Mathematical analysis of some open quantum systems : the repeated interaction systems

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Open Systems

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Goal: understand the asymptotic ($t \to +\infty$) behaviour of the system $\mathcal{S}$ (asymptotic state, thermodynamical properties).

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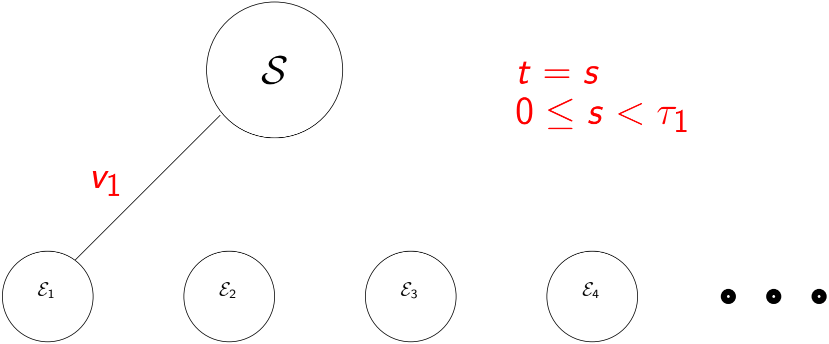
Both at the same time : Repeated Interaction Systems (start with work of Attal-Pautrat)

Repeated Interaction Systems (RIS)



$t = s$
$0 \leq s < \tau_1$

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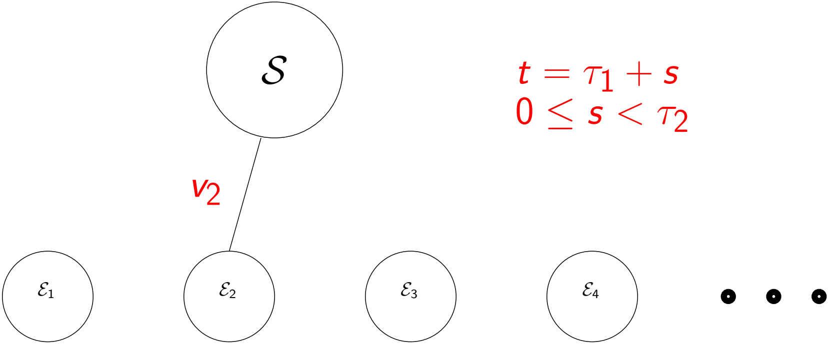

$$t = \tau_1 + s$$
$$0 \leq s < \tau_2$$

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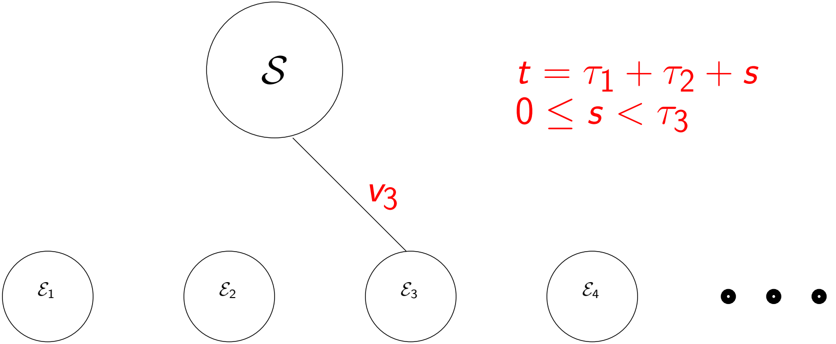
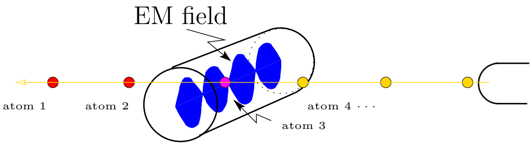


$t = \tau_1 + \tau_2 + s$

$0 \leq s < \tau_3$

A concrete example

Physics: One-atom maser (Walther et al '85, Haroche et al '92)



- $\mathcal{S}=$ one mode of the electromagnetic field in a cavity.
- $\mathcal{E}_k=$ k-th atom interacting with the field.
- $\mathcal{C}$: beam of atoms sent into the cavity.

Plan

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Hamiltonian description

A "small" system $\mathcal{S}$:

- Quantum system governed by some hamiltonian $h_{\mathcal{S}}$ acting on $\mathfrak{h}_{\mathcal{S}}$.

