Repeated Interaction Quantum Systems

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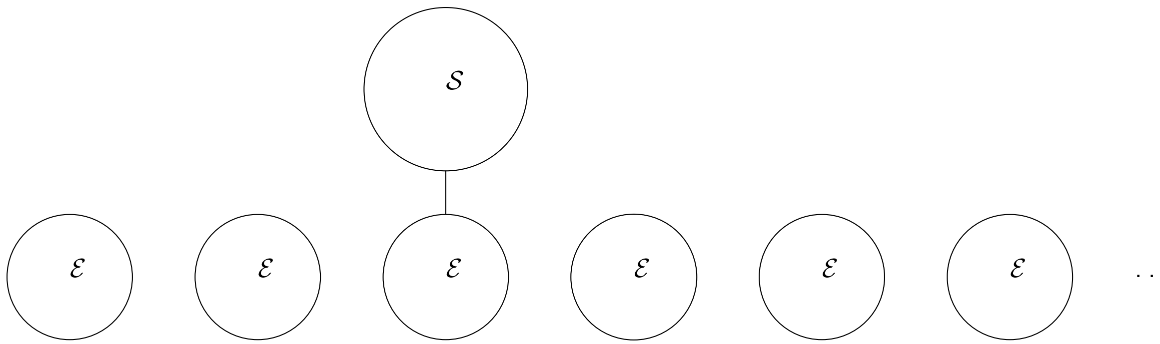
INTRODUCTION

General idea: a small system $\mathcal{S}$ interacts in a successive way with the elements $\mathcal{E}$ of a chain $\mathcal{C}$ (each during a time τ).



Some motivations:

- Quantum optics: models of radiation-matter coupling (e.g. a beam $\mathcal{C}$ of two level atoms $\mathcal{E}$ interacting with a single mode $\mathcal{S}$ of a field in a cavity).
- In some regimes of the interaction time τ and the coupling strenght λ: model a field of quantum noises interacting with $\mathcal{S}$ (Attal-Pautrat), or a Markovian effective evolution of $\mathcal{S}$ (Attal-Joye).

THE R.I. MODEL

1) The system $\mathcal{S}$ and the element $\mathcal{E}$

$\mathcal{S}$ (resp. $\mathcal{E}$) is a W^*-dynamical system $(\mathfrak{M}_\#, \tau_\#^t)$ acting on $\mathcal{H}_\#$ ($\# = \mathcal{S}, \mathcal{E}$), where dim $\mathcal{H}_\mathcal{S} < +\infty$.

$\Omega_\# \in \mathcal{H}_\#$ is a cyclic and separating vector for $\mathfrak{M}_\#$ representing a $\tau_\#^t$-invariant state (e.g. some KMS state).

$L_\#$ is the standard Liouvillean, i.e. the unique s.a. operator on $\mathcal{H}_\#$ such that

$$\forall A \in \mathfrak{M}_\#, \quad \tau_\#^t(A) = \mathrm{e}^{\mathrm{i}tL_\#} A \mathrm{e}^{-\mathrm{i}tL_\#}, \qquad \text{and} \qquad L_\#\Omega_\# = 0.$$

Example: 2-level system with hamiltonian $h = \begin{pmatrix} 0 & 0 \\ 0 & E \end{pmatrix}$, and reference state is the equilibrium state at inverse temperature β, i.e. $\omega_\beta(A) = Z_\beta^{-1}\mathrm{Tr}(\mathrm{e}^{-\beta h}A)$.

$\mathfrak{M} = M_2(\mathbb{C}) \otimes \mathbb{1}$ acting on $\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2$, $L = h \otimes \mathbb{1} - \mathbb{1} \otimes h$,

$\Omega = \frac{1}{\sqrt{1+\mathrm{e}^{-\beta E}}}(\phi_1 \otimes \phi_1 + \mathrm{e}^{-\beta E/2}\phi_2 \otimes \phi_2)$.

