameter symplectic group. Then $r(t)$ is implementable if and only if $\|q(t)\|_2 < \infty$, i.e. $q(t) \in \mathcal{B}^2(\mathfrak{h})$, for all t and $\lim_{t \to 0} \|q(t)\|_2 = 0$.*

Since a Bogoliubov implementer of a symplectic map r is unique up to a phase, if $r(t)$ is implementable then there exists $c(t) \in \mathbb{C}$, $|c(t)| = 1$, such that $U(t) = c(t) U_{\mathrm{nat}}(t)$, and where $U_{\mathrm{nat}}(t)$ is the natural Bogoliubov implementer of $r(t)$. $c(t)$ will be called the natural cocycle for $U(t)$. Obviously a Bogoliubov Hamiltonian is also defined up to a constant. Let us describe more precisely the two natural choices we will consider.

The first one would amount to take $c = 0$ in (3.3). Since we have no explicit formula for H in the general situation, we need for a definition which bypasses it. Let us come back to the case where $\dim(\mathfrak{h})$ is finite and denote by H_I the Bogoliubov Hamiltonian with $c = 0$. Using (3.10) we have

$$\mathrm{e}^{\mathrm{i}tH_I} = \det(\bar{p}(t) \mathrm{e}^{\mathrm{i}t\bar{h}})^{-1/2} \mathrm{e}^{-\frac{1}{2}a_2^*(l(t))} \Gamma((p(t)^{-1})^*) \mathrm{e}^{-\frac{1}{2}a_2(l(t))}.$$

The operator h is defined from $p(t)$ by $\frac{\mathrm{d}}{\mathrm{d}t} p(t) \lceil_{t=0} = \mathrm{i}h$, and the determinant makes sense provided $p(t)\mathrm{e}^{-\mathrm{i}th} - 1$ is trace class. In the general case, this leads to the following:

Definition 3.12. *Let $t \mapsto r(t)$ be an implementable symplectic group. We say that it is of type I if and only if there exists a self-adjoint operator h on $\mathfrak{h}$ such that:*

i) there exists a dense subspace $\mathfrak{h}_0 \subset \mathfrak{h}$ such that, for any $f \in \mathfrak{h}_0$, $p(t)f$ is differentiable at 0 with $\frac{\mathrm{d}}{\mathrm{d}t}p(t)f\lceil_{t=0} = \mathrm{i}hf$,

ii) $p(t)\mathrm{e}^{-\mathrm{i}th} - 1 \in \mathcal{B}^1(\mathfrak{h})$ for all $t \in \mathbb{R}$,

iii) $\lim_{t\to 0} \|p(t)\mathrm{e}^{-\mathrm{i}th} - 1\|_1 = 0$.

In this case, h is defined uniquely, and we set

$$U_I(t) := \det(\bar{p}(t)\mathrm{e}^{\mathrm{i}t\bar{h}})^{-1/2}\mathrm{e}^{-\frac{1}{2}a_2^*(k(t))}\Gamma((p(t)^{-1})^*)\mathrm{e}^{-\frac{1}{2}a_2(l(t))}, \tag{3.11}$$

$$c_I(t) := \det(\bar{p}(t)\mathrm{e}^{\mathrm{i}t\bar{h}})^{-1/2}\det(1-k(t)^*k(t))^{-1/4}. \tag{3.12}$$

The operator $H_I = \frac{1}{\mathrm{i}}\frac{\mathrm{d}}{\mathrm{d}t}U_I(t)\lceil_{t=0}$ is called a Bogoliubov Hamiltonian of type I.

The second choice is simpler to define. This is the one which corresponds to have 0 as ground state energy of H.

Definition 3.13. *An implementable symplectic group $r(t)$ is of type II if and only if it has a bounded from below Bogoliubov Hamiltonian (and hence all its Bogoliubov Hamiltonians are bounded from below).*

If $r(t)$ is of type II, we define the Bogoliubov Hamiltonian of type II to be the unique associated Bogoliubov Hamiltonian whose infimum of spectrum is 0. We denote it by H_{II}. The corresponding Bogoliubov unitary dynamics is denoted $U_{II}(t) = \mathrm{e}^{\mathrm{i}tH_{II}}$.

As we have seen in the previous section, for a finite number of degrees of freedom, all one-parameter symplectic groups are implementable. They are always of type I and they are of type II iff the classical Hamiltonian is positive. As we shall see, the situation is completely different in the general case.

3.2.4 Generators of type I and type II symplectic one-parameter groups

Contrary to the finite dimensional case, we do not have necessary and sufficient conditions in general for the existence of type I or type II Bogoliubov Hamiltonians. In this section, we give some sufficient conditions on the generator a of the symplectic group $r(t)$ which guarantee that $r(t)$ is implementable, i.e. a Bogoliubov Hamiltonian is well defined, is of type I, resp. type II.

The first result concerns the implementability of $r(t)$ and is essentially due to Berezin [Be] (Section III.6, Lemma 3).

Theorem 3.14. *[BD2] Suppose that the assumption of Proposition 3.6 is satisfied. Define*

$$w(t) := -\mathrm{i}\int_0^t \mathrm{e}^{\mathrm{i}\tau h}v\mathrm{e}^{\mathrm{i}\tau\bar{h}}\mathrm{d}\tau. \tag{3.13}$$

Suppose also that for all t, $w(t) \in \mathcal{B}^2(\mathfrak{h})$, the function $t \mapsto \|w(t)\|_2$ is locally integrable on $\mathbb{R}$ and continuous at $t = 0$. Then $r(t)$ is implementable.

Obviously if v itself is Hilbert-Schmidt, so that $a_2(v)$ and $a_2^*(v)$ are defined as operators and not only as quadratic forms, then the above Theorem applies.

Our next result concerns the existence of type I Hamiltonians. Again, it can be traced back to [Be] (Section III.6, Lemma 4).

