Thermal relaxation in a quantum cavity

(Collaboration with C.A. Pillet)

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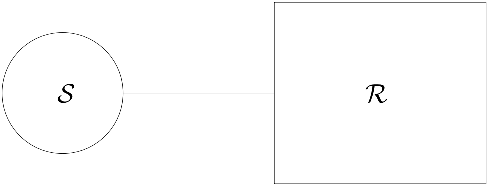
Mathematical models of Quantum Field Theory, Ecole Polytechnique, 07-08 December 2010

Open Systems

A "small" (or confined) system $\mathcal{S}$ interacts with an environment $\mathcal{R}$.



Goal: understand the asymptotic ($t \to +\infty$) behaviour of the system $\mathcal{S}$ (asymptotic state, thermodynamical properties).

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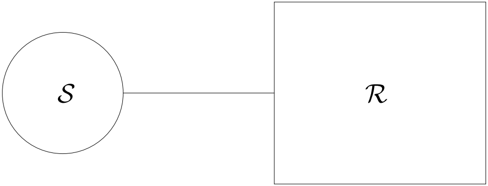

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2 approaches: Hamiltonian / Markovian

- Hamiltonian: full description, spectral analysis, scattering theory. Restrictions: perturbative results, $\mathcal{S}$ finite dimensional.
- Markovian: effective description of $\mathcal{S}$, obtained by weak-coupling type limits or if $\mathcal{S}$ undergoes stochastic forces (Langevin equation).

Repeated Interaction Quantum Systems (RIQS)

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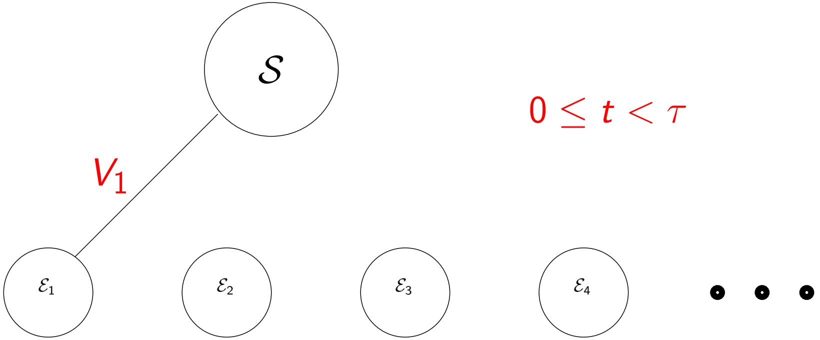

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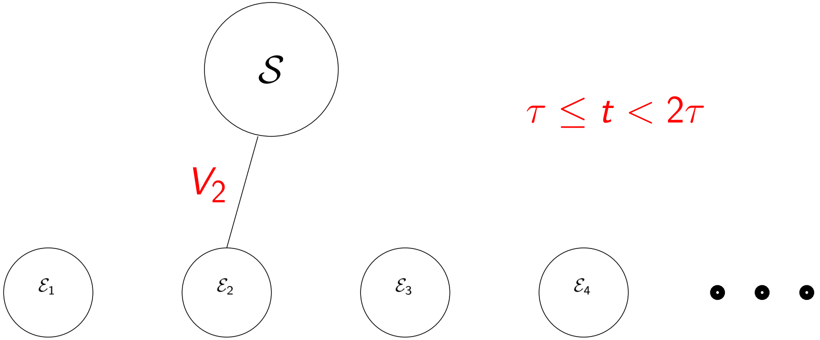