A chain $\mathcal{C}$ of quantum sub-systems $\mathcal{E}_k$ $(k = 1, 2, \ldots)$:

- $\mathcal{C} = \mathcal{E}_1 + \mathcal{E}_2 + \cdots$
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For $t \in [t_{n-1}, t_n[$, $t_n = \tau_1 + \cdots + \tau_n$:

- $\mathcal{S}$ interacts with $\mathcal{E}_n$,
- $\mathcal{E}_k$ evolves freely for $k \neq n$,

i.e. the full system is governed by

$$\tilde{h}_n = h_{\mathcal{S}} + h_{\mathcal{E}_n} + v_n + \sum_{k \neq n} h_{\mathcal{E}_k} = h_n + \sum_{k \neq n} h_{\mathcal{E}_k}.$$

The repeated interaction dynamics

Data:

1. Full Hamiltonian: $h_n = h_{\mathcal{S}} \otimes 1\!\!1_{\mathcal{E}_n} + 1\!\!1_{\mathcal{S}} \otimes h_{\mathcal{E}_n} + v_n$.
2. Initial state of $\mathcal{S}$: density matrix $\rho \in \mathcal{J}_1(\mathfrak{h}_{\mathcal{S}})$.
3. Initial state of $\mathcal{E}_n$: $\rho_{\mathcal{E}_n}$ (invariant state for the free dynamics of $\mathcal{E}_n$, e.g. Gibbs state at some inverse temperature β_n).

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After 0 interaction, the state of the total system is

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After 1 interaction, the state of the total system is

$$\rho^{\rm tot}(1) := \quad \mathrm{e}^{-i\tau_1 \tilde{h}_1} \left(\rho \otimes \bigotimes_{k\geq 1} \rho_{\mathcal{E}_k} \right) \mathrm{e}^{i\tau_1 \tilde{h}_1}$$

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After 2 interactions, the state of the total system is

$$\rho^{\mathrm{tot}}(2) := \mathrm{e}^{-i\tau_2\tilde{h}_2}\mathrm{e}^{-i\tau_1\tilde{h}_1}\left(\rho \otimes \bigotimes_{k\geq 1} \rho_{\mathcal{E}_k}\right)\mathrm{e}^{i\tau_1\tilde{h}_1}\mathrm{e}^{i\tau_2\tilde{h}_2}$$

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After n interactions, the state of the total system is

$$\rho^{\mathrm{tot}}(n) := \mathrm{e}^{-i\tau_n \tilde{h}_n} \cdots \mathrm{e}^{-i\tau_2 \tilde{h}_2} \mathrm{e}^{-i\tau_1 \tilde{h}_1} \left(\rho \otimes \bigotimes_{k\geq 1} \rho_{\mathcal{E}_k}\right) \mathrm{e}^{i\tau_1 \tilde{h}_1} \mathrm{e}^{i\tau_2 \tilde{h}_2} \cdots \mathrm{e}^{i\tau_n \tilde{h}_n}.$$

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It is the unique state of $\mathcal{S}$ such that

$$\forall A \in \mathcal{B}(\mathfrak{h}_{\mathcal{S}}), \quad \mathrm{Tr}\left(\rho^{\mathrm{tot}}(n)\ A \otimes 1\!\!1_{\mathcal{C}}\right) = \mathrm{Tr}_{\mathcal{S}}\left(\rho(n) A\right).$$

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If $\mathcal{S}$ is in the state ρ before the n-th interaction, after it it is in the state

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The "repeated interaction" structure induces a markovian behaviour:

$$\forall n, \quad \rho(n) = \mathcal{L}_n(\rho(n-1)).$$

$\Longrightarrow$ One has to understand $\mathcal{L}_n \circ \cdots \circ \mathcal{L}_1$ as $n \to \infty$.

Some questions about RIS

Large time behaviour:

- Existence of the limit $\lim_{n \to +\infty} \rho(n) = \rho_+$?
- Several situations : ideal (identical interactions, equilibrium), random (non-equilibrium).

Thermodynamical properties:

- Energy variation (external work, power delivered to the system)?
- In the non equilibrium case : fluxes?
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Concrete examples?

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Spectrum of a RDM

A key ingredient will be the spectral analysis of the RDM : existence of invariant state, spectral gap,... For example, in the ideal case $\rho(n) = \mathcal{L}_n \circ \cdots \circ \mathcal{L}_1(\rho) = \mathcal{L}^n(\rho)$.