Theorem 3.15. *[BD2] Suppose that a satisfies the assumptions of Proposition 3.6, $w(t)$ is defined as in (3.13) and satisifes the assumptions of Theorem 3.14. If moreover $\bar{v}w(t)$ is trace class and $t \mapsto \|\bar{v}w(t)\|_1$ is locally integrable and continuous at zero, then $r(t)$ is of type I.*

Again, the above theorem applies when v is Hilbert-Schmidt. Formally, it is easy to see that the Bogoliubov Hamiltonian of type I is indeed given by (3.3) with $c = 0$. We can make this precise if v is Hilbert-Schmidt.

Theorem 3.16. *[BD2] Suppose v is Hilbert Schmidt. Then*

1. *The operator $H_I = \mathrm{d}\Gamma(h) + \frac{1}{2}(a_2^*(v) + a_2(v))$ is essentially selfadjoint on $\mathcal{D} := \Gamma_+^{\mathrm{fin}}(\mathfrak{h}) \cap \mathrm{Dom}(\mathrm{d}\Gamma(h))$.*

2. *U_I and c_I defined as in (3.11) and (3.12) satisfy*

$$\begin{aligned} U_I(t) &= \mathrm{e}^{\frac{\mathrm{i}}{2}\mathrm{Tr}(\int_0^t q(s)v\bar{p}(s)^{-1}\mathrm{d}s)}\mathrm{e}^{-\frac{1}{2}a_2^*(k(t))}\Gamma((p(t)^{-1})^*)\mathrm{e}^{-\frac{1}{2}a_2(l(t))}, \\ c_I(t) &= \mathrm{e}^{\frac{\mathrm{i}}{2}\mathrm{Re}(\mathrm{Tr}(\int_0^t q(s)v\bar{p}(s)^{-1}\mathrm{d}s))}. \end{aligned}$$

3. *H_I is the Bogoliubov Hamiltonian of type I: that is $\mathrm{e}^{\mathrm{i}tH_I} = U_I(t)$.*

4. *Let H be a Bogoliubov Hamiltonian associated with $r(t)$. Then the vaccum $\Omega \in \mathrm{Dom}(H)$ and $H = H_I$ iff $\langle\Omega|H\Omega\rangle = 0$.*

Remark 3.1. *Since $\Gamma_+^{\mathrm{fin}}(\mathfrak{h}) \subset \mathrm{Dom}(a_2^*(v) + a_2(v))$, the operator H_I is therefore essentially selfadjoint on $\mathrm{Dom}(\mathrm{d}\Gamma(h)) \cap \mathrm{Dom}(a_2^*(v) + a_2(v))$. The strategy of the proof for the essential selfadjointness comes from [Be] (Theorem 6.1) and goes back to Carleman [Ca]. However, the proof in [Be] is not completely rigorous. A similar result has also been proven in [IH] when h is bounded.*

Finally we give a result which allows to consider $\frac{1}{2}\,(a_2^*(v) + a_2(v))$ as a perturbation of $\mathrm{d}\Gamma(h)$. In this case both type I and type II Bogoliubov Hamiltonians exist.

Theorem 3.17. *[BD2] Let h be a positive selfadjoint operator on $\mathfrak{h}$ and suppose that $(h^{-1/2} \otimes h^{-1/2})v \in \mathfrak{h} \otimes_{\mathrm{s}} \mathfrak{h}$ and $h^{-1/2}v \in B(\mathfrak{h})$. Then $a_2(v) + a_2^*(v)$ is $\mathrm{d}\Gamma(h)$ bounded with relative bound less than $2\|(h^{-1/2} \otimes h^{-1/2})v\|$.*

As a consequence, if moreover $\|(h^{-1/2} \otimes h^{-1/2})v\| < 1$ then the operator (3.3) is selfadjoint on $\mathrm{Dom}(\mathrm{d}\Gamma(h))$ and bounded from below. In particular, the symplectic group $r(t)$ generated by a given by (3.6) is both of type I and type II, and (3.3) coincides with the type I Bogoliubov Hamiltonian H_I.

Note that we used different meanings of v in the two conditions on v: 2-particle vector or operator on $\mathfrak{h}$. Note also that these conditions resemble the condition one can find for the Van-Hove and the generalized spin-boson Hamiltonians (see e.g. [D, DJ1] and Section 3.1), where a perturbation linear in the annihilation and creation operators, instead of quadratic, is involved.

To end this section, we consider the simplest "infinite dimensional" case. Namely, $\mathfrak{h} := \ell^2(\mathbb{N})$ with its canonical basis $(e_n)_{n\in\mathbb{N}}$ and h and v are both diagonal, i.e.

$$h := \sum_n h_n |e_n\rangle\langle e_n|, \qquad v := \sum_n v_n |e_n\rangle\langle \bar{e}_n|, \tag{3.14}$$

and where the h_n are real numbers so that h is selfadjoint. We can identify $\mathfrak{h} \oplus \mathfrak{h}$ with $\oplus_n \mathbb{C}^2$ and the operator a with $\oplus_n \mathrm{i}\begin{bmatrix} h_n & -v_n \\ \bar{v}_n & -h_n \end{bmatrix}$.

Note that (3.14) is equivalent to assuming that $hv = v\bar{h}$ and that there exists a basis of eigenvectors of h.

Our goal is to describe, in this simple situation, what are the one parameter symplectic groups $t \mapsto r(t)$ which are implementable, which are those of type I, and those of type II. More precisely, using the previous theorems, we have

Theorem 3.18. *[BD2] Consider on $\ell^2(\mathbb{N})$ the operators h and v defined by (3.14).*

(i) $t \mapsto r(t)$ defines a strongly continuous one parameter group of symplectic maps if and only if v is h-bounded with relative bound strictly less than one, i.e. there exists $\alpha \in [0, 1[$ and $\beta \geq 0$ such that for all $n \in \mathbb{N}$, $|v_n| \leq \alpha|h_n| + \beta$.

(ii) $t \mapsto r(t)$ is implementable if and only if $\sum \frac{|v_n|^2}{1+h_n^2} < +\infty$.

(iii) $t \mapsto r(t)$ is of type I if and only if $\sum \frac{|v_n|^2}{1+|h_n|} < +\infty$.