$\tau \leq t < 2\tau$

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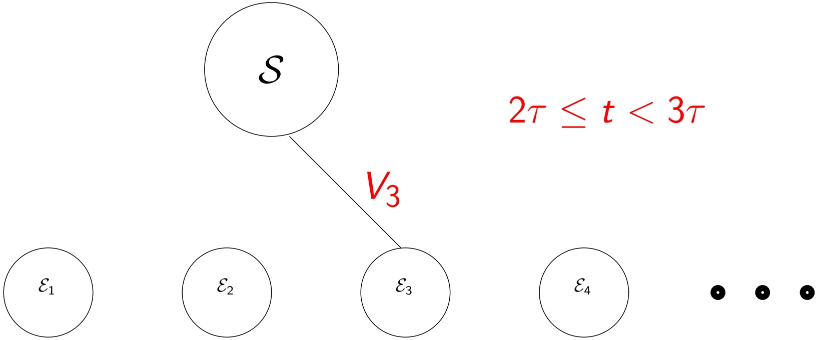

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For $t \in [(n-1)\tau, n\tau[$:

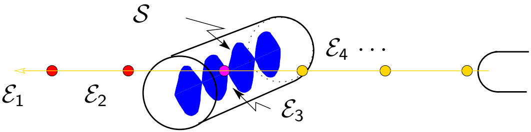
- $\mathcal{S}$ interacts with $\mathcal{E}_n$,
- $\mathcal{E}_k$ evolves freely for $k \neq n$,

i.e. the full system is governed by

$$\widetilde{H}_n = H_\mathcal{S} + H_{\mathcal{E},n} + V_n + \sum_{k \neq n} H_{\mathcal{E},k} = H_n + \sum_{k \neq n} H_{\mathcal{E},k}.$$

Some motivations

1. Physics: "One-atom masers" (Walther et al '85, Haroche et al '92)



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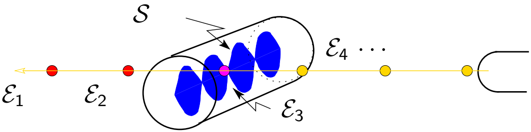

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2. Mathematics: Because of their particular structure (they are both Hamiltonian and Markovian), develop our understanding of open quantum systems, e.g. small system of infinite dimension, large coupling constant.

Mathematical model of the one-atom maser

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$$\mathcal{H}_{\mathcal{S}} = \Gamma_s(\mathbb{C}), \quad H_{\mathcal{S}} = \omega a^* a = \omega N.$$

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This is the Jaynes-Cummings hamiltonian (dipole interaction in the rotating-wave approximation).

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After 1 interaction, the state of the total system is

$$\rho_1^{\mathrm{tot}} := \qquad \mathrm{e}^{-i\tau H_1} \left(\rho \otimes \bigotimes_{k\geq 1} \rho_\beta\right) \mathrm{e}^{i\tau H_1}$$

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After 2 interactions, the state of the total system is

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The reduced dynamics map

If $\mathcal{S}$ is in the state ρ before some interaction, right after it it is in the state

$$\mathcal{L}_\beta(\rho) := \mathrm{Tr}_\mathcal{E}\left(\mathrm{e}^{-i\tau H}\rho\otimes\rho_\beta\,\mathrm{e}^{i\tau H}\right),$$

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Main difficulty: Perturbation theory doesn't work.
When $\lambda = 0$, $\mathcal{L}_\beta(\rho) = \mathrm{e}^{-i\tau H_S}\rho\,\mathrm{e}^{i\tau H_S}$. Hence
$\mathrm{sp}(\mathcal{L}_\beta) = \{\mathrm{e}^{i\omega\tau(n-m)},\ n,m\in\mathbb{Z}\}$: pure point spectrum, but all the eigenvalues, and in particular 1, are infinitely degenerate!

Jaynes-Cummings Hamiltonian and Rabi oscillations

If there are n photons in the cavity, the probability for the atom to make a transition $|-\rangle \to |+\rangle$ is a periodic function of time

$$P(t) = \left|\langle n-1,+|\, \mathrm{e}^{-itH}\, |n,-\rangle\right| = \left(1 - \frac{\Delta^2}{\nu_n^2}\right) \sin^2\left(\frac{\nu_n t}{2}\right),$$

with frequency

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Conclusion: If the field is in state $|n\rangle$ before an interaction and τ is a multiple of the Rabi period $T_n := \dfrac{2\pi}{\nu_n}$, after this interaction it can not be in state $|n-1\rangle$: there is a decoupling between the "energy levels" $n-1$ and n.