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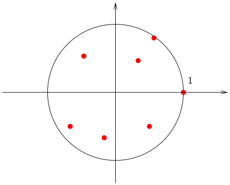
The $\mathcal{L}_n$ are completely positive and trace preserving maps on $\mathcal{J}_1(\mathfrak{h}_{\mathcal{S}})$.

Consequence:
$\mathrm{Spec}(\mathcal{L}_n) \subset \{z \in \mathbb{C} \,|\, |z| \leq 1\}$,
1 is in the spectrum.



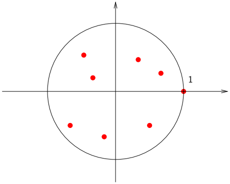
Ideal interactions

Take all the interactions identical, i.e. $\mathfrak{h}_{\mathcal{E}_k} \equiv \mathfrak{h}_{\mathcal{E}}$, $h_{\mathcal{E}_k} \equiv h_{\mathcal{E}}$, $\tau_k \equiv \tau$, $v_k \equiv v$, $\rho_{\mathcal{E}_k} \equiv \rho_{\mathcal{E}}$. Hence $\mathcal{L}_k \equiv \mathcal{L}$.

Ergodic assumption (E):

$\mathrm{Spec}(\mathcal{L}) \cap S^1 = \{1\}$,

1 is a simple eigenvalue.



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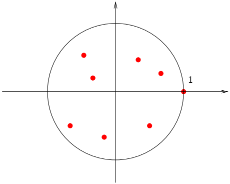

Theorem (B.-Joye-Merkli '06)

Let dim $\mathfrak{h}_{\mathcal{S}} < \infty$. If (E) is satisfied, there exist $C, \alpha > 0$ s.t. for any initial state ρ

$$\|\rho(n) - \rho_+\|_1 \leq C\mathrm{e}^{-\alpha n}, \qquad \forall n \in \mathbb{N},$$

where ρ_+ is the (unique) invariant state of $\mathcal{L}$.

Random interactions

We allow some fluctuations w.r.t. ideal situation (interaction time, temperature): $\mathcal{L} = \mathcal{L}(\omega_0)$ random variable with values in RDM (CP, trace preserving maps on $\mathfrak{h}_{\mathcal{S}}$) over a probability space $(\Omega_0, \mathcal{F}, \mathrm{p})$.

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Product of i.i.d. RDMs: $\Omega = \Omega_0^{\mathbb{N}^*}$, $\mathrm{d}\mathbb{P} = \prod_{n\geq 1} \mathrm{dp}$ and $\omega = (\omega_n)_{n\geq 1}$.

$\Rightarrow$ Understand $\rho(n,\omega) = (\mathcal{L}(\omega_n) \circ \cdots \circ \mathcal{L}(\omega_1))\,(\rho)$.

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$\Rightarrow$ Understand $\rho(n,\omega) = (\mathcal{L}(\omega_n) \circ \cdots \circ \mathcal{L}(\omega_1))\,(\rho)$.

Theorem (B.-Joye-Merkli '08)

Let $\dim \mathfrak{h}_{\mathcal{S}} < \infty$. *If* $\mathrm{p}(\mathcal{L}(\omega_0)$ *satisfies (E))* > 0, *then*

1. $\mathbb{E}(\mathcal{L})$ *satisfies (E).*
2. *For any* $\rho \in \mathcal{J}_1(\mathcal{H}_{\mathcal{S}})$, $\displaystyle \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \rho(n,\omega) = \rho_+$, *a.e.* $\omega \in \Omega$, *where* ρ_+ *is the unique invariant state of* $\mathbb{E}(\mathcal{L})$.