(iv) $t \mapsto r(t)$ is of type II if and only if $h_n \geq |v_n|$ for all n and $\sum \frac{|v_n|^2}{h_n+h_n^2} < +\infty$.

Remark 3.2. *The classical hamilotnian is positive if and only if $h_n \geq |v_n|$ for all n. (iv) therefore shows that there are examples where the quantum Bogoliubov Hamiltonian is unbounded below although the classical Hamiltonian is positive: when $\sum \frac{|v_n|^2}{h_n+h_n^2}$ diverges. Moreover, (ii) shows that this is due to the small eigenvalues of h. It may thus be interpreted as a kind of infrared catastroph as we mentioned in the introduction.*

On the other hand, (ii) and (iii) show that there are Bogoliubov Hamiltonians which are not of type I. In other words, in order to express them in terms of creation and annihilation operators one needs to add an infinite constant, i.e. perform some renormalization. Moreover, (ii) and (iii) also show that this is due to the large eigenvalues of h, i.e. to ultraviolet divergencies.

Chapter 4

Some results about random matrices

This chapter contains two results conserning random matrices but which are of very different nature. In the first section we consider i.i.d. matrices and study the convergence of the product of a large number of them. In the second section we consider n by n real matrices whose entries are non-degenerate random variables that are independent but non necessarily identically distributed, and investigate how the probability that such a matrix is singular behaves when n is large. The results presented in these two sections come respectively from the articles [BJM2] and [BG].

4.1 Product of random contractions

In this section we study products of infinitely many i.i.d. random matrices. The matrices we consider satisfy two basic properties, reflecting the fact that they describe the dynamics of random quantum or classical dynamical systems. The first property is that they are contractions for some fixed norm. It reflects the fact that the underlying dynamics is in a certain sense norm-preserving. The second property is that there is a deterministic invariant vector. This represents a normalization of the dynamics.

Our first motivation comes from the study of RI systems. In the Liouvillian description, we showed in Section 1.1 that the study of observables on the system $\mathcal{S}$, and more generally of instantaneous observables, reduces to the analysis of the product $M_1 \cdots M_n$ of the corresponding RDO's. The general properties of RDO's (Proposition 1.1) make them fit into the above framework: they are contractions because the underlying dynamics is unitary, and the fact that they have a deterministic invariant vector simply traduces that the identity observable is always invariant. Another important example of systems falling into this category are Markov chains in a *random environment*, i.e. products of *random stochastic matrices* (with the $\| \cdot \|_\infty$ norm and the invariant vector $(1, \ldots, 1)^t$).

Let $M(\omega)$ be a random matrix on $\mathbb{C}^d$, with underlying probability space $(\Omega, \mathcal{F}, \mathrm{p})$. We say that $M(\omega)$ is a *random reduced dynamics operator* (RRDO) if

1. There exists a norm $|||\cdot|||$ on $\mathbb{C}^d$ such that, for almost all ω, $M(\omega)$ is a contraction on $\mathbb{C}^d$ endowed with the norm $|||\cdot|||$.

2. There is a vector $\psi_{\mathcal{S}}$, constant in ω, such that $M(\omega)\psi_{\mathcal{S}} = \psi_{\mathcal{S}}$, for almost all ω.

We shall normalize $\psi_{\mathcal{S}}$ such that $\|\psi_{\mathcal{S}}\| = 1$ where $\|\cdot\|$ denotes the euclidean norm. (In view of Chapter 1, one should have in mind that $\psi_{\mathcal{S}}$ is the reference vector of $\mathcal{S}$.)
To an RRDO $M(\omega)$, we associate the (iid) *random reduced dynamics process* (RRDP)

$$\Phi_n(\tilde{\omega}) := M(\omega_1)\cdots M(\omega_n), \qquad \tilde{\omega} = (\omega_n)_n \in \tilde{\Omega} := \Omega^{\mathbb{N}^*}. \tag{4.1}$$

We will show that Φ_n has a decomposition into an exponentially decaying part and a generically fluctuating part (Theorems 4.1 and 4.2). To identify these parts, we proceed as follows. It follows from 1. and 2. that the spectrum of an RRDO $M(\omega)$ must lie inside the closed complex unit disk, and that 1 is an eigenvalue (with eigenvector $\psi_{\mathcal{S}}$). Let $P_1(\omega)$ denote the spectral projection of $M(\omega)$ corresponding to the eigenvalue 1 ($\dim P_1(\omega) \geq 1$), and let $P_1^*(\omega)$ be its adjoint operator. Define

$$\psi(\omega) := P_1(\omega)^*\psi_{\mathcal{S}}, \tag{4.2}$$

and set $P(\omega) = |\psi_{\mathcal{S}}\rangle\langle\psi(\omega)|$. We put $Q(\omega) = \mathbb{1} - P(\omega)$. Note that the vector $\psi(\omega)$ is normalized as $\langle\psi_{\mathcal{S}}, \psi(\omega)\rangle = 1$. We decompose $M(\omega)$ as

$$M(\omega) = P(\omega) + Q(\omega)M(\omega)Q(\omega) =: P(\omega) + M_Q(\omega). \tag{4.3}$$

Note that if $\{1\}$ is a simple eigenvalue of $M(\omega)$ then this is nothing but the spectral decomposition of the matrix $M(\omega)$. Taking into account this decomposition, we obtain

$$\Phi_n(\tilde{\omega}) := M(\omega_1)\cdots M(\omega_n) = |\psi_{\mathcal{S}}\rangle\langle\psi_n(\tilde{\omega})| + M_Q(\omega_1)\cdots M_Q(\omega_n),$$

where $\psi_n(\tilde{\omega})$ is the Markov process

$$\psi_n(\tilde{\omega}) = M^*(\omega_n)\cdots M^*(\omega_2)\psi(\omega_1), \tag{4.4}$$