Rabi resonances

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3 possible situations (depending on the arithmetic properties of ξ and η): $R(\xi, \eta)$ is empty, a singlet or infinite.

Genericically: $R(\xi, \eta)$ is empty = no resonance. We now restrict (for the talk) to this non-resonant situation.

Ergodicity and mixing (I)

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Definition

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1. ergodic if for any $\mu \ll \rho$ $\mathrm{w}-\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mathcal{L}_\beta^n(\mu) = \rho$,

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1. ergodic if for any $\mu \ll \rho$ $\mathrm{w}-\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \mathcal{L}_\beta^n(\mu) = \rho$,
2. mixing if for any $\mu \ll \rho$ $\mathrm{w}-\lim_{n\to\infty} \mathcal{L}_\beta^n(\mu) = \rho$.

Ergodicity and mixing (I)

The support $s(\rho)$ of a state is the orthogonal projection on the closure of $\mathrm{Ran}(\rho)$.
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$$|\mathcal{L}_\beta^n(\mu)(A) - \rho(A)| \leq C_{A,\mu}\mathrm{e}^{-\alpha n}, \quad \forall n \in \mathbb{N}.$$

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To understand the ergodic properties of $\mathcal{L}_\beta$, the main issue is to understand its peripheral spectrum, i.e. $\mathrm{sp}(\mathcal{L}_\beta) \cap S^1$.
In particular, the invariant states are the possible ergodic states.

Spectral analysis of $\mathcal{L}_\beta$

1. Use gauge symmetry: $[H, a^*a + b^*b] = [H_\mathcal{E}, \rho_\beta] = 0$

$$\Rightarrow \quad \mathcal{L}_\beta(\mathrm{e}^{-i\theta a^*a} X \mathrm{e}^{i\theta a^*a}) = \mathrm{e}^{-i\theta a^*a} \mathcal{L}_\beta(X) \mathrm{e}^{i\theta a^*a}.$$

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Corollary: the subspaces $E_k = \{\rho = \sum_n p_n |n+k\rangle\langle n|\}$ of $\mathcal{J}_1$ are globally invariant.

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3. A Perron-Frobenius type lemma (Shrader '2000) for completely positive maps on trace ideals $\mathcal{J}_p$:

$$\mathcal{L}_\beta(X) = \mathrm{e}^{i\theta} X \;\Rightarrow \mathcal{L}_\beta(|X|) = |X| \text{ where } |X| = \sqrt{X^*X}.$$

Ergodicity and mixing (II)

Proposition

If $R(\xi, \eta) = \emptyset$, 1 is the only eigenvalue of $\mathcal{L}_\beta$ on S^1 and it is simple. The unique invariant state is

$$\rho_{\mathcal{S}}^{\beta^*} = \frac{\mathrm{e}^{-\beta^* H_{\mathcal{S}}}}{\mathrm{Tr}(\mathrm{e}^{-\beta^* H_{\mathcal{S}}})}.$$

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Remarks:

1) There is a weak form of decoherence.
2) Numerically it seems that $\rho_{\mathcal{S}}^{\beta^*}$ is not only ergodic but also mixing.
3) If $R(\xi,\eta)\neq\emptyset$ the multiplicity of 1 increases (one invariant state per "sector").

Quasi-resonances

Recall: for diagonal states

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where $D(n) = \frac{1}{1+\mathrm{e}^{-\beta\omega_0}} \sin^2(\pi\sqrt{\xi n + \eta}) \frac{\xi n}{\xi n + \eta}$.