2) The chain $\mathcal{C}$

$\mathcal{H}_\mathcal{C} := \otimes_{m\geq 1}\mathcal{H}_\mathcal{E}$ w.r.t. $\Omega_\mathcal{E}$, i.e. the completion of

$$Span\{\otimes_{m\geq 1}\psi_m|\psi_m \in \mathcal{H}_\mathcal{E}, \psi_m = \Omega_\mathcal{E} \text{ for } m > M\}$$

with $\langle \otimes_m\psi_m| \otimes_m \phi_m\rangle_\mathcal{C} := \prod_m \langle\psi_m|\phi_m\rangle_\mathcal{E}$.

$\mathfrak{M}_\mathcal{C} := \otimes_{m\geq 1}\mathfrak{M}_\mathcal{E}$, i.e. the weak closure of

$$Span\{\otimes_{m\geq 1}A_m|A_m \in \mathfrak{M}_\mathcal{E}, A_m = \mathbb{1}_\mathcal{E} \text{ for } m > M\}.$$

3) The interaction $\mathcal{S}$-$\mathcal{E}$

It is specified by a selfadjoint operator $V \in \mathfrak{M}_\mathcal{S} \otimes \mathfrak{M}_\mathcal{E}$. The interacting dynamics between $\mathcal{S}$ and $\mathcal{E}$ is the automorphism group $\mathrm{e}^{\mathrm{i}tL} \cdot \mathrm{e}^{-\mathrm{i}tL}$ of $\mathfrak{M}_\mathcal{S} \otimes \mathfrak{M}_\mathcal{E}$ where

$$L := L_\mathcal{S} \otimes \mathbb{1} + \mathbb{1} \otimes L_\mathcal{E} + V = L_0 + V.$$

4) The RI dynamics

For $m \geq 1$, let $\tilde{L}_m := L_m + \sum_{k \neq m} L_{\mathcal{E},k}$, as an operator on $\mathcal{H} := \mathcal{H}_{\mathcal{S}} \otimes \mathcal{H}_{\mathcal{C}}$.
It implements the dynamics when $\mathcal{S}$ interacts with the m^{th} element.

For $t \in \mathbb{R}^+$, $t := m(t)\tau + s(t)$ with $s(t) \in [0, \tau[$, and $A \in \mathfrak{M} := \mathfrak{M}_{\mathcal{S}} \otimes \mathfrak{M}_{\mathcal{C}}$,

$$\alpha_{\mathrm{RI}}^t(A) := \mathrm{e}^{\mathrm{i}\tau \tilde{L}_1} \cdots \mathrm{e}^{\mathrm{i}\tau \tilde{L}_{m(t)}} \mathrm{e}^{\mathrm{i}s(t)\tilde{L}_{m(t)+1}} A \mathrm{e}^{-\mathrm{i}s(t)\tilde{L}_{m(t)+1}} \mathrm{e}^{-\mathrm{i}\tau \tilde{L}_{m(t)}} \cdots \mathrm{e}^{-\mathrm{i}\tau \tilde{L}_1}.$$

5) Instantaneous observables

If $A \in \mathfrak{M}_{\mathcal{E}}$ and $m \geq 1$, $\mathfrak{M}_{\mathcal{C}} \ni \theta_m(A) := \underbrace{\mathbb{1} \otimes \cdots \otimes \mathbb{1}}_{m-1} \otimes A \otimes \mathbb{1} \otimes \cdots$

An instantaneous observable is of the form $A_{\mathcal{S}} \otimes \theta_{m(t)+1}(A_{\mathcal{E}})$.

When $A_{\mathcal{E}} = \mathbb{1}$, it is simply an observable on the small system $\mathcal{S}$.

GOAL: Given an initial state (normal) ω, understand the large time behaviour of

$$E(t, \omega) := \omega\left(\alpha_{\mathrm{RI}}^t(A_{\mathcal{S}} \otimes \theta_{m(t)+1}(A_{\mathcal{E}}))\right).$$

THE REDUCED DYNAMICS

Idea: during an interaction $\mathcal{S}$ feels an effective dynamics.

$P := \mathbb{1}_{\mathcal{S}} \otimes |\Omega_{\mathcal{C}}\rangle\langle\Omega_{\mathcal{C}}|$. We identify $\mathcal{H}_{\mathcal{S}}$ and $P\mathcal{H}$.