As in Section 1.4, let $\mathcal{M}_{(E)}$ be the set of RRDOs whose spectrum on the complex unit circle consists only of a simple eigenvalue $\{1\}$, and define the probability measure $\mathrm{d}\mathbb{P}$ on $\tilde{\Omega}$ by

$$\mathrm{d}\mathbb{P} = \Pi_{j\geq 1}\mathrm{dp}_j, \quad \text{where} \quad \mathrm{dp}_j \equiv \mathrm{dp}, \ \ \forall j \in \mathbb{N}^*.$$

Theorem 4.1 (Decaying process)**.** *[BJM2] Let $M(\omega)$ be a random reduced dynamics operator. Suppose that $\mathrm{p}(M(\omega) \in \mathcal{M}_{(E)}) > 0$. Then there exist a set $\Omega_1 \subset \tilde{\Omega}$ and a constant $\alpha > 0$ s.t. $\mathbb{P}(\Omega_1) = 1$ and for any $\tilde{\omega} \in \Omega_1$ there exists $C_{\tilde{\omega}}$ so that for any $n \in \mathbb{N}^*$,*

$$\|M_Q(\omega_1)\cdots M_Q(\omega_n)\| \leq C_{\tilde{\omega}}\mathrm{e}^{-\alpha n}.$$

Remark 4.1. *1. Any stochastic matrix whose entries are all nonzero belongs to $\mathcal{M}_{(E)}$.*

2. The choice (4.2) ensures that $\psi(\omega)$ is an eigenvector of $M^*(\omega)$ and defines a bona fide *random variable [Az]. Other choices of measurable $\psi(\omega)$ which are bounded in ω lead to different decompositions of $M(\omega)$, and can be useful as well. For instance, if $M(\omega)$ is a bistochastic matrix, then one can take for $\psi(\omega)$ the $M^*(\omega)$-invariant and ω-independent vector $(1,\ldots,1)^t$.*

The next result concerns the asymptotics of the Markov process (4.4). Denote by $P_{1,\mathbb{E}[M]}$ the spectral projection of $\mathbb{E}[M]$ onto the eigenvalue $\{1\}$.

Theorem 4.2 (Fluctuating process)**.** *[BJM2] Let $M(\omega)$ be a random reduced dynamics operator. Suppose that $\mathrm{p}(M(\omega) \in \mathcal{M}_{(E)}) > 0$. Then we have $\mathbb{E}[M] \in \mathcal{M}_{(E)}$. Moreover, there exists a set $\Omega_2 \subset \tilde{\Omega}$ s.t. $\mathbb{P}(\Omega_2) = 1$ and, for all $\tilde{\omega} \in \Omega_2$,*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \psi_n(\tilde{\omega}) = \psi_\infty,$$

where

$$\psi_\infty = (\mathbb{1} - \mathbb{E}[M_Q(\omega)]^*)^{-1}\, \mathbb{E}[\psi(\omega)] = P^*_{1,\mathbb{E}[M(\omega)]} \mathbb{E}[\psi(\omega)] = P^*_{1,\mathbb{E}[M(\omega)]} \psi_{\mathcal{S}}. \tag{4.5}$$

Remark 4.2. *It follows from the last equality in (4.5) that the ergodic average limit of $\psi_n(\tilde{\omega})$ does not depend on the particular choice of $\psi(\omega)$.*

Combining the above two theorems we obtain the following

Theorem 4.3. *[BJM2] Let $M(\omega)$ be a random reduced dynamics operator. Suppose $\mathrm{p}(M(\omega) \in \mathcal{M}_{(E)}) > 0$. Then there exists $\Omega_3 \subset \tilde{\Omega}$ s.t. $\mathbb{P}(\Omega_3) = 1$ and, for all $\tilde{\omega} \in \Omega_3$,*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} M(\omega_1) \cdots M(\omega_n) = P_{1,\mathbb{E}[M(\omega)]} = |\psi_{\mathcal{S}}\rangle\langle\psi_\infty|,$$

where ψ_∞ is defined in (4.5).

If one can choose $\psi(\omega) \equiv \psi$ to be independent of ω, e.g. when $M(\omega)$ is a bistochastic matrix, then one actually has $\psi_n(\tilde{\omega}) = \psi$ for all $n, \tilde{\omega}$, and hence the following better convergence result holds.

Theorem 4.4. *[BJM2] Let $M(\omega)$ be a random reduced dynamics operator. Suppose that $\mathrm{p}(M(\omega) \in \mathcal{M}_{(E)}) > 0$ and there exists $\psi \in \mathbb{C}^d$ such that $M(\omega)^*\psi = \psi$ for all ω. Then there exists a set $\Omega_4 \subset \tilde{\Omega}$, and a constant $\alpha > 0$, s.t. $\mathbb{P}(\Omega_4) = 1$ and for any $\tilde{\omega} \in \Omega_4$, there exists $C_{\tilde{\omega}}$ so that for any $n \in \mathbb{N}^*$,*

$$\left\| M(\omega_1) \cdots M(\omega_n) - \frac{|\psi_{\mathcal{S}}\rangle\langle\psi|}{\langle\psi, \psi_{\mathcal{S}}\rangle} \right\| \leq C_{\tilde{\omega}} \mathrm{e}^{-\alpha n}.$$

Results on the convergence (of some kind) of products of random stochastic matrices, and non-negative matrices, are numerous, see e.g. [KS, Mu, Se] and references therein. However, these results mostly concern properties of the limiting distribution, if it exists, in terms of the properties of the distribution of the random matrices. Those results rely heavily on the positivity of the entries of the considered matrices. When studying convergence in distribution, the order of the factors in the product does not matter, and usually products of the form

$$\Xi_n = M_n M_{n-1} \cdots M_1$$

as studied as well (compare with (4.1)). Our techniques still yield results for them that are stronger than those for products (4.1). Namely, we have

Theorem 4.5. *[BJM2] Suppose* $\mathrm{p}(M(\omega) \in \mathcal{M}_{(E)}) > 0$. *Then there exist* $\alpha > 0$, *a random vector* $\psi_\infty(\tilde{\omega})$ *and* $\Omega_5 \subset \tilde{\Omega}$ *with* $\mathbb{P}(\Omega_5) = 1$ *such that for any* $\tilde{\omega} \in \Omega_5$ *there exists* $C_{\tilde{\omega}} > 0$ *so that*

$$\left\| \Xi_n(\tilde{\omega}) - |\psi_{\mathcal{S}}\rangle\langle\psi_\infty(\tilde{\omega})| \right\| \leq C_{\tilde{\omega}} e^{-\alpha n}, \quad \forall n \in \mathbb{N}^*.$$