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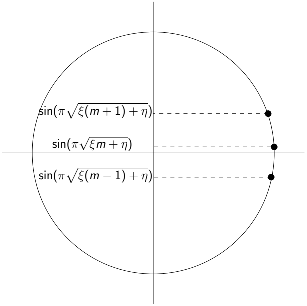
We call $m \in \mathbb{N}^*$ a quasi-resonance if $D(m) < D(m \pm 1)$.



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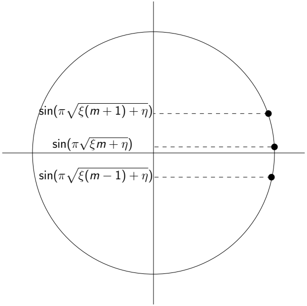

If $(m_k)_k$ denotes the sequence of quasi-resonances, we have $D(m_k) = O(k^{-2})$.

Metastable sates

Let $\mathcal{L}_\beta^0 = \mathbb{1} - \nabla^* D_0(N) \mathrm{e}^{-\beta\omega_0 N} \nabla \mathrm{e}^{\beta\omega_0 N}$ where

$$D_0(n) = \begin{cases} 0 & \text{if } n \in \{m_1, \ldots\}, \\ D(n) & \text{otherwise.} \end{cases}$$

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Then $\mathcal{L}_\beta = \mathcal{L}_\beta^0 + \mathcal{T}$ where $\mathcal{T}$ is of trace class and 1 is an infinitely degenerate eigenvalue of $\mathcal{L}_\beta^0$.
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The eigenstates of $\mathcal{L}_\beta^0$ are metastable states.
$\Rightarrow$ There are infinitely many metastable states with arbitrarily large lifetimes. Hence we can not expect exponential mixing.

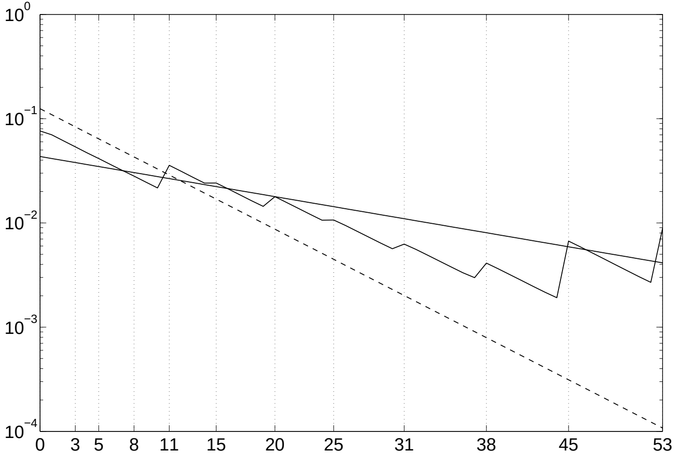
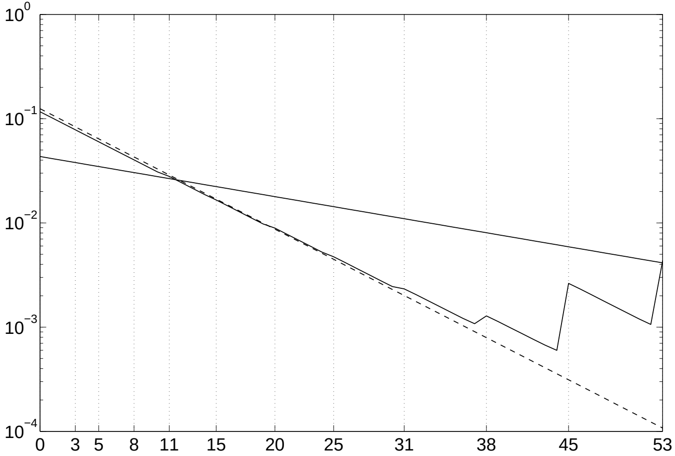


Figure: Cooling the cavity: 5000 interactions.



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Some questions

1. Prove mixing.
2. Estimate on the mixing rate?
3. Random interaction time $\Rightarrow$ convergence is better?
4. Non-equilibrium situation?