If moreover there exists ρ_+ *s.t.* $\mathcal{L}(\omega_0)(\rho_+) = \rho_+$ *for a.e.* ω_0, *i.e. there is a deterministic invariant state, then there exists* $\alpha > 0$ *s.t. for any* $\rho \in \mathcal{J}_1(\mathcal{H}_{\mathcal{S}})$ *and for a.e.* $\omega \in \Omega$, *there exists* $C(\omega) > 0$

$$\|\rho(n,\omega) - \rho_+\|_1 \leq C(\omega)\mathrm{e}^{-\alpha n}, \quad \forall n \in \mathbb{N}.$$

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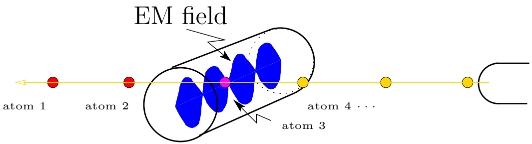
The one-aom maser

Physics: One-atom maser (Walther et al ’85, Haroche et al ’92)



- $\mathcal{S}$= one mode of the electromagnetic field in a cavity.
- $\mathcal{E}_k$= k-th atom interacting with the field.
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Mathematical model of the one-atom maser

1. The field in the cavity: (a harmonic oscillator)
$$\mathfrak{h}_{\mathcal{S}} = \Gamma_s(\mathbb{C}), \quad h_{\mathcal{S}} = \omega a^* a = \omega N.$$
Denote by $|n\rangle$ the eigenstates of $h_{\mathcal{S}}$: $h_{\mathcal{S}}|n\rangle = n\omega|n\rangle$.
2. The atoms: 2-level atoms.
$$\mathfrak{h}_{\mathcal{E}} = \mathbb{C}^2, \quad h_{\mathcal{E}} = \begin{pmatrix} 0 & 0 \\ 0 & \omega_0 \end{pmatrix}.$$
We denote by $|-\rangle$, $|+\rangle$ the eigenstates of $\mathcal{E}$.
If $b = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is the annihilation operator on $\mathbb{C}^2$ ($b|+\rangle = |-\rangle$ and $b|-\rangle = 0$), we have $h_{\mathcal{E}} = \omega_0 b^* b$.
3. The interaction: dipole interaction in the rotating-wave approximation, i.e. $v = \frac{\lambda}{2}(a \otimes b^* + a^* \otimes b)$.

This is the Jaynes-Cummings hamiltonian.

The repeated interaction dynamics.

1. Full Hamiltonian: $h = h_{\mathcal{S}} \otimes \mathbb{1}_{\mathcal{E}} + \mathbb{1}_{\mathcal{S}} \otimes h_{\mathcal{E}} + v$.
2. Initial state of $\mathcal{S}$: density matrix $\rho \in \mathcal{J}_1(\mathfrak{h}_{\mathcal{S}})$.
3. Initial state of $\mathcal{E}$: $\rho_\beta =$ equilibrium state at temperature β^{-1}, i.e. $\rho_\beta = \frac{\mathrm{e}^{-\beta h_{\mathcal{E}}}}{\mathrm{Tr}(\mathrm{e}^{-\beta h_{\mathcal{E}}})}$.

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We are at equilibrium : $\rho(n) = \mathcal{L}^n(\rho)$.

Conclusion: we have to understand the spectrum of $\mathcal{L}$.

Main difficulty: Perturbation theory doesn't work.

When $\lambda = 0$, $\mathcal{L}(\rho) = \mathrm{e}^{-i\tau h_\mathcal{S}}\rho\,\mathrm{e}^{i\tau h_\mathcal{S}}$. Hence $\mathrm{sp}(\mathcal{L}) = \{\mathrm{e}^{i\omega\tau(n-m)},\ n, m \in \mathbb{N}\}$: pure point spectrum (possibly dense in S^1), but all the eigenvalues, and in particular 1, are infinitely degenerate!

Jaynes-Cummings Hamiltonian and Rabi oscillations

If there are n photons in the cavity, the probability for the atom to make a transition $|-\rangle \to |+\rangle$ is a periodic function of time

$$P(t) = \left|\langle n-1, +|\, \mathrm{e}^{-ith} \,|n, -\rangle\right|^2 = \left(1 - \frac{\Delta^2}{\nu_n^2}\right) \sin^2\left(\frac{\nu_n t}{2}\right),$$

with frequency

$$\nu_n := \sqrt{\lambda^2 n + (\omega - \omega_0)^2} = \sqrt{\lambda^2 n + \Delta^2}.$$

($\lambda = $ 1-photon Rabi frequency in a cavity where $\Delta = 0$).

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Conclusion: If the field is in state $|n\rangle$ before an interaction and τ is a multiple of the Rabi period $T_n := \frac{2\pi}{\nu_n}$, after this interaction it can not be in state $|n-1\rangle$: there is a decoupling between the $n-1$ and n photon states.