Remark 4.3. *1. Of course,* $\mathbb{E}(\psi_\infty(\tilde{\omega})) = \psi_\infty$ *as expected.*

2. A direct consequence of the above theorem is that the top Lyapunov exponent $\gamma_1(\tilde{\omega})$ *associated to this process is zero and is almost surely of multiplicity one.*

Comments on related results

As we already mentioned, results on the convergence (of some kind) of products of random stochastic matrices, and non-negative matrices, are numerous and mostly concern properties of the limiting distribution, if it exists, in terms of the properties of the distribution of the random matrices. The techniques used to obtain those results rely heavily on the positivity of matrix elements. Random matrix products have been heavily studied also for matrices in $Gl_d(\mathbb{R})$, see e.g. [Arn, G, K]. Again, the main focus of these works is on the study of the properties of the limiting distribution, if it exists, and on the properties of the Lyapunov exponents. In this case, the group property of the set of invertible matrices $Gl_d(\mathbb{R})$ plays a prominent role in the derivation of the results.

By contrast, besides conditions 1. and 2. defining RRDOs, we do not require our matrices to be real valued, to have positive entries, or to be invertible. Moreover, we are concerned here with the limiting properties of the products only, not with the limiting distribution. On the one hand, we get a.s. convergence results for the products Ξ_n (Theorem 4.5). On the other hand, in order to eliminate the unavoidable fluctuations in the products Φ_n, we resort to a limit in an average sense, the Cesaro limit (Theorem 4.3).

Getting informations on the fluctuations of the process around its limiting value is certainly an interesting and important issue. It amounts to getting informations about the law of the vector valued random variable ψ_∞ of Theorem 4.5, which is quite difficult in general. There are recent partial results about aspects of the law of such random vectors in case they are obtained by means of matrices belonging to some subgroups of $Gl_d(\mathbb{R})$ satisfying certain irreducibility conditions, see e.g. [DGL]. However, these results do not apply to our situation.

Random ergodic theorems for products $\Phi_n(\tilde{\omega})$ have been obtained in a more general framework in [BS]. They prove the almost sure existence of a Cesaro limit for these products. However, the identification of the limit is not provided by this general result. This identification relies on the detailed properties of the matrices involved, and in particular in the separation of a fluctuating part from a decaying part in the dynamical process. In this respect, and having in mind the application to RI systems, the contribution of [BJM2] consists in identifying completely the Cesaro limit of products of RRDO.

4.2 Singularity of large random matrices with independent entries

Our second result on random matrices is of a completely different nature. We consider real matrices $M_n = (a_{ij})_{1 \leq i,j \leq n}$ whose entries are non-degenerate random variables that are inde-

pendent but non necessarily identically distributed, and we are interested in the probability that such a matrix is singular in the limit of large n.

The study of the singularity of random matrices goes back, at least, to Komlós who showed in [Ko1] that $\mathbb{P}(M_n \text{ is singular}) = o(1)$ for independent and identically distributed (iid) Bernoulli entries, namely $a_{ij} = 0, 1$ with probability $1/2$. Using Sperner's Lemma, Komlós noticed that the probability was $O(n^{-1/2})$ [Bo]. For iid Bernoulli entries, the conjecture is that $\mathbb{P}(M_n \text{ is singular}) = (c + o(1))^n$ with $c = \frac{1}{2}$. Such an exponential behaviour has been successively obtained and improved in [KKoS, TV1, TV2] up to $c = \frac{3}{4}$.

If one turns to general entries, Komlós proved in [Ko2] that $\mathbb{P}(M_n \text{ is singular}) = o(1)$ for independent *and identically distributed* non degenerate random variables. Furthermore, as pointed out by Tao and Vu in [TV1, Section 8], it follows from their analysis that $\mathbb{P}(M_n \text{ is singular}) = o(1)$ for independent non degenerate entries, provided the following uniform non-degeneracy property holds:

(UND) There exists $\rho \in]0, \frac{1}{2}[$ such that for any $i, j = 1, \cdots, n$, $\mathbb{P}(a_{ij} > x_{ij}^+) > \rho$ and $\mathbb{P}(a_{ij} < x_{ij}^-) > \rho$ for some real numbers $x_{ij}^- < x_{ij}^+$.

The result proven in [BG] improves the estimate on $\mathbb{P}(M_n \text{ is singular})$ under the same non-degeneracy hypothesis.

Proposition 4.6. *[BG] Let M_n be an $n \times n$ matrix whose coefficients are independent random variables satisfying* **(UND)**. *Then $\mathbb{P}(M_n$ is singular$) = O(1/\sqrt{n})$.*

The approach of [BG] is to use a Bernoulli decomposition of arbitrary non degenerate random variables as developed in [AGKW].

Lemma 4.7. *[AGKW, BG] Let M_n be an $n \times n$ matrix whose coefficients are independent random variables satisfying* **(UND)**. *We can decompose the entries of the matrix M_n as follows: For all i, j, there exist two independent random variables w_{ij} and ϵ_{ij} and functions $f_{ij} :]0, 1[\to \mathbb{R}$ and $\delta_{ij} :]0, 1[\to]0, +\infty[$ such that*
1. ϵ_{ij} is a Bernoulli random variable with parameter $p_{ij} \in]0, 1[$;
2. w_{ij} has the uniform distribution in $]0, 1[$;
3. $a_{ij} = f_{ij}(w_{ij}) + \delta_{ij}(w_{ij})\epsilon_{ij}$.
Moreover, $p_{ij} \in]\rho, 1 - \rho[$ for all i, j.