Definition: $M : A\Omega_{\mathcal{S}} \mapsto P\alpha_{\mathrm{RI}}^{\tau}(A \otimes \mathbb{1})\Omega_{\mathcal{S}} \otimes \Omega_{\mathcal{C}}$ is the reduced dynamics operator.

Remark: $M\Omega_{\mathcal{S}} = \Omega_{\mathcal{S}}$.

To analyse M, the main tool is the C-Liouvillean.

<u>C-LIOUVILLEAN</u> (Jaksic and Pillet: spectral approach to NESS)

J, Δ : modular data associated to $(\mathfrak{M}_{\mathcal{S}} \otimes \mathfrak{M}_{\mathcal{E}},\, \Omega_{\mathcal{S}} \otimes \Omega_{\mathcal{E}})$, i.e.

$$J\Delta^{1/2}A(\Omega_{\mathcal{S}} \otimes \Omega_{\mathcal{E}}) = A^*(\Omega_{\mathcal{S}} \otimes \Omega_{\mathcal{E}}), \quad \forall A \in \mathfrak{M}_{\mathcal{S}} \otimes \mathfrak{M}_{\mathcal{E}}.$$

They satisfy $J\Delta^{1/2} = \Delta^{-1/2}J$ and $J(\mathfrak{M}_{\mathcal{S}} \otimes \mathfrak{M}_{\mathcal{E}})J = (\mathfrak{M}_{\mathcal{S}} \otimes \mathfrak{M}_{\mathcal{E}})'$.

(H1)	$\Delta^{1/2} V \Delta^{-1/2} \in \mathfrak{M}_{\mathcal{S}} \otimes \mathfrak{M}_{\mathcal{E}}$.

Definition: The C-Liouvillean is the (non selfadjoint) operator

$$K := L - J\Delta^{1/2} V \Delta^{-1/2} J = L_0 + V - J\Delta^{1/2} V \Delta^{-1/2} J.$$

It satisfies: $1)\quad \mathrm{e}^{\mathrm{i}tK} A \mathrm{e}^{-\mathrm{i}tK} = \mathrm{e}^{\mathrm{i}tL} A \mathrm{e}^{-\mathrm{i}tL}, \quad \forall A \in \mathfrak{M}_{\mathcal{S}} \otimes \mathfrak{M}_{\mathcal{E}}$.

$2)\quad K(\Omega_{\mathcal{S}} \otimes \Omega_{\mathcal{E}}) = 0.$

We immediately get $M = P\mathrm{e}^{\mathrm{i}\tau K} P$.

PROP: $\exists C > 0$ s.t. $\|M^n\|_{\mathcal{B}(\mathcal{H}_{\mathcal{S}})} \leq C, \forall n \in \mathbb{N}$.

CORO: sp$(M) \subset \{z \in \mathbb{C}, |z| \leq 1\}$ and sp$(M) \cap S^1$ consists in semisimple eigenvalues.

Remark: When $V = 0$, $M \equiv \mathrm{e}^{\mathrm{i}\tau L_{\mathcal{S}}}$.

(H2)	sp$(M) \cap S^1 = \{1\}$ and it is non degenerate.

If $\Omega_{\mathcal{S}}^*$ is the unique eigenvector of M^* s.t. $\langle \Omega_{\mathcal{S}}^* | \Omega_{\mathcal{S}} \rangle = 1$ then

$$\lim_{n \to \infty} M^n = |\Omega_{\mathcal{S}}\rangle\langle \Omega_{\mathcal{S}}^*|.$$

THE ASYMPTOTIC STATE

Definition: $\omega_+(\cdot) := \langle \Omega_{\mathcal{S}}^* | \cdot \Omega_{\mathcal{S}} \rangle$.