Ergodicity

$n > 0$ is called a Rabi resonance if $\exists k \in \mathbb{N},\ \tau = kT_n$.

$R =$ set of Rabi resonances. The cavity splits into independant "sectors" each time there is a resonance.

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Proposition (B.-Pillet '09)

If $R = \emptyset$, 1 is the only eigenvalue of $\mathcal{L}$ on S^1 and it is simple. The invariant state is $\rho_{\mathcal{S},\beta^*}$, the Gibbs state of $\mathcal{S}$ at inverse temp. $\beta^* = \frac{\omega_0}{\omega}\beta$.

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Remarks:
1) Numerically it seems that $\rho_{\mathcal{S},\beta^*}$ is not only ergodic but also mixing.
2) 3 possible situations R is empty, a singlet or infinite. Generically: R is empty = no resonance. If $R \neq \emptyset$ the multiplicity of 1 increases (one invariant state per sector).

The tight binding model

- $\mathcal{S}=$ one electron in the tight binding approximation + constant electric field, i.e.

$$\mathfrak{h}_{\mathcal{S}}=\ell^2(\mathbb{Z}) \quad \text{and} \quad h_{\mathcal{S}}=-\Delta-FX.$$

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Bloch oscillations prevent a current from being set up.

Idea: contact with a thermal environment will lead to a steady current (via scattering mechanisms).

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$$\mathfrak{h}_{\mathcal{S}} = \ell^2(\mathbb{Z}) \quad \text{and} \quad h_{\mathcal{S}} = -\Delta - FX.$$

  Bloch oscillations prevent a current from being set up.

  Idea: contact with a thermal environment will lead to a steady current (via scattering mechanisms).

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- $\mathcal{E}$ = 2-level systems (E = Bohr frequency).
- Let $T = \sum_{k\in\mathbb{Z}} |k+1\rangle\langle k| = \mathrm{e}^{-iP}$.

  $v = \lambda(T \otimes b^* + T^* \otimes b)$. (If $F > 0$, T acts as an annihilation operator.)

RI dynamics of the tight binding model

Questions: transport properties of the electron, e.g.

$$\frac{\mathrm{Tr}(X\rho(n))}{n\tau} \overset{?}{\to} v, \quad \mathrm{Tr}((X - vn\tau)^2\rho(n)) \sim ?$$

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Fact: The dynamics induced by the RDM $\mathcal{L}$ corresponds to

Free dynamics of $\mathcal{S}$ $+$ random walk

More precisely $\mathrm{e}^{i\tau h_{\mathcal{S}}}\mathcal{L}(\rho)\mathrm{e}^{-i\tau h_{\mathcal{S}}} = p_- T^{-1}\rho T + p_0\,\rho + p_+ T\rho T^{-1}$, where $p_- + p_0 + p_+ = 1$ are explicit.

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Assumptions: $F > 0$, $\lambda \neq 0$ and $\omega\tau \notin 2\pi\mathbb{Z}$ (so that $p_0 \neq 1$).

Drift and diffusion

Theorem (B.-De Bièvre-Pillet '11)

If $\mathrm{Tr}\left(X^2\rho\right) < +\infty$, *then*

$$\lim_{n\to\infty} \frac{\mathrm{Tr}(X\rho(n))}{n\tau} = v, \quad \lim_{n\to\infty} \frac{\mathrm{Tr}((X - vn\tau)^2\rho(n))}{n\tau} = 2D,$$

where v and D are explicit.

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where v *and* D *are explicit.*

Remark: One actually proves the following CLT : for any $f \in C_{\mathrm{b}}(\mathbb{R})$,

$$\lim_{n\to\infty} \mathrm{Tr}\left(f\left(\frac{X - vn\tau}{\sqrt{2Dn\tau}}\right)\rho(n)\right) = \int f(x)\,\mathrm{e}^{-x^2/2}\,\frac{\mathrm{d}x}{\sqrt{2\pi}},$$

as well as a Large Deviation Principle.