This Bernoulli decomposition allows to extend results known for Bernoulli variables to the general case of independent non degenerate random variables. Proposition 4.6 is an illustration of it by extending Komlós's argument to independent random variables satisfying the Property **(UND)**. (It is however not clear whether results in [TV1, TV2], and in particular Halàsz-type arguments, could be extended in a similar way.) The approach of [BG] is the following:

1. Decompose the a_{ij} as in Lemma 4.7;

2. Since the w_{ij} and ϵ_{ij} are independent r.v., do the conditioning with respect to the variables w_{ij}, so that, given the w_{ij}'s, M_n becomes a sum of a constant matrix and of a random matrix with Bernoulli entries with probabilities $(1 - p_{ij}, p_{ij})$ and amplitudes $\delta_{ij}(w_{ij})$;

3. Estimate, with respect to the Bernoulli variables ϵ_{ij}, the probability that M_n is singular following the strategy of [Bo];

4. Take the expectation value with respect to the variables w_{ij}.

Bibliography

[AGKW] Aizenman M., Germinet F., Klein A., Warzel S., *On Bernoulli decompositions for random variables, concentration bounds and spectral localization*, Prob. Th. and Rel. Fields., **143**, 219-238 (2009).

[Ar] Araï A., *Ground state of the massless Nelson model without infrared cutoff in a Non-Fock representation*, Rev. Math. Phys, **13**, 1075-1094 (2001).

[A] Araki H., *On quasifree states of the canonical commutation relations (II)*, Publ. RIMS, Kyoto Univ., **7**, 121-152 (1971).

[AH] Araki H., Ho, T.G., *Asymptotic time evolution of a partitioned infinite two-sided isotropic XY-chain*, Proc. Steklov Inst. Math., **228**, 191-204 (2000).

[AW] Araki H., Wyss, W., *Representations of canonical anticommutation relations*, Helv. Phys. Acta, **37**, 136-159 (1964).

[AY] Araki H., Yamagami S., *On quasi-equivalence of quasifree states of the canonical commutation relations*, Publ. RIMS, Kyoto Univ., **18**, 283-338 (1982).

[Arn] Arnold L., Random Dynamical Systems, Springer, 2003.

[AJPP] Aschbacher, W., Jakšić, V., Pautrat, Y., Pillet, C.A., *Transport properties of quasi-free fermions*, J. Math. Phys., **48**, 032101 (2007).

[AP] Aschbacher, W., Pillet, C.-A., *Non-equilibrium steady states of the XY chain*, J. Stat. Phys. **112**, 1153-1175 (2003).

[AJ1] Attal S., Joye A., *Weak Coupling and Continuous Limits for Repeated Quantum Interaction*, J. Stat. Phys. **126**, 1241-1283 (2007).

[AJ2] Attal S., Joye A., *The Langevin Equation for a Quantum Heat Bath*, J. Func. Anal. **247**, 253-288 (2007).

[AJP] Attal S., Joye A., Pillet C.-A. (Eds.), Open Quantum Systems I-III, Lecture Notes in Mathematics, volumes 1880-1882, Springer Verlag, 2006.

[APa] Attal S., Pautrat Y., *From repeated to continuous quantum interactions*, Ann. Henri Poincaré, **7**, 59-104 (2006).

[ABGK] Avron, J.E., Bachmann, S., Graf, G.M. and Klich, I., *Fredholm Determinants and the Statistics of Charge Transport*, Comm. Math. Phys., **280**, 807-829 (2008).

[Az] Azoff E.A., *Borel Measurability in Linear Algebra*, Proc. Am. Math. Soc., **42** (2), 346-350 (1974).

[BFS] Bach V., Fröhlich J., Sigal I.M., *Return to equilibrium*, J. Math. Phys. **41**, vol 6, 3985-4060 (2000).

[Ba] Bayfield, J.E., Quantum evolution. An Introduction to Time-Dependent Quantum Mechanics. Wiley, New York (1999).

[BS] Beck A., Schwartz J.T., *A Vector-Valued Random Ergodic Theorem*, Proc. Am. Math. Soc., **8**, no.6, 1049-1059 (1957).

[Be] Berezin E. A., The method of second quantization, Academic Press, 1966.

[Bo] Bollobas B., Random graphs, Academic Press 1985.

[BR] Bratteli O., Robinson D.W., Operator Algebras and Quantum Statistical Mechanics, volumes 1 and 2, Texts and Monographs in Physics, Springer Verlag, (1996).

[BRH] Brune M., Raimond J.M., Haroche S., *Theory of the Rydberg-atom two-photon micromaser*, Phys. Rev. A, **35**, vol 1, 154-163 (1987).

[BDP] Bruneau L., De Bièvre S., Pillet C.-A., *Scattering induced current in a tight binding band*, to appear in J. Math. Phys., arXiv 1012.2657 (2010).

[BD1] Bruneau L., Dereziński J., *Pauli-Fierz Hamiltonians defined as quadratic forms*, Reports in Math. Phys., **54**, 169-199 (2004).

[BD2] Bruneau L., Dereziński J., *Bogoliubov Hamiltonians and one-parameter groups of Bogoliubov transformations*, J. Math. Phys., **48**, vol 2, 022101 (2007).

[BG] Bruneau L., Germinet F., *On the singularity of random matrices with independent entries*, Proc. Amer. Math. Soc., **137**, vol 3, 787-792 (2009).

[BJM1] Bruneau L., Joye A., Merkli M., *Asymptotics of repeated quantum interaction systems*, J. Func. Anal. **239**, 310-344 (2006).

[BJM2] Bruneau L., Joye A., Merkli M., *Infinite products of random matrices and repeated interaction quantum dynamics*, Ann. Inst. Henri Poincaré Probab. Stat., **46**, vol 2, 442-464 (2010).

[BJM3] Bruneau L., Joye A., Merkli M., *Random repeated interaction quantum systems*, Comm. Math. Phys., **284**, 553-581 (2008).

[BJM4] Bruneau L., Joye A., Merkli M., *Repeated and continuous interactions in open quantum systems*, Ann. Henri Poincaré, **10**, vol 7, 1251-1284 (2010).