THEOREM 1 Suppose (H1) and (H2) hold. Then, there exists $\gamma > 0$ such that for any normal state ω, $A_{\mathcal{S}} \in \mathfrak{M}_{\mathcal{S}}$ and $A_{\mathcal{E}} \in \mathfrak{M}_{\mathcal{E}}$, $\exists\, C > 0$ s.t.

$$|E(t,\omega) - E_+(t)| \leq C \mathrm{e}^{-\gamma t}, \qquad \forall t \geq 0$$

where $E_+(t)$ is the τ-periodic function

$$E_+(t) := \omega_+(P \mathrm{e}^{\mathrm{i}s(t)L}(A_{\mathcal{S}} \otimes A_{\mathcal{E}})\mathrm{e}^{-\mathrm{i}s(t)L} P).$$

In particular, $\quad |E(n\tau,\omega) - \omega_+(A_{\mathcal{S}})\langle \Omega_{\mathcal{E}} | A_{\mathcal{E}} \Omega_{\mathcal{E}} \rangle| \leq C \mathrm{e}^{-\gamma\tau n}$.

Some remarks:

$1)$ dim $\mathcal{H}_{\mathcal{S}} < +\infty \Rightarrow$ on $\mathcal{S}$ any initial state is normal.

$2)$ ω_+ does not depend on the reference state $\Omega_{\mathcal{S}}$.

$3)$ What happens if (H2) is not satisfied?

$a)$ if sp$(M) \cap S^1 \neq \{1\}$, then one has

$$\left| \frac{1}{t} \sum_{m=0}^{m(t)} E(m\tau + s(t), \omega) - \frac{E_+(t)}{\tau} \right| \leq C t^{-1}.$$

$b)$ if 1 is degenerate, then there is a τ-periodic function $E_\infty(t, \omega)$ such that $E(t, \omega) \sim E_\infty(t, \omega)$.

EXAMPLE: SPIN-SPIN

Description of $\mathcal{S}$ and $\mathcal{E}$

$\mathcal{S}$ and $\mathcal{E}$ are $2-$level systems with hamiltonian $h_{\#} = \begin{pmatrix} 0 & 0 \\ 0 & E_{\#} \end{pmatrix}$.

The reference state of $\mathcal{S}$ is the tracial state (for convenience) and the one of $\mathcal{E}$ is the equilibrium state at inverse temperature β.

The interaction

Let $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \in \mathbb{C}$. We define

$$V := \begin{pmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{pmatrix} \otimes \mathbb{1}_{\mathbb{C}^2} \otimes \underbrace{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}_{\text{``creation'' op.}} \otimes \mathbb{1}_{\mathbb{C}^2}.$$

The interacting Liouvillean is then $L_\lambda := L_{\mathcal{S}} \otimes \mathbb{1} + \mathbb{1} \otimes L_{\mathcal{E}} + \lambda V$.

We denote by M_λ the corresponding reduced dynamics.

Remark: $M_0 = \mathrm{e}^{\mathrm{i}\tau L_{\mathcal{S}}} \Rightarrow \mathrm{sp}(M_0) = \{1, \mathrm{e}^{2\mathrm{i}\tau E_{\mathcal{S}}}, \mathrm{e}^{-2\mathrm{i}\tau E_{\mathcal{S}}}\}$ and 1 is twice degenerate.

(Assu 1)	$\mathbf{b} \neq 0$ and $\tau(E_{\mathcal{E}} - E_{\mathcal{S}}) \notin 2\pi\mathbb{Z}$
(Assu 2)	$\mathbf{c} \neq 0$ and $\tau(E_{\mathcal{E}} + E_{\mathcal{S}}) \notin 2\pi\mathbb{Z}$

THEOREM Suppose $\tau E_{\mathcal{S}} \notin \pi\mathbb{Z}$ and (Assu 1) or (Assu 2) holds. Then for any $0 < |\lambda| < \Lambda_0$, M_λ satisfies (H2). The asymptotic state is

$$\omega_{+,\lambda}(A) = \frac{1}{\alpha_1 + \alpha_2} \langle \alpha_1 \psi_1 \otimes \psi_1 + \alpha_2 \psi_2 \otimes \psi_2, A(\psi_1 \otimes \psi_1 + \psi_2 \otimes \psi_2) \rangle + O(\lambda^2),$$

where

$$\alpha_1 := |\mathbf{b}|^2 \mathrm{sinc}^2\left(\frac{\tau(E_{\mathcal{E}} - E_{\mathcal{S}})}{2}\right) + \mathrm{e}^{-\beta E_{\mathcal{E}}} |\mathbf{c}|^2 \mathrm{sinc}^2\left(\frac{\tau(E_{\mathcal{E}} + E_{\mathcal{S}})}{2}\right),$$

$$\alpha_2 := \mathrm{e}^{-\beta E_{\mathcal{E}}} |\mathbf{b}|^2 \mathrm{sinc}^2\left(\frac{\tau(E_{\mathcal{E}} - E_{\mathcal{S}})}{2}\right) + |\mathbf{c}|^2 \mathrm{sinc}^2\left(\frac{\tau(E_{\mathcal{E}} + E_{\mathcal{S}})}{2}\right).$$