Plan

Energy variation

The total Hamiltonian is time-dependent $\Rightarrow$ the total energy is usually not conserved.
During the n-th interaction the energy is constant, formally given by

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When one switches from interaction n to interaction $n+1$, there is an energy jump (external work):

$$\begin{aligned}\delta W(n) &:= \mathrm{Tr}\big(\rho^{\mathrm{tot}}(n)\times(h_{n+1}-h_n)\big) = \mathrm{Tr}\big(\rho^{\mathrm{tot}}(n)\times(v_{n+1}-v_n)\big)\\ &= \mathrm{Tr}_{\mathcal{S},\mathcal{E}_{n+1}}\left[\rho(n)\otimes\rho_{\mathcal{E}_{n+1}}\ v_{n+1}\right]\\ &\qquad -\mathrm{Tr}_{\mathcal{S},\mathcal{E}_n}\left[\rho(n-1)\otimes\rho_{\mathcal{E}_n}\ \mathrm{e}^{\mathrm{i}\tau_n h_n} v_n \mathrm{e}^{-\mathrm{i}\tau_n h_n}\right].\end{aligned}$$

Energy variation

In the ideal case, one easily gets

Proposition (B.-Joye-Merkli '06)

If Assumption (E) is satisfied,

$$\Delta W := \lim_{N\to\infty} \frac{1}{N\tau} \sum_{n=1}^{N} \delta W(n) = \frac{1}{\tau} \mathrm{Tr}_{\mathcal{S},\mathcal{E}} \left(\rho_+ \otimes \rho_{\mathcal{E}} \left(v - \mathrm{e}^{i\tau h} v \mathrm{e}^{-i\tau h}\right)\right).$$

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In the random case we have,

Proposition (B.-Joye-Merkli '08)

If $\mathrm{p}(\mathcal{L}(\omega_0)$ *satisfies (E))* > 0, *then*

$$\Delta W := \lim_{N\to\infty} \frac{1}{t_N(\omega)} \sum_{n=1}^{N} \delta W(n) = \frac{\mathbb{E}\left(\mathrm{Tr}_{\mathcal{S},\mathcal{E}} \left(\rho_+ \otimes \rho_{\mathcal{E}} \left(v - \mathrm{e}^{i\tau h} v \mathrm{e}^{-i\tau h}\right)\right)\right)}{\mathbb{E}(\tau)},$$

where ρ_+ *is the unique invariant state of* $\mathbb{E}(\mathcal{L})$.

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We assume that the $\rho_{\mathcal{E}_n}$ are Gibbs states at inverse temperature β_n.

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Relative entropy: $\mathrm{Ent}(\rho|\rho_0) = \mathrm{Tr}(\rho \log \rho - \rho \log \rho_0) \geq 0$.

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Theorem (B.-Joye-Merkli '06 -'08)

1) Ideal case: if (E) is satisfied, then

$$\Delta S := \lim_{n\to\infty} \frac{\mathrm{Ent}(\rho^{\mathrm{tot}}(n)|\rho_0) - \mathrm{Ent}(\rho^{\mathrm{tot}}(0)|\rho_0)}{n\tau} = \beta \Delta W.$$

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2) Random case: if $\mathrm{p}(\mathcal{L}(\omega_0)$ *satisfies (E))* > 0*, then*

$$\begin{aligned} \Delta S &:= \lim_{n\to\infty} \frac{\mathrm{Ent}(\rho^{\mathrm{tot}}(n,\omega)|\rho_0) - \mathrm{Ent}(\rho^{\mathrm{tot}}(0,\omega)|\rho_0)}{t_n(\omega)} \\ &= \frac{\mathbb{E}\left(\beta\, \mathrm{Tr}_{\mathcal{S},\mathcal{E}}\left(\rho_+ \otimes \rho_{\mathcal{E}}\left(v - \mathrm{e}^{i\tau h} v \mathrm{e}^{-i\tau h}\right)\right)\right)}{\mathbb{E}(\tau)}. \end{aligned}$$

In particular, if β *is not random we still have* $\Delta S = \beta \Delta W$.

Some remarks and perspectives

- RIS have also been studied in various limiting regimes: weak coupling, continuous interactions,... (Attal-Pautrat, Attal-Joye, Pellegrini).
- We can also add an extra reservoir : leaky RIS (B.-Joye-Merkli '10).
- Linear response theory and fluctuation symmetries in RIS
- Study the correlations in the chain after the interaction
- One-atom maser + losses (important to allow initially excited 2-level atoms)
- In the one-atom maser, the relaxation is slow (not exponential) due to metastable states with arbitrarily long lifetime. What about random interaction times? Does it enhance the relaxation speed?
- Tight-binding model with scattering in momentum.