[BP] Bruneau L., Pillet C.-A., *Thermal relaxation of a QED cavity*, J. Stat. Phys., **134**, vol 5-6, 1071-1095 (2009).

[Ca] Carleman, Les fonctions quasi-analytiques, Gauthier-Villars, Paris, 1926.

[CJM] Cornean, H.D., Jensen, A., Moldoveanu, V., *A rigorous proof of the Landauer-Büttiker formula*, J. Math. Phys., **46**, 042106 (2005).

[CDG] Cohen-Tannoudji, C., Dupont-Roc, J., Grinberg, G., Atom-Photon Interactions. Wiley, New York (1992).

[CNZ] Cornean, H.D., Neidhardt, H., Zagrebnov, V., *The effect of time dependent coupling on non-equilibrium steady states*, Ann. Henri Poincaré, **10**, vol 1, 61-93 (2009).

[dR] de Roeck, W., *Large deviation generating function for currents in the Pauli-Fierz model*, Rev. Math. Phys., **21**, 549-585 (2009).

[DGL] De Saporta B., Guivarc'h Y., LePage E., *On the multidimensional stochastic equation* $Y(n+1) = a(n)Y(n) + b(n)$, C.R.A.S., **339**(7), 499-502, (2004).

[Da1] Davies E.B. *Markovian master equations*, Comm. Math. Phys., **39**, 91-110 (1974).

[Da2] Davies E.B. *Markovian master equations II*, Math. Ann., **219**, 147-158 (1976).

[D] Dereziński J., *Van Hove Hamiltonians-Exactly solvable models of the infrared and ultraviolet problem*, Ann. Inst. H. Poincaré, **4**, 713-738 (2003).

[DF] Dereziński J., Früboes R., *Stationary van Hove limit*, J. Math. Phys, **46**, 063511 (2005).

[DG1] Dereziński J., Gérard C., *Asymptotic completeness in quantum field theory. Massive Pauli-Fierz Hamiltonians*, Rev. Math. Phys., **11**, 383-450 (1999).

[DG2] Dereziński J., Gérard C., *Scattering theory of infrared divergent Pauli-Fierz Hamiltonians*, Ann. Henri Poincaré, **5**, 523-577 (2004).

[DJ1] Dereziński J., Jakšić V., *Spectral theory of Pauli-Fierz operators*, J. Func. Anal., **180**, 243-327 (2001).

[DJ2] Dereziński J., Jakšić, V., *On the nature of Fermi Golden Rule for open quantum systems*, J. Stat. Phys., **116**, 411-423 (2004).

[Du] Dutra, S.M., Cavity Quantum Electrodynamics. Wiley, New York (2005).

[FJM] Filipowicz P., Javanainen J., Meystre P., *Theory of a microscopic maser*, Phys. Rev. A, **34**, vol 4, 3077-3087 (1986).

[F] Friedrichs K. O. Mathematical aspects of the quantum theory of field, New York, 1953.

[FMU] Fröhlich, J., Merkli, M., Ueltschi, D., *Dissipative transport: thermal contacts and tunneling junctions*, Ann. Henri Poincaré, **4**, 897 (2004).

[GGM] Georgescu V., Gérard C., Møller J.S., *Spectral Theory of Massless Pauli-Fierz Models*, Comm. Math. Phys., **249**, 29-78 (2004).

[Ge] Gérard C., *On the existence of ground states for massless Pauli-Fierz Hamiltonians*, Ann. Henri Poincaré, **1**, 443-459 (2000).

[GeHPS] Gérard C., Hiroshima F., Panati A., Suzuki A., *Infrared problem for the Nelson model on static space-times*, preprint arXiv 1004.5204 (2010).

[G-al] Gleyzes S., Kuhr S., Guerlin C., Bemu J., Deleglise S., Hoff U.B., Brune M., Raimond J.-M., Haroche S., *Quantum jumps of light recording the birth and death of a photon in a cavity*, Nature, **446**, 297-300 (2007).

[GLL] Griesemer M., Lieb E.H., Loss M., *Ground states in non-relativistic quantum electrodynamics*, Invent. Math., **145**, 557-595 (2001).

[G] Guivarc'h Y., *Limit Theorem for Random Walks and Products of Random Matrices*, in the Proceedings of the CIMPA-TIFR School on Probability measures on Groups, Recent Directions and Trends, sept. 2002, TIFR Mumbai.

[HVV] Hainzl C., Vougalter V., Vugalter S.A., *Enhanced binding in non-relativistic QED*, Comm. Math. Phys., **223**, 13-26 (2003).

[HS] Hiroshima F., Spohn H., *Ground state degeneracy of the Pauli-Fierz Hamiltonian including spin*, Adv. Theor. Math. Phys., **5**, 1091-1104 (2001).

[HP] Hudson R.L., Parthasaraty K.R. *Quantum Ito's formula and stochastic evolution*, Comm. Math. Phys. **93**, 301-323 (1984).

[IH] Ito K. R., Hiroshima F., *Local exponents and infinitesimal generators of canonical transformations on Boson Fock spaces*, Infin. Dimens. Anal. Quantum Probab. Relat. Top., **7**, 547-571 (2004).

[JOP1] Jakšić, V., Ogata, Y., Pillet, C.-A., *Linear response theory for thermally driven quantum open systems*, J. Stat. Phys., **123**, 547-569 (2006).

[JOP2] Jakšić, V., Ogata, Y., Pillet, C.-A., *The Green-Kubo formula and the Onsager reciprocity relations in quantum statistical mechanics*, Comm. Math. Phys., **265**, 721-738 (2006).

[JOP3] Jakšić, V., Ogata, Y., Pillet, C.-A., *The Green-Kubo formula for the spin-fermion system*, Comm. Math. Phys., **268**, 369-401 (2006).

[JOP4] Jakšić, V., Ogata, Y., Pillet, C.-A., *The Green-Kubo formula for locally interacting fermionic open systems*, Ann. Henri Poincaré, **8**, vol 6, 1013-1036 (2007).