ENERGY AND ENTROPY

Energy variation

Formally, the energy at time t is $\alpha_{\mathrm{RI}}^{t}(\tilde{L}_{m(t)+1})$ (usually not well defined).

However, energy variation makes sense:

- $\alpha_{\mathrm{RI}}^{m\tau+s}(\tilde{L}_{m+1}) - \alpha_{\mathrm{RI}}^{m\tau}(\tilde{L}_{m+1}) = 0, \quad \forall s \in [0,\tau[,$
- there is an energy jump as time passes $m\tau$

$$j(m) := \alpha_{\mathrm{RI}}^{m\tau+s}(\tilde{L}_{m+1}) - \alpha_{\mathrm{RI}}^{(m-1)\tau+s'}(\tilde{L}_{m}) = \alpha_{\mathrm{RI}}^{m\tau}(V_{m+1} - V_m),$$

where $V_m := (\mathbb{1}_{\mathcal{S}} \otimes \theta_m)(V)$.

THEOREM 1 implies $|\omega(j(m)) - \omega_+(j_+)| \leq C\mathrm{e}^{-\gamma m}$, where

$$j_+ := P(V - \mathrm{e}^{\mathrm{i}\tau L} V \mathrm{e}^{-\mathrm{i}\tau L})P.$$

Definition: $\mathrm{d}E_+ := \frac{1}{\tau}\omega_+(j_+)$ is the asymptotic energy variation.

The total energy variation is then

$$\Delta E(t) := \sum_{m=1}^{m(t)} j(m).$$

PROP: If ω is a normal state,

$$\left| \frac{\omega(\Delta E(t))}{t} - \mathrm{d}E_+ \right| \leq \frac{C}{t}.$$

Entropy production

We assume $\Omega_\#$ is $(\tau_\#^t, \beta_\#)$-KMS. The reference state ω_0 is associated to $\Omega_{\mathcal{S}} \otimes \Omega_{\mathcal{E}} \otimes \cdots$

$\mathrm{Ent}(\omega|\omega_0)$ is the relative entropy of ω wrt ω_0 (generalisation of $\mathrm{Tr}(\rho(\log\rho - \log\rho_0))$). It is always positive.

PROP Suppose ω is a normal state with $\mathrm{Ent}(\omega|\omega_0) < +\infty$. Then,

$$(i) \quad \lim_{t\to+\infty} \left[\mathrm{Ent}(\omega \circ \alpha_{\mathrm{RI}}^{t+\tau}|\omega_0) - \mathrm{Ent}(\omega \circ \alpha_{\mathrm{RI}}^{t}|\omega_0)\right] = \beta_{\mathcal{E}}\omega_+(j_+),$$

$$(ii) \quad \left|\frac{\mathrm{Ent}(\omega \circ \alpha_{\mathrm{RI}}^{t}|\omega_0)}{t} - \frac{\beta_{\mathcal{E}}\omega_+(j_+)}{\tau}\right| \le \frac{C}{t}.$$

Corollary The asymptotic energy variation $\mathrm{d}E_+$ is positive.

Definition: $\mathrm{d}S_+ := \frac{\beta_{\mathcal{E}}\omega_+(j_+)}{\tau}$ is the asymptotic entropy production.

THEOREM 2 (*Asymptotic 2^{nd} law*): $\mathrm{d}E_+ = T_{\mathcal{E}}\mathrm{d}S_+$ where $T_{\mathcal{E}} := \beta_{\mathcal{E}}^{-1}$.

In the example, if both (Assu 1) and (Assu 2) hold, then the asymptotic entropy production is strictly positive.