[JPP] Jakšić, V., Pautrat, Y., Pillet, C.-A., *Central limit theorem for locally interacting Fermi gas*, Comm. Math. Phys., **285**, 175-217 (2009).

[JP1] Jakšić V., Pillet C.-A., *On a model for quantum friction. III. Ergodic properties of the spin-boson system*, Comm. Math. Phys., **178**, no. 3, 627-651 (1996).

[JP2] Jakšić V., Pillet C.-A., *Non-equilibrium steady states of finite quantum systems coupled to thermal reservoirs*, Comm. Math. Phys. **226**, 131-162 (2002).

[JP3] Jakšić, V., Pillet, C.-A., *A note on the entropy production formula*, Advances in differential equations and mathematical physics (Birmingham, AL, 2002), 175-180, Contemp. Math., **327**, Amer. Math. Soc., Providence, RI, 2003.

[KKoS] Kahn J., Komlós J., Szemerédi E., *On the probability that a random* ± 1 *matrix is singular*, J. Amer. Math. Soc., **8**, 223-240 (1995).

[KS] Kesten H., Spitzer F., *Convergence in Distribution of Products of Random Matrices*, Z. Wahrscheinlichkeitstheorie verw. Gebiete, **67**, 363-386 (1984).

[K] Kifer Y., Liu P.-D., Random Dynamics, in *Handbook of Dynamical Systems*, vol1B, Hasselblatt and Katok Edts, 379-499, North-Holland, 2006.

[Ko1] Komlós J., *On the determinant of* $(0, 1)$ *matrices*, Studia. Sci. Math. Hungar., **2**, 7-22 (1967).

[Ko2] Komlós J., *On the determinant of random matrices*, Studia. Sci. Math. Hungar., **3**, 387-399 (1968).

[MS] Matsui T., Shimada Y., *On quasifree representations of infinite dimensional symplectic group*, J. Func. Anal., **215**, 67-102 (2004).

[MMS] Merkli, M., Mück, M., Sigal, I.M., *Theory of non-equilibrium stationary states as a theory of resonances*, Ann. Henri Poincaré, **8**, 1539 (2007).

[MWM] Meschede D., Walther H., Müller G., *One-atom maser*, Phys. Rev. Lett., **54**, 551-554 (1993).

[Mö] Möller J.-S., *The polaron revisited*, Rev. Math. Phys., **18**, 485-517 (2006).

[Mu] Mukherjea A., Topics in Products of Random Matrices, TIFR, Mumbai, 2000.

[NPe] Nechita I., Pellegrini C., *Random repeated quantum interactions and random invariant states*, to appear in Prob. Th. and Rel. Fields.

[N] Nenciu, G., *Independent electron model for open quantum systems: Landauer-Büttiker formula and strict positivity of the entropy production*, J. Math. Phys., 48, 033302 (2007).

[OLBC] Olver, W.J., Lozier, D.W., Boisvert, R.F. and Clark, C.W.: NIST Handbook of Mathematical Functions. Cambridge University Press (2010).

[Pe1] Pellegrini C., *Existence, uniqueness and approximation of a stochastic Schrödinger equation: the Diffusive case*, Ann. of Proba., **36** (6), 2332-2353 (2008).

[Pe2] Pellegrini C., *Existence, uniqueness and approximation of the jump-type stochastic Schrödinger equation for two-level systems*, Stochastic process and their Applications, **120** (9), 1722-1747 (2010).

[P] Pillet C.-A., *Quantum Dynamical Systems*, in "Open Quantum Systems", Vol. I: The Hamiltonian Approach, S.Attal, A.Joye and C.A.Pillet Eds. Springer Lecture Notes in Mathematics, 1880, (2006), p. 107-182.

[RH] Raimond, J.-M., Haroche, S., *Monitoring the decoherence of mesoscopic quantum superpositions in a cavity*, Séminaire Poincaré, **2**, 25 (2005).

[RS4] Reed M., Simon B., Methods of Modern Mathematical Physics, vol 4. Academic Press, San Diego, (1978).

[Ro] Rosen L., *The $(\phi^{2n})_2$ Quantum Field Theory: High Order Estimates*, Comm. Pure Appl. Math., **24**, 417-457 (1971).

[R] Ruelle, D., *Natural nonequilibrium states in quantum statistical mechanics*, J. Stat. Phys., **98**, 57 (2000).

[Ru1] Ruijsenaars S.N.M., *On Bogoliubov transformations for systems of relativistic charged particles*, J. Math. Phys., **18**, 517-526 (1976).

[Ru2] Ruijsenaars S.N.M., *On Bogoliubov transforms II. The general case*, Ann. Phys., **116**, 105-134 (1978).

[Sch] Schrader, R. *Perron-Frobenius theory for positive maps on trace ideals*, Fields Inst. Commun., **30**, (2001).

[Se] Seneta E., Non-negative Matrices and Markov Chains, Springer series in Statistics, 2000.

[Sh] Shale D., *Linear symmetries of free boson fields*, Trans. Amer. Math. Soc., **103**, 149-167 (1962).

[Sp] SpohnH., *Ground state of a quantum particle coupled to a scalar Bose field*, Lett. Math. Phys., **44**, 9-16 (1998).

[TV1] Tao T., Vu V., *On Random ± 1 matrices: Singularity and Determinant*, Random Structures and Algorithms, **28**, 1-23 (2006).

[TV2] Tao T., Vu V., *On the singularity of random Bernoulli matrices*, J. Amer. Math. Soc., **20**, 603-628 (2007).

[VAS] Vogel, K., Akulin, V.M., Schleich, W.P., *Quantum State Engineering of the Radiation Field*, Phys. Rev. Lett., **71**, no.12, 1816–1819 (1993).

[WVHW] Weidinger M., Varcoe B.T.H., Heerlein R., Walther H., *Trapping states in micromaser*, Phys. Rev. Lett., **82**, 3795-3798 (1999).

[WBKM] Wellens, T., Buchleitner, A., Kümmerer, B., Maassen, H., *Quantum State Preparation via Asymptotic Completeness* Phys. Rev. Lett., **85**, no.16, 3361-3364 (